

Counting Plane Rational Curves: Old and New Approaches

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Abstract

These notes are intended as an easy-to-read supplement to some of the background material presented in my talks on enumerative geometry. In particular, the numbers n_3 and n_4 of plane rational cubics through eight points and of plane rational quartics through eleven points are determined via the classical approach of counting curves. The computation of the latter number also illustrates my topological approach to counting the zeros of a fixed vector bundle section that lie in the main stratum of a compact space. The arguments used in the computation of the number n_4 extend easily to counting plane curves with two or three nodes, for example. Finally, an inductive formula for the number n_d of plane degree- d rational curves passing through $3d-1$ points is derived via the modern approach of counting stable maps. This method is far simpler.

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1 Introduction

Enumerative geometry of algebraic varieties is a field of mathematics that dates back to the nineteenth century. The general goal of this subject is to determine the number of geometric objects that satisfy pre-specified geometric conditions. The objects are typically (complex) curves in a smooth algebraic manifold. Such curves are usually required to represent the given homology class, to have certain singularities, and to satisfy various contact conditions with respect to a collection of subvarieties. One of the most well-known examples of an enumerative problem is

Question 1.1 *If d is a positive integer, what is the number n_d of degree- d rational curves that pass through $3d-1$ points in general position in the complex projective plane \mathbb{P}^2 ?*

Since the number of (complex) lines through any two distinct points is one, $n_1 = 1$. A little bit of algebraic geometry and topology gives $n_2 = 1$ and $n_3 = 12$; see Section 2. It is far harder to find that $n_4 = 620$, but this number was computed as early as the middle of the nineteenth century; see [Ze, p378]. We give a “classical-style” computation of this number in Section 3. Along the way, we determine the number of plane quartics that pass through 12 points and have two nodes and the number of plane quartics that pass through 11 points and have a cusp and a simple node; see Table 1. The derivations of Subsections 3.2-3.4 easily extend to counting arbitrary-degree plane curves with two nodes, a node and a cusp, and with three nodes; see Table 2 for explicit formulas. These curves are of course not rational in general. Subsections 3.3 and 3.4 also illustrate our approach to determining the number of zeros of a fixed vector bundle section that lie in the main stratum of a space. This approach is one of the two main tools that we have applied to a number of enumerative problems; see [Z1] and [Z2], for example.

The higher-degree numbers n_d remained unknown until the early 1990s, when a recursive formula for the numbers n_d was announced in [KoMa] and [RuT]:

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2 \frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}. \quad (1.1)$$

We describe the argument of the latter paper in Section 4. It can also be used to solve the natural generalization of Question 1.1 to the higher-dimensional projective spaces; see Section 10 in [RuT].

Remark: A derivation of (1.1), which is classical in spirit, appears in [Ra2] and is based on [Ra1]. The approach of Section 3 is more direct and involves no blowups.

Subsection 2.3 and Section 3, which are not used in Section 4, assume some familiarity with cohomology groups and chern classes. All other non-elementary terms, including those used in Question 1.1, are described in Appendix A. A different (and far more extensive) introduction to enumerative geometry, as well as to its relations with physics, is given in [Ka].

2 The Low-Degree Numbers

2.1 The Degree-One Number

We start by computing the number n_1 topologically. Throughout these notes, we will use the homogeneous coordinates $[X, Y, Z]$ on the complex projective plane of Question 1.1, i.e. we take

$$\mathbb{P}^2 = \{(X, Y, Z) \in \mathbb{C}^3 : (X, Y, Z) \neq (0, 0, 0)\} / \mathbb{C}^* = \{[X, Y, Z] : (X, Y, Z) \in \mathbb{C}^3 - (0, 0, 0)\}.$$

In this section, we use the following lemma.

Lemma 2.1 *If $\gamma \rightarrow \mathbb{P}^2$ is the tautological line bundle, d is positive integer, and $s \in \Gamma(\mathbb{P}^2; \gamma^{*\otimes d})$ is transverse to the zero set, the set $s^{-1}(0)$ is a smooth two-dimensional submanifold of \mathbb{P}^2 of genus*

$$g(s^{-1}(0)) = \binom{d-1}{2}.$$

This lemma is proved in Subsection A.3. It can easily be verified directly in the $d=1$ and $d=2$ cases.

A line, or degree-one curve, in \mathbb{P}^2 is the quotient by the \mathbb{C}^* -action of the zero set of a nonzero homogeneous polynomial

$$s_{a_{100}, a_{010}, a_{001}} \equiv a_{100}X + a_{010}Y + a_{001}Z$$

of degree one on $\mathbb{C}^3 - \{0\}$. In other words, a degree-one curve in \mathbb{P}^2 has the form

$$\mathcal{C} = \mathcal{C}_{a_{100}, a_{010}, a_{001}} = \{[X, Y, Z] \in \mathbb{P}^2 : a_{100}X + a_{010}Y + a_{001}Z = 0\}$$

for some $(a_{100}, a_{010}, a_{001}) \in \mathbb{C}^3 - \{0\}$. Furthermore,

$$\mathcal{C}_{a_{100}, a_{010}, a_{001}} = \mathcal{C}_{b_{100}, b_{010}, b_{001}} \iff (a_{100}, a_{010}, a_{001}) = \lambda(b_{100}, b_{010}, b_{001}) \text{ for some } \lambda \in \mathbb{C}^*.$$

Thus, the space of all degree-one curves in \mathbb{P}^2 is

$$\mathcal{D}_1 = \{(a_{100}, a_{010}, a_{001}) : (a_{100}, a_{010}, a_{001}) \neq (0, 0, 0)\} / \mathbb{C}^* \approx \mathbb{P}^2.$$

A homogeneous polynomial $s = a_{100}X + a_{010}Y + a_{001}Z$ of degree one on \mathbb{C}^3 determines a section $s_{a_{100}, a_{010}, a_{001}}$ of the bundle $\gamma^* \rightarrow \mathbb{P}^2$. If $(a_{100}, a_{010}, a_{001}) \neq (0, 0, 0)$, this section is transverse to

the zero set. Thus, by Lemma 2.1, for all $[a_{100}, a_{010}, a_{001}] \in \mathcal{D}_1$ the genus of $\mathcal{C}_{a_{100}, a_{010}, a_{001}}$ is zero, i.e. this is a rational curve.

Finally, let $p_1 = [X_1, Y_1, Z_1]$ and $p_2 = [X_2, Y_2, Z_2]$ be two distinct points in \mathbb{P}^2 . The curve $\mathcal{C}_{a_{100}, a_{010}, a_{001}}$ passes through the point p_i if and only if $s_{a_{100}, a_{010}, a_{001}}(p_i) = 0$. Thus, the number n_1 is the number of elements $[a_{100}, a_{010}, a_{001}] \in \mathcal{D}_1$ such that

$$\begin{cases} a_{100}X_1 + a_{010}Y_1 + a_{001}Z_1 = 0; \\ a_{100}X_2 + a_{010}Y_2 + a_{001}Z_2 = 0. \end{cases} \quad (2.1)$$

The solution of each of these equations on \mathcal{D}_1 is a line. Since $[X_1, Y_1, Z_1] \neq [X_2, Y_2, Z_2]$, the two lines are distinct. Since two lines in a plane, or \mathbb{P}^2 , intersect in a single point, $n_1 = 1$. Stated differently, $n_1 = 1$ because the space of solutions of the system (2.1) in $(a_{100}, a_{010}, a_{001}) \in \mathbb{C}^3$ is a line through the origin.

2.2 The Degree-Two Number

The computation of the number n_2 is very similar. A degree-two curve in \mathbb{P}^2 is described by a nonzero degree-two homogeneous polynomial

$$s_{a_{2,0,0}, a_{1,1,0}, a_{1,0,1}, a_{0,2,0}, a_{0,1,1}, a_{0,0,2}} = \sum_{j+k+l=2} a_{jkl} X^j Y^k Z^l.$$

Thus, the space of degree-two curves in \mathbb{P}^2 is

$$\mathcal{D}_2 = \{(a_{2,0,0}, a_{1,1,0}, a_{1,0,1}, a_{0,2,0}, a_{0,1,1}, a_{0,0,2}) \in \mathbb{C}^6 - \{0\}\} / \mathbb{C}^* \approx \mathbb{P}^5.$$

If $p_i = [X_i, Y_i, Z_i]$ for $i = 1, \dots, 5$ are five points in \mathbb{P}^2 , the subset of conics that pass through these points is the set of elements $[(a_{jkl})_{j+k+l=2}] \in \mathcal{D}_2$ such that

$$\sum_{j+k+l=2} a_{jkl} X_i^j Y_i^k Z_i^l = 0 \quad \text{for } i = 1, \dots, 5. \quad (2.2)$$

Each of these five linear equations determines a hyperplane H_i in \mathcal{D}_2 .

We assume that the five points p_i do not lie on any pair of lines in \mathbb{P}^2 . Then by Lemma 2.1, every conic passing through the five points p_i is smooth and of genus zero. It follows that any two distinct conics \mathcal{C}_1 and \mathcal{C}_2 passing through the five points p_i must intersect at most $2 \cdot 2 = 4$ points; see Lemma A.5. Thus, the system (2.2) of five equations must have at most one solution \mathcal{D}_2 , and such a solution represents a plane rational conic through the five points in \mathbb{P}^2 . On the other hand, the five hyperplanes H_i in \mathcal{D}_2 must have at least a point in common, since the poincare dual of a hyperplane generates $H^*(\mathbb{P}^n; \mathbb{Z})$. In simpler terms, the solution space of the system (2.2) of five linear homogeneous equations on \mathbb{C}^6 must contain a line through the origin. We conclude that $n_2 = 1$.

2.3 The Degree-Three Number

Computing the number n_3 requires a bit more care. Similarly to the previous two subsections, the space of cubics in \mathbb{P}^2 is described by

$$\mathcal{D}_3 = \{(a_{jkl})_{j+k+l=3} \in \mathbb{C}^{10} - \{0\}\} / \mathbb{C}^* \approx \mathbb{P}^9.$$

For a generic $\underline{a} \in \mathcal{D}_3$, the section $s_{\underline{a}}$ of the bundle $\gamma^{*\otimes 3} \rightarrow \mathbb{P}^2$ is transverse to the zero set. Thus, by Lemma 2.1, a typical cubic is smooth and of genus one, not zero.

Let $p_i = [X_i, Y_i, Z_i]$ for $i = 1, \dots, 8$ be eight points in \mathbb{P}^2 that do not lie on the union of any line and any conic in \mathbb{P}^2 . It can then be shown that if the cubic $\mathcal{C}_{\underline{a}}$ passes through these eight points, the section $s_{\underline{a}}$ has at most one singular point. In such a case, the curve $\mathcal{C}_{\underline{a}}$ is a sphere with two points identified. In other words, a circle on a torus collapses to a point. This fact is immediate from the algebraic-geometry point of view, but can also be checked directly. Thus, the number n_3 is the number of plane cubics that pass through the eight points p_1, \dots, p_8 and have a singular point. This singular point will be a simple node; see Figure 1 on page 8.

As in the previous two subsections, the space H_i of elements $\underline{a} \in \mathcal{D}_3$ such that $p_i \in \mathcal{C}_{\underline{a}}$ is a hyperplane. With our assumption on the eight points, the eight hyperplanes intersect transversally, and thus

$$\mathcal{D} \equiv \bigcap_{i=1}^{i=8} H_i \approx \mathbb{P}^1.$$

In simpler words, the eight equations analogous to (2.2) are linearly independent. Thus, the space of solution of the corresponding system of equations on \mathbb{C}^{10} is a plane through the origin, which corresponds to a line \mathbb{P}^1 in $\mathcal{D}_3 \approx \mathbb{P}^9$.

By the above, we need to determine the cardinality of the set

$$\mathcal{Z} = \{([\underline{a}], x) \in \mathcal{S} : ds_{\underline{a}}|_x = 0\}, \quad \text{where } \mathcal{S} = \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^2 : s_{\underline{a}}(x) = 0\}.$$

An element of the subspace \mathcal{S} of $\mathcal{D} \times \mathbb{P}^2$ is a cubic through the eight points p_1, \dots, p_8 with a choice of a point on it. Such an element $([\underline{a}], x)$ lies in \mathcal{Z} if $s_{\underline{a}}$ is not transverse to the zero set at x .

Let $\pi_0, \pi_1 : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{D}, \mathbb{P}^2$ be the two projection maps. If $\gamma_{\mathcal{D}} \rightarrow \mathcal{D}$ and $\gamma_{\mathbb{P}^2} \rightarrow \mathbb{P}^2$ are the tautological line bundles, we set

$$\gamma_0 = \pi_0^* \gamma_{\mathcal{D}} \rightarrow \mathcal{D} \times \mathbb{P}^2 \quad \text{and} \quad \gamma_1 = \pi_1^* \gamma_{\mathbb{P}^2} \rightarrow \mathcal{D} \times \mathbb{P}^2.$$

A homogeneous polynomial in three variables of degree d induces a section of the bundle $\gamma^{*\otimes d} \rightarrow \mathbb{P}^2$. For the same reason, the map

$$\{\underline{a} \in \mathbb{C}^2 : [\underline{a}] \in \mathcal{D}\} \times \mathbb{P}^2 \rightarrow \gamma_{\mathbb{P}^2}^{*\otimes 3}, \quad (\underline{a}, x) \rightarrow s_{\underline{a}}(x),$$

induces a section ψ_0 of the line bundle $\gamma_0^* \otimes \gamma_1^{*\otimes 3} \rightarrow \mathcal{D} \times \mathbb{P}^2$. This section is transverse to the zero set. Thus, $\mathcal{S} = \psi_0^{-1}(0)$ is a smooth submanifold of $\mathcal{D} \times \mathbb{P}^2$; see Lemma 2.2 below.

If $([\underline{a}], x) \in \mathcal{S}$, $s_{\underline{a}}(x) = 0$, and thus $ds_{\underline{a}}|_x$ is well-defined. The map

$$\{([\underline{a}], x) \in \mathcal{S} : ds_{\underline{a}}|_x = 0\} \rightarrow \gamma_0^* \otimes \gamma_1^{*\otimes 3} \otimes T^* \mathbb{P}^2, \quad (\underline{a}, x) \rightarrow ds_{\underline{a}}|_x,$$

induces a section ψ_1 of the vector bundle $\gamma_0^* \otimes \gamma_1^{*\otimes 3} \otimes \pi_1^* T^* \mathbb{P}^2 \rightarrow \mathcal{S}$. This section is transverse to the zero set. Thus, by Lemma 2.2,

$$\begin{aligned} n_3 &= |\mathcal{Z}| = |\psi_1^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^{*\otimes 3} \otimes \pi_1^* T^* \mathbb{P}^2), [\mathcal{S}] \rangle \\ &= \langle c_2(\gamma_0^* \otimes \gamma_1^{*\otimes 3} \otimes \pi_1^* T^* \mathbb{P}^2) \text{PD}_{\mathcal{D} \times \mathbb{P}^2}([\mathcal{S}]), [\mathcal{D} \times \mathbb{P}^2] \rangle \\ &= \langle (3ya + 3a^2)(y + 3a), [\mathcal{D} \times \mathbb{P}^2] \rangle = 12, \end{aligned}$$

where $y = \pi_0^* c_1(\gamma_{\mathcal{D}}^*)$ and $a = \pi_1^* c_1(\gamma_{\mathbb{P}^2}^*)$.

Lemma 2.2 *If M is a compact oriented manifold, $V \rightarrow M$ is an oriented vector bundle, and $\psi \in \Gamma(M; V)$ is transverse to the zero set, the space $\psi^{-1}(0)$ is a smooth oriented submanifold of M and*

$$PD_M([\psi^{-1}(0)]) = e(V) \in H^*(M; \mathbb{Z}),$$

where $e(V)$ is the euler class of V .

This lemma is a standard fact in differential topology; see Sections 9-12 of [MiSt]. It implies that if the dimension of M and the rank of V are the same, the set $s^{-1}(0)$ is finite and its signed cardinality is given by

$$\pm |s^{-1}(0)| = \langle e(V), [M] \rangle.$$

In fact, this is the only case of Lemma 2.2 we would have needed if we extended the section ψ_1 over the entire space $\mathcal{D} \times \mathbb{P}^2$ by using the canonical connection of the hermitian holomorphic vector bundle $\gamma \rightarrow \mathbb{P}^2$; see [GriH].

3 The Degree-Four Number

3.1 Summary

In this section we use the general approach of Subsection 2.3 to compute the number n_4 . Since the genus of a smooth plane quartic is three by Lemma 2.1, we will need to determine the number of quartics that pass through 11 points in \mathbb{P}^2 and have three nodes. This number is one-sixth the cardinality of the set

$$\tilde{\mathcal{N}}_3 \equiv \{([\underline{a}], x_1, x_2, x_3) \in \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2 : x_i \neq x_j \ \forall i \neq j; \ s_{\underline{a}}(x_i) = 0, \ ds_{\underline{a}}|_{x_i} = 0 \ \forall i = 1, 2, 3\},$$

where $\mathcal{D} \approx \mathbb{P}^3$ is the space of quartics that pass through the eleven chosen points and $\mathbb{P}_i^2 = \mathbb{P}^2$.

Similarly to Subsection 2.3, each of the sections

$$\varphi_i([\underline{a}], x_i) = (s_{\underline{a}}(x_i), ds_{\underline{a}}|_{x_i}) \in \gamma_0^* \otimes \gamma_i^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_i^{*\otimes 4} \otimes T\mathbb{P}_i^2$$

is transverse to the zero set over $\mathcal{D} \times \mathbb{P}_i^2$. However, the section

$$\varphi \equiv \varphi_1 \oplus \varphi_2 \oplus \varphi_3$$

is not transverse to the zero set over

$$M \equiv \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2.$$

For example, the zero set of φ contains the two-dimensional space

$$\{([\underline{a}], x, x, x) : s_{\underline{a}}(x) = 0, \ ds_{\underline{a}}|_x = 0\}.$$

Thus, $|\tilde{\mathcal{N}}_3|$ is not the euler class of the bundle

$$V \equiv \bigoplus_{i=1}^{i=3} (\gamma_0^* \otimes \gamma_i^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_i^{*\otimes 4} \otimes T^*\mathbb{P}_i^2) \rightarrow M \equiv \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P}_2^2 \times \mathbb{P}_3^2.$$

set	singularities	#pts	card.
\mathcal{N}_1	1 node	13	27
$\mathcal{N}_{1,1}$	1 node on a fixed line	12	9
\mathcal{K}_1	1 cusp	12	72
$\mathcal{K}_{1,1}$	1 cusp on a fixed line	11	20
\mathcal{T}_1	1 tacnode	11	200
\mathcal{N}_2	2 nodes	12	225
$\mathcal{N}_{2,1}$	2 nodes, one on a fixed line	11	170
\mathcal{K}_2	1 node and 1 cusp	11	840
\mathcal{N}_3	3 nodes	11	675

Table 1: Some Characteristic Numbers of Plane Quartics

On the other hand, φ is transverse to the zero set over the “main stratum” of M :

$$M^0 \equiv \{([\underline{a}], x_1, x_2, x_3) \in M : x_i \neq x_j \ \forall i \neq j\}.$$

Thus, $|\tilde{\mathcal{N}}_3|$ is the euler class of the bundle V minus the φ -contribution to $e(V)$ from the “boundary” of M :

$$|\tilde{\mathcal{N}}_3| = \langle e(V), M \rangle - \mathcal{C}_{\partial M}(\varphi), \quad \text{where} \quad \partial M = M - M^0.$$

The number $\mathcal{C}_{\partial M}(\varphi)$ is the signed number of zeros of the bundle section $\varphi + \nu$, for a small generic perturbation ν , that lie near ∂M . If $\partial M = \sqcup_i \mathcal{Z}_i$ is a stratification of ∂M ,

$$\mathcal{C}_{\partial M}(\varphi) = \sum_i \mathcal{C}_{\mathcal{Z}_i}(\varphi).$$

If this stratification is sufficiently fine, each of the numbers $\mathcal{C}_{\mathcal{Z}_i}(\varphi)$ is a certain multiple of the number of zeros of an affine bundle map between vector bundles over $\tilde{\mathcal{Z}}_i$. The latter number can be computed through a reductive procedure, described in detail in [Z1] and [Z2] and implemented in the relevant cases in Subsections 3.3 and 3.4 below.

In order to simplify the computation of $|\tilde{\mathcal{N}}_3|$, we will essentially be adding one point at a time. This computation will require knowing the numbers of plane quartics with various one- and two-point singularities. These numbers, along with $|\mathcal{N}_3|$, are given in Table 1. For example, according to this table, the cardinality of the set $\mathcal{N}_{2,1}$ of plane quartics that pass through 11 points in general position and have two nodes, one of which lies on a fixed general line, is 170. Figure 1 shows a simple node, a simple cusp, and a simple tacnode. If s is a section of $\gamma^{*\otimes d}$ and $x \in s^{-1}(0)$ is a node of $s^{-1}(0)$, then $ds|_x = 0$. We describe the analogous cuspidal and tacnodal condition on s in the next subsection. All numbers in Table 1 are computed in Subsections 3.2-3.4.

Finally, we note that a plane quartic that has 3 nodes and passes through 11 points is either irreducible, in which case it is rational, or a union of a smooth cubic, passing through 9 of the points, and a line, passing through the remaining 2 points. By the same argument as in Subsections 2.1 and 2.2, the number of plane cubics passing through 9 points in general position is 1. Thus, by the

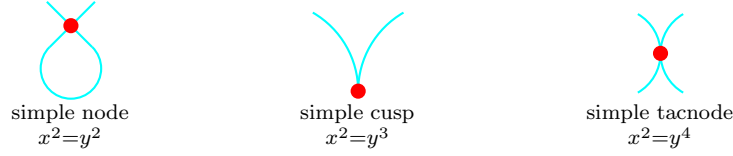


Figure 1: Simple Node, Simple Cusp, and Simple Tacnode

last row of Table 1, the number of *rational* quartics passing through 11 points in general position in \mathbb{P}^2 is

$$n_4 = 675 - \binom{11}{2} \cdot 1 \cdot 1 = 620.$$

The computations of Subsections 3.2-3.4 generalize easily to plane curves of arbitrary degree, essentially by replacing 4 by d everywhere. The results and the arguments are summarized in Table 2 and in Subsection 3.5, respectively.

3.2 Quartics with One Singular Point

Throughout the rest of Section 3, we denote by p_1, \dots, p_{13} thirteen points in general position in \mathbb{P}^2 and by $\mathcal{D}_4 \approx \mathbb{P}^{14}$ the space of plane quartics. In this subsection, we compute the first five numbers in Table 1.

Lemma 3.1 *The number $|\mathcal{N}_1|$ of plane quartics that have a node and pass through 13 points in general position is 27. The number $|\mathcal{N}_{1,1}|$ of plane quartics that have a node on a fixed general line and pass through 12 points in general position is 9.*

Proof: (1) Let $\mathcal{D} \approx \mathbb{P}^1 \subset \mathcal{D}_4$ denote the subspace of plane quartics that pass through the points p_1, \dots, p_{13} . With notation as in Subsection 2.3, let

$$\begin{aligned} \mathcal{N}_1 &= \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^2 : \varphi([\underline{a}], x) = 0\}, \quad \text{where} \\ \varphi &\in \Gamma(\mathcal{D} \times \mathbb{P}^2; \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), \quad \varphi([\underline{a}], x) = (s_{\underline{a}}(x), ds_{\underline{a}}|_x). \end{aligned}$$

Since the section φ is transverse to the zero set, by Lemma 2.2,

$$\begin{aligned} |\mathcal{N}_1| &= |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), \mathcal{D} \times \mathbb{P}^2 \rangle \\ &= \langle c_1(\gamma_0^* \otimes \gamma_1^{*\otimes 4}) c_2(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), \mathcal{D} \times \mathbb{P}^2 \rangle \\ &= \langle (y+4a)(y^2+5ya+7a^2), \mathcal{D} \times \mathbb{P}^2 \rangle = 27. \end{aligned}$$

(2) Let $\mathcal{D} \approx \mathbb{P}^2 \subset \mathcal{D}_4$ denote the subspace of plane quartics that pass through the points p_1, \dots, p_{12} . Let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a general line in \mathbb{P}^2 . We put

$$\begin{aligned} \mathcal{N}_{1,1} &= \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^1 : \varphi([\underline{a}], x) = 0\}, \quad \text{where} \\ \varphi &\in \Gamma(\mathcal{D} \times \mathbb{P}^1; \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2|_{\mathbb{P}^1}), \quad \varphi([\underline{a}], x) = (s_{\underline{a}}(x), ds_{\underline{a}}|_x). \end{aligned}$$

Since the section φ is transverse to the zero set, by Lemma 2.2,

$$\begin{aligned} |\mathcal{N}_{1,1}| &= |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), \mathcal{D} \times \mathbb{P}^1 \rangle \\ &= \langle c_1(\gamma_0^* \otimes \gamma_1^{*\otimes 4})c_2(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), \mathcal{D} \times \mathbb{P}^1 \rangle \\ &= \langle (y+4a)(y^2+5ya+7a^2), \mathcal{D} \times \mathbb{P}^1 \rangle = 9. \end{aligned}$$

Lemma 3.2 *The number $|\mathcal{K}_1|$ of plane quartics that have a cusp and pass through 12 points in general position is 72. The number $|\mathcal{K}_{1,1}|$ of plane quartics that have a cusp on a fixed general line and pass through 11 points in general position is 20.*

Proof: (1) Let $\mathcal{D} \approx \mathbb{P}^2$ be as in (2) of the proof of Lemma 3.1. We put

$$\mathcal{N}'_1 = \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^2 : s_{\underline{a}}(x) = 0, ds_{\underline{a}}|_x = 0\}.$$

If $([\underline{a}], x) \in \mathcal{N}'_1$, we denote by

$$H_{\underline{a},x} \in \Gamma(\mathcal{N}'_1; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2))$$

the Hessian of $s_{\underline{a}}$ at x , i.e. the total second derivative of $s_{\underline{a}}$ at x . Let

$$\begin{aligned} \mathcal{K}_1 &= \{([\underline{a}], x) \in \mathcal{N}'_1 : \varphi([\underline{a}], x) = 0\}, \quad \text{where} \\ \varphi &\in \Gamma(\mathcal{N}'_1; (\gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}), \quad \varphi([\underline{a}], x) = \det H_{\underline{a},x}. \end{aligned}$$

Since the section φ is transverse to the zero set, by Lemma 2.2,

$$\begin{aligned} |\mathcal{K}_1| &= |\varphi^{-1}(0)| = \langle e((\gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}), \mathcal{N}'_1 \rangle \\ &= 2\langle y+a, \mathcal{N}'_1 \rangle = 2(|\mathcal{N}'_1| + |\mathcal{N}_{1,1}|) = 2(27+9) = 72. \end{aligned}$$

(2) Similarly, let $\mathcal{D} \approx \mathbb{P}^3 \subset \mathcal{D}_4$ denote the subspace of plane quartics that pass through the points p_1, \dots, p_{11} . Let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a general line in \mathbb{P}^2 . We put

$$\begin{aligned} \mathcal{N}'_{1,1} &= \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^1 : s_{\underline{a}}(x) = 0, ds_{\underline{a}}|_x = 0\}; \\ \mathcal{K}_{1,1} &= \{([\underline{a}], x) \in \mathcal{N}'_{1,1} : \det H_{\underline{a},x} = 0\}. \end{aligned}$$

Then, by Lemma 2.2,

$$\begin{aligned} |\mathcal{K}_{1,1}| &= \langle e((\gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}), \mathcal{N}'_{1,1} \rangle \\ &= 2\langle y+a, \mathcal{N}'_{1,1} \rangle = 2(|\mathcal{N}'_{1,1}| + \langle a, \mathcal{N}'_{1,1} \rangle) = 2(9+1) = 20. \end{aligned}$$

Note the number $\langle a, \mathcal{N}'_{1,1} \rangle$ of plane quartics that pass through 11 points and have a node at a fixed twelfth point is 1, since all conditions on $\underline{a} \in \mathcal{D}_4$ are linear, as in Subsections 2.1 and 2.2.

Lemma 3.3 *The number $|\mathcal{T}_1|$ of plane quartics that have a tacnode and pass through 11 points in general position is 200.*

Proof: Let $\mathcal{D} \approx \mathbb{P}^3$ be as in (2) of the proof of Lemma 3.2. We put

$$\mathcal{N}''_1 = \{([\underline{a}], x) \in \mathcal{D} \times \mathbb{P}^2 : s_{\underline{a}}(x) = 0, ds_{\underline{a}}|_x = 0\}, \quad M = \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}''_1}.$$

We denote by $\gamma \longrightarrow M$ the tautological line bundle and by

$$\tilde{H}_{\cdot, \cdot} \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2))$$

the bundle map induced by $H_{\cdot, \cdot}$. Let

$$\begin{aligned} \mathcal{K}'_1 &= \{([\underline{a}], x) \in M : \tilde{H}_{\underline{a}, x} = 0\}, & \mathcal{T}_1 &= \{([\underline{a}], x) \in \mathcal{K}'_1 : \varphi(\underline{a}, x) = 0\}, \\ \text{where } \varphi &\in \Gamma(M; \text{Hom}(\gamma^{\otimes 3}, \gamma_0^* \otimes \gamma_1^{*\otimes 4})), & \varphi([\underline{a}], x) &= \mathcal{D}_{\underline{a}, x}^3, \end{aligned}$$

and $\mathcal{D}_{\underline{a}, x}^3$ is the third derivative of $s_{\underline{a}}$ at x . Let $\lambda = c_1(\gamma^*)$. Since the sections φ and $\tilde{H}_{\cdot, \cdot}$ are transverse to the zero set, by Lemma 2.2,

$$\begin{aligned} |\mathcal{T}_1| &= |\varphi^{-1}(0)| = \langle e(\gamma^{*\otimes 3} \otimes \gamma_0^* \otimes \gamma_1^{*\otimes 4}), \mathcal{K}'_1 \rangle \\ &= \langle e(\gamma^{*\otimes 3} \otimes \gamma_0^* \otimes \gamma_1^{*\otimes 4})e(\gamma^* \otimes \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2), M \rangle \\ &= \langle 3\lambda^3 + (7y+19a)\lambda^2 + (5y^2+28ya+41a^2)\lambda, M \rangle \\ &= \langle 5y^2 + 7ya + 2a^2, \mathcal{N}'_1 \rangle \\ &= 5|\mathcal{N}_1| + 7|\mathcal{N}_{1,1}| + 2\langle a, \mathcal{N}'_{1,1} \rangle = 5 \cdot 27 + 7 \cdot 9 + 2 \cdot 1 = 200. \end{aligned}$$

3.3 Quartics with Two Singular Points

In this subsection, we compute the three numbers of Table 1 that involve two-point singularities. As the relevant bundle sections are no longer transverse everywhere, each of these numbers is the euler class of the corresponding vector bundle minus the contribution from the ‘‘boundary’’ for the given bundle section.

Suppose $E, V \longrightarrow M$ are vector bundle such that $\dim M + \text{rk } E = \text{rk } V$ and

$$\alpha \in \Gamma(M; \text{Hom}(E, V)).$$

If $\nu \in \Gamma(M; V)$ is a generic section, the affine bundle map

$$\psi_{\alpha, \nu}: E \longrightarrow V, \quad \psi_{\alpha, \nu}(m; e) = \alpha(m; e) + \nu(m),$$

has a finite number of transverse zeros. By Lemma 3.14 in [Z1] and Proposition 2.18A in [Z2], the signed cardinality of $\psi_{\alpha, \nu}^{-1}(0)$ is independent of the choice of ν . We denote this cardinality by $N(\alpha)$.

Lemma 3.4 *The number $|\mathcal{N}_2|$ of plane quartics that have two nodes and pass through 12 points in general position is 225. The number $|\mathcal{N}_{2,1}|$ of plane quartics that have two nodes, one of which lies on a fixed general line, and pass through 11 points in general position is 170.*

Proof: (1) Let $\mathcal{N}'_1 \subset \mathcal{D} \times \mathbb{P}_1^2$ be defined as in (1) of the proof of Lemma 3.2. We put

$$M = \mathcal{N}'_1 \times \mathbb{P}_2^2, \quad M^0 = \{([\underline{a}], x_1, x_2) \in M : x_1 \neq x_2\}, \quad \partial M = M - M^0, \quad \tilde{\mathcal{N}}_2 = \varphi^{-1}(0) \cap M^0,$$

$$\text{where } \varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}_2^2), \quad \varphi([\underline{a}], x_1, x_2) = (s_{\underline{a}}(x_2), ds_{\underline{a}}|_{x_2}), \quad \gamma_2 = \pi_2^* \gamma_{\mathbb{P}_2^2},$$

and $\pi_2: M \longrightarrow \mathbb{P}_2^2$ is the projection onto the last component. Since $\varphi|_{M^0}$ is transverse to the zero set,

$$\begin{aligned} |\tilde{\mathcal{N}}_2| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}_2^2), M \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= \langle (y+4a_2)(y^2+5ya_2+7a_2^2), \mathcal{N}'_1 \times \mathbb{P}_2^2 \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 27\langle y, \mathcal{N}'_1 \rangle - \mathcal{C}_{\partial M}(\varphi) = 27|\mathcal{N}_1| - \mathcal{C}_{\partial M}(\varphi) = 27 \cdot 27 - \mathcal{C}_{\partial M}(\varphi), \end{aligned} \tag{3.1}$$

where $a_2 = \pi_2^* c_1(\gamma_{\mathbb{P}^2}^*)$. In order to determine $\mathcal{C}_{\partial M}(\varphi)$, we split ∂M into two strata:

$$\mathcal{Z}_1 = \{([\underline{a}], x, x) : ([\underline{a}], x) \in \mathcal{N}'_1 - \mathcal{K}_1\}, \quad \mathcal{Z}_0 = \{([\underline{a}], x, x) : ([\underline{a}], x) \in \mathcal{K}_1\}.$$

With appropriate identifications, for some $C \in C(\mathcal{N}'_1; \mathbb{R}^+)$,

$$|\varphi([\underline{a}], x, v) - H_{\underline{a}, x} v| \leq C([\underline{a}], x) |v|^2 \quad \forall ([\underline{a}], x, x) \in \partial M, \quad v \in \text{Norm}_M \partial M|_{([\underline{a}], x, x)} \approx T_x \mathbb{P}^2. \quad (3.2)$$

By definition of the set \mathcal{K}_1 ,

$$|H_{\underline{a}, x} v| \geq C([\underline{a}], x)^{-1} |v| \quad \forall ([\underline{a}], x) \in \mathcal{N}'_1 - \mathcal{K}_1, \quad v \in T_x \mathbb{P}^2. \quad (3.3)$$

By (3.2), (3.3), and a rescaling and cobordism argument as in Subsection 3.1 of [Z1],

$$\begin{aligned} \mathcal{C}_{\mathcal{Z}_1}(\varphi) &= N(\alpha), \quad \text{where} \\ \alpha &\in \Gamma(\mathcal{N}'_1; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\underline{a}], x; v) = (0, H_{\underline{a}, x} v). \end{aligned} \quad (3.4)$$

On the other hand, suppose $([\underline{a}], x) \in \mathcal{K}_1$. We denote by $\mathcal{L}_{([\underline{a}], x)} \subset T\mathbb{P}^2$ the kernel of $H_{\underline{a}, x}$ and by $\mathcal{L}_{([\underline{a}], x)}^\perp$ its orthogonal complement. Let $N_{([\underline{a}], x)}$ be the normal bundle of \mathcal{K}_1 in \mathcal{N}'_1 at $([\underline{a}], x)$. Then, with appropriate identifications, for some $\beta_2, \beta_3 \in \mathbb{C}^*$, $\beta_4 \in \mathbb{C}$, and $C \in \mathbb{R}^+$,

$$\begin{aligned} |\varphi([\underline{a}], x; u, v, w) - \alpha_0(u, v, w)| &\leq C(|v|^4 + |w|^2) \quad \forall u \in N_{([\underline{a}], x)}, \quad v \in \mathcal{L}_{([\underline{a}], x)}, \quad w \in \mathcal{L}_{([\underline{a}], x)}^\perp, \\ \text{where } \alpha_0(u, v, w) &= \left(\frac{1}{2}uv^2 + \frac{1}{3}\beta_3v^3, uv + \beta_3v^2 + \beta_4v^3, \beta_2w\right). \end{aligned} \quad (3.5)$$

Here β_2 is the second derivative of $s_{\underline{a}}$ at x along $\mathcal{L}_{([\underline{a}], x)}^\perp$ and $2\beta_3$ is the third derivative of $s_{\underline{a}}$ at x along $\mathcal{L}_{([\underline{a}], x)}$. Since the polynomial α_0 is three-to-one near the origin, it follows from (3.5) that each point of $\mathcal{Z}_0 \approx \mathcal{K}_1$ contributes 3 to $\mathcal{C}_{\mathcal{Z}_0}(\varphi)$. From (3.4) and Lemmas 3.2 and 3.5, we conclude that

$$\mathcal{C}_{\partial M}(\varphi) = \mathcal{C}_{\mathcal{Z}_1}(\varphi) + \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 63 + 3|\mathcal{K}_1| = 63 + 3 \cdot 72 = 279. \quad (3.6)$$

The first claim of the lemma follows from (3.1) and (3.6), since $\mathcal{N}_2 = \tilde{\mathcal{N}}_2/S_2$, where S_2 is the symmetric group on two elements.

(2) Similarly, let $\mathcal{N}'_{1,1} \subset \mathcal{D} \times \mathbb{P}^2_1$ be defined as in (2) of the proof of Lemma 3.2. We put

$$\begin{aligned} M &= \mathcal{N}'_{1,1} \times \mathbb{P}^2_2, \quad M^0 = \{([\underline{a}], x_1, x_2) \in M : x_1 \neq x_2\}, \quad \partial M = M - M^0, \quad \mathcal{N}_{2,1} = \varphi^{-1}(0) \cap M^0, \\ \text{where } \varphi &\in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}^2_2), \quad \varphi([\underline{a}], x_1, x_2) = (s_{\underline{a}}(x_2), ds_{\underline{a}}|_{x_2}). \end{aligned}$$

Since $\varphi|_{M^0}$ is transverse to the zero set,

$$\begin{aligned} |\mathcal{N}_{2,1}| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}^2_2), M \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 27 \langle y, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\partial M}(\varphi) = 27|\mathcal{N}_{1,1}| - \mathcal{C}_{\partial M}(\varphi) = 27 \cdot 9 - \mathcal{C}_{\partial M}(\varphi). \end{aligned} \quad (3.7)$$

We split ∂M into two strata:

$$\mathcal{Z}_1 = \{([\underline{a}], x, x) : ([\underline{a}], x) \in \mathcal{N}'_{1,1} - \mathcal{K}_{1,1}\}, \quad \mathcal{Z}_0 = \{([\underline{a}], x, x) : ([\underline{a}], x) \in \mathcal{K}_{1,1}\}.$$

By the same argument as in (1) above,

$$\begin{aligned} \mathcal{C}_{\mathcal{Z}_1}(\varphi) &= N(\alpha), \quad \text{where} \\ \alpha &\in \Gamma(\mathcal{N}'_{1,1}; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\underline{a}], x; v) = (0, H_{\underline{a}, x} v), \end{aligned}$$

while $\mathcal{C}_{\mathcal{Z}_0}(\varphi) = 3|\mathcal{K}_{1,1}|$. Using Lemmas 3.2 and 3.5, we conclude that

$$\mathcal{C}_{\partial M}(\varphi) = \mathcal{C}_{\mathcal{Z}_1}(\varphi) + \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 13 + 3 \cdot 20 = 73. \quad (3.8)$$

The second claim of the lemma follows immediately from (3.7) and (3.8).

Lemma 3.5 *If $\mathcal{N}'_1 \subset \mathcal{D} \times \mathbb{P}^2$ is as in (1) of the proof of Lemma 3.2 and*

$$\alpha \in \Gamma(\mathcal{N}'_1; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\underline{a}], x; v) = (0, H_{\underline{a},x}v),$$

then $N(\alpha) = 63$. If $\mathcal{N}'_{1,1} \subset \mathcal{D} \times \mathbb{P}^2$ is as in (2) of the proof of Lemma 3.2 and

$$\alpha \in \Gamma(\mathcal{N}'_{1,1}; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)), \quad \alpha([\underline{a}], x; v) = (0, H_{\underline{a},x}v),$$

then $N(\alpha) = 13$.

Proof: (1) We put

$$M = \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}'_1}, \quad \partial M = \{([\underline{a}], x) \in M : \tilde{H}_{\underline{a},x} = 0\} \approx \mathcal{K}_1,$$

where $\tilde{H}_{\cdot, \cdot}$ is as in the proof of Lemma 3.3. Let

$$\tilde{\alpha} = (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2))$$

be the section induced by α . By Lemma 3.14 in [Z1] or Proposition 2.18A in [Z2],

$$\begin{aligned} N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)c(T\mathbb{P}^2)^{-1}, \mathcal{N}'_1 \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3y + 6a, \mathcal{N}'_1 \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (3|\mathcal{N}'_1| + 6|\mathcal{N}_{1,1}|) - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp), \end{aligned} \quad (3.9)$$

where $\tilde{\alpha}^\perp$ is the composition of the linear bundle map $\tilde{\alpha}$ with the quotient projection map

$$\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2 \longrightarrow (\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2) / \mathcal{C}\nu,$$

for a generic nonvanishing section ν . The claim (3.9) can in fact be easily seen directly from the definition of $N(\alpha)$. Since the section \tilde{H} is transverse to the zero set, so is the section $\tilde{\alpha}^\perp$ if ν is generic. Thus,

$$\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) = \pm |\tilde{\alpha}^{-1}(0)| = |\mathcal{K}_1|. \quad (3.10)$$

The first claim of the lemma follows from (3.9) and (3.10), along with Lemmas 3.1 and 3.2.

(2) Similarly, we put

$$\begin{aligned} M &= \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}'_{1,1}}, \quad \partial M = \{([\underline{a}], x) \in M : \tilde{H}_{\underline{a},x} = 0\} \approx \mathcal{K}_{1,1}, \\ \tilde{\alpha} &= (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)). \end{aligned}$$

By Lemma 3.14 in [Z1] or Proposition 2.18A in [Z2],

$$\begin{aligned} N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)c(T\mathbb{P}^2)^{-1}, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3y + 6a, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (3|\mathcal{N}_{1,1}| + 6\langle a, \mathcal{N}_{1,1} \rangle) - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp). \end{aligned} \quad (3.11)$$

As in (1), $\tilde{\alpha}^\perp$ is transverse to the zero, and thus

$$\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) = \pm |\tilde{\alpha}^{-1}(0)| = |\mathcal{K}_{1,1}|. \quad (3.12)$$

The second claim of the lemma follows from (3.11) and (3.12), along with Lemmas 3.1 and 3.2.

Lemma 3.6 *The number $|\mathcal{K}_2|$ of plane quartics that have one node and one cusp and pass through 11 points in general position is 840.*

Proof: Let $\mathcal{N}_1'' \subset \mathcal{D} \times \mathbb{P}_1^2$ and $\mathcal{K}'_1 \subset \mathbb{P}T\mathbb{P}_1^2|_{\mathcal{N}_1''}$ be as in the proof of Lemma 3.3. We denote by

$$\tilde{\pi}_1: \mathbb{P}T\mathbb{P}_1^2|_{\mathcal{N}_1''} \longrightarrow \mathbb{P}_1^2$$

the composition of the bundle projection $\mathbb{P}T\mathbb{P}_1^2|_{\mathcal{N}_1''} \longrightarrow \mathcal{N}_1''$ with π_1 . We put

$$M = \mathcal{K}'_1 \times \mathbb{P}_2^2, \quad M^0 = \{([\underline{a}], x_1, x_2) \in M: \tilde{\pi}_1([\underline{a}], x_1) \neq x_2\}, \quad \partial M = M - M^0, \quad \mathcal{K}_2 = \varphi^{-1}(0) \cap M^0,$$

where $\varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}_2^2)$, $\varphi([\underline{a}], x_1, x_2) = (s_{\underline{a}}(x_2), ds_{\underline{a}}|_{x_2})$, $\gamma_2 = \pi_2^* \gamma_{\mathbb{P}_2^2}$.

Since $\varphi|_{M^0}$ is transverse to the zero set, similarly to (3.1),

$$\begin{aligned} |\mathcal{K}_2| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle (y+4a_2)(y^2+5ya_2+7a_2^2), \mathcal{K}'_1 \times \mathbb{P}_2^2 \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 27 \langle y, \mathcal{K}'_1 \rangle - \mathcal{C}_{\partial M}(\varphi) = 27|\mathcal{K}_1| - \mathcal{C}_{\partial M}(\varphi) = 27 \cdot 72 - \mathcal{C}_{\partial M}(\varphi). \end{aligned} \quad (3.13)$$

We split ∂M into two strata:

$$\mathcal{Z}_1 = \{([\underline{a}], x, \tilde{\pi}_1([\underline{a}], x)): ([\underline{a}], x) \in \mathcal{K}'_1 - \mathcal{T}_1\}, \quad \mathcal{Z}_0 = \{([\underline{a}], x, \tilde{\pi}_1([\underline{a}], x)): ([\underline{a}], x) \in \mathcal{T}_1\}.$$

Let $\gamma^\perp \longrightarrow \mathcal{K}'_1$ be the orthogonal complement of γ in $\pi^*T\mathbb{P}^2$. We define the bundle map

$$\begin{aligned} \alpha &\in \Gamma(\mathcal{K}'_1; \text{Hom}(\gamma^{\otimes 2} \oplus \gamma^\perp, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}_1^2)) \quad \text{by} \\ \alpha(\tilde{v}, w) &= (0, \frac{1}{2}D_{\underline{a},x}^3 \tilde{v}, \tilde{H}_{\underline{a},x} w) \in \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^* \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^{\perp*}. \end{aligned}$$

Note that by definition of the set \mathcal{T}_1 , for some $C \in C(\mathcal{K}'_1; \mathbb{R}^+)$,

$$|\alpha_{[\underline{a}],x}(\tilde{v}, w)| \geq C([\underline{a}], x)^{-1} (|\tilde{v}| + |w|) \quad \forall ([\underline{a}], x) \in \mathcal{K}'_1 - \mathcal{T}_1, (\tilde{v}, w) \in (\gamma^{\otimes 2} \oplus \gamma^\perp)|_{([\underline{a}],x)}. \quad (3.14)$$

On the other hand, with appropriate identifications,

$$\begin{aligned} |\varphi([\underline{a}], x, v, w) - \alpha_{[\underline{a}],x}(v^{\otimes 2}, w)| &\leq C([\underline{a}], x)(|v|^3 + |w|^2) \\ \forall ([\underline{a}], x, \tilde{\pi}_1([\underline{a}], x)) &\in \partial M, v \in \gamma_{([\underline{a}],x)}, w \in \gamma_{([\underline{a}],x)}^\perp. \end{aligned} \quad (3.15)$$

Since the bundle map

$$T\mathbb{P}^2 = \gamma \oplus \gamma^\perp \longrightarrow \gamma^{\otimes 2} \oplus \gamma^\perp, \quad (v, w) \longrightarrow (v^{\otimes 2}, w),$$

is two-to-one, outside of the proper subbundle γ^\perp ,

$$\mathcal{C}_{\mathcal{Z}_1}(\varphi) = 2 \cdot N(\alpha), \quad (3.16)$$

by (3.14), (3.15), and a rescaling and cobordism argument as in Subsection 3.1 of [Z1]. Suppose next that $([\underline{a}], x) \in \mathcal{T}_1$. Let $N_{([\underline{a}],x)}$ be the normal bundle of \mathcal{T}_1 in \mathcal{K}'_1 at $([\underline{a}], x)$. Then, with appropriate identifications, for some $\beta_2, \beta_4 \in \mathbb{C}^*$ and $C \in \mathbb{R}^+$,

$$\begin{aligned} |\varphi([\underline{a}], x; u, v, w) - \alpha_0(u, v, w)| &\leq C(|v|^5 + |w|^2) \quad \forall u \in N_{([\underline{a}],x)}, v \in \gamma_{([\underline{a}],x)}, w \in \gamma_{([\underline{a}],x)}^\perp, \\ \text{where } \alpha_0(u, v, w) &= \left(\frac{1}{6}uv^3 + \frac{1}{4}\beta_4v^4, \frac{1}{2}uv^2 + \beta_4v^3, \beta_2w\right). \end{aligned} \quad (3.17)$$

Since the polynomial α_0 is four-to-one near the origin, it follows from (3.17) that each point of $\mathcal{Z}_0 \approx \mathcal{T}_1$ contributes 4 to $\mathcal{C}_{\mathcal{Z}_0}(\varphi)$. From (3.16) and Lemmas 3.3 and 3.7, we conclude that

$$\mathcal{C}_{\partial M}(\varphi) = \mathcal{C}_{\mathcal{Z}_1}(\varphi) + \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 2 \cdot 152 + 4|\mathcal{T}_1| = 2 \cdot 152 + 4 \cdot 200 = 1104. \quad (3.18)$$

The lemma follows from (3.13) and (3.18).

Lemma 3.7 *If $\mathcal{K}'_1 \subset \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}''_1}$ is as in the proof of Lemma 3.3 and*

$$\begin{aligned} \alpha &\in \Gamma(\mathcal{K}'_1; \text{Hom}(\gamma^{\otimes 2} \oplus \gamma^\perp, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2)), \\ \alpha(v, w) &= (0, \frac{1}{2}D_{\underline{a}, x}^3 v, \tilde{H}_{\underline{a}, x} w) \in \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^* \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^{\perp*}, \end{aligned}$$

then $N(\alpha) = 152$.

Proof: Since the linear map

$$\alpha: \gamma^\perp \longrightarrow \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^{\perp*}$$

is an isomorphism over \mathcal{K}'_1 ,

$$N(\alpha) = N(\tilde{\alpha}), \quad \text{where} \quad (3.19)$$

$$\tilde{\alpha} \in \Gamma(\mathcal{K}'_1; \text{Hom}(\gamma^{\otimes 2}, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^*)), \quad \alpha([\underline{a}], x; v, w) = (0, \frac{1}{2}D_{\underline{a}, x}^3 v).$$

Similarly to the proof of the Lemma 3.5,

$$\begin{aligned} N(\tilde{\alpha}) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes \gamma^*)c(\gamma^{\otimes 2})^{-1}, \mathcal{K}'_1 \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3\lambda^3 + (8y + 23a)\lambda^2 + (7y^2 + 41ya + 61a^2)\lambda, \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}''_1} \rangle - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp) \\ &= \langle 7y^2 + 17ya + 10a^2, \mathcal{N}''_1 \rangle - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp) \\ &= (7|\mathcal{N}_1| + 17|\mathcal{N}_{1,1}| + 10\langle a, \mathcal{N}'_{1,1} \rangle) - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp) = 7 \cdot 27 + 17 \cdot 9 + 10 \cdot 1 - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp). \end{aligned} \quad (3.20)$$

Since the section D^3 is transverse to the zero set, so is the section $\tilde{\alpha}^\perp$ if ν is generic. Thus,

$$\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) = \pm |\tilde{\alpha}^{-1}(0)| = |\mathcal{T}_1|. \quad (3.21)$$

The lemma follows from (3.19)-(3.21) along with Lemma 3.3.

3.4 Quartics with Three Simple Nodes

In this subsection we compute the last number of Table 1. We start with the following structural lemma.

Lemma 3.8 *Let $\mathcal{N}''_1 \subset \mathcal{D} \times \mathbb{P}^2_1$ be as in the proof of Lemma 3.3 and let*

$$\begin{aligned} \tilde{\mathcal{N}}'_{2,0} &= \{([\underline{a}], x_1, x_2) \in \mathcal{N}''_1 \times \mathbb{P}^2_2: x_1 \neq x_2, \varphi_2([\underline{a}], x_1, x_2) = 0\}, \quad \text{where} \\ \varphi_2 &\in \Gamma(\mathcal{N}''_1 \times \mathbb{P}^2_2; \gamma_0^* \otimes \gamma_2^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes 4} \otimes T^*\mathbb{P}^2_2), \quad \varphi_2([\underline{a}], x_1, x_2) = (s_{\underline{a}}(x_2), ds_{\underline{a}}|_{x_2}). \end{aligned}$$

If $\tilde{\mathcal{N}}'_2$ is the closure of $\tilde{\mathcal{N}}'_{2,0}$ in $\mathcal{N}''_1 \times \mathbb{P}^2_2$, then

$$\partial \tilde{\mathcal{N}}'_2 \equiv \tilde{\mathcal{N}}'_2 - \tilde{\mathcal{N}}'_{2,0} = \{([\underline{a}], x, x) \in \mathcal{N}''_1 \times \mathbb{P}^2_2: ([\underline{a}], x) \in \mathcal{T}_1\}.$$

Proof: We will only show that if $([\underline{a}], x_1, x_2) \in \partial\tilde{\mathcal{N}}'_2$, then $x_1 = x_2$ and $([\underline{a}], x_1) \in \mathcal{T}_1$. The converse follows from the proofs of Lemmas 3.9 and 3.10. Suppose $([\underline{a}], x_1, x_2) \in \partial\tilde{\mathcal{N}}'_2$. Since the section φ_2 is continuous, $x_2 = x_1$ by definition of $\partial\tilde{\mathcal{N}}'_2$. If $([\underline{a}], x_1) \in \mathcal{N}''_1 - \mathcal{K}'_1$, by (3.2) and (3.3) and with appropriate identifications,

$$|\varphi_2([\underline{a}], x_1, v)| \geq C([\underline{a}], x_1)^{-1}|v|$$

for all $v \in T_{x_1}\mathbb{P}^2_1$ sufficiently small. Thus, $([\underline{a}], x_1, x_1)$ is not in the closure of $\tilde{\mathcal{N}}_{2;0}$. Suppose next that $(\underline{a}, x_1) \in \mathcal{K}'_1 - \mathcal{T}_1$. Then, by (3.5),

$$|\varphi_2([\underline{a}], x_1, u, v, w)| \geq C([\underline{a}], x_1)^{-1}(|v|^3 + |w|) \quad (3.22)$$

for all $u \in N_{(\underline{a}, x_1)}$, $v \in \mathcal{L}_{(\underline{a}, x_1)}$, and $w \in \mathcal{L}^\perp_{(\underline{a}, x_1)}$ sufficiently small. In this case, N is the normal bundle of \mathcal{K}'_1 , viewed as a submanifold of \mathcal{N}''_1 , in \mathcal{N}''_1 , while the line bundles \mathcal{L} and \mathcal{L}^\perp over \mathcal{K}'_1 are defined as in (1) of the proof of Lemma 3.4. From (3.22), we conclude that $([\underline{a}], x_1, x_1)$ is not in the closure of $\tilde{\mathcal{N}}_{2;0}$.

Lemma 3.9 *The number $|\mathcal{N}_3|$ of plane quartics that have three nodes and pass through 11 points in general position is 675.*

Proof: With notation as in the statement of Lemma 3.8, let

$$M = \tilde{\mathcal{N}}'_2 \times \mathbb{P}^2_3, \quad M^0 = \{([\underline{a}], x_1, x_2, x_3) \in M : x_3 \neq x_1, x_2\}, \quad \partial M = M - M^0, \quad \tilde{\mathcal{N}}_3 = \varphi_3^{-1}(0) \cap M^0,$$

where $\varphi_3 \in \Gamma(M; \gamma_0^* \otimes \gamma_3^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_3^{*\otimes 4} \otimes T^*\mathbb{P}^2_3)$, $\varphi_3([\underline{a}], x_1, x_2, x_3) = (s_{\underline{a}}(x_3), ds_{\underline{a}}|_{x_3})$, $\gamma_3 = \pi_3^* \gamma_{\mathbb{P}^2_3}$,

and $\pi_3: M \rightarrow \mathbb{P}^2_3$ is the projection onto the last component. Since $\varphi_3|_{M^0}$ is transverse to the zero set,

$$\begin{aligned} |\tilde{\mathcal{N}}_3| &= \pm |\varphi_3^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_3^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_3^{*\otimes 4} \otimes T^*\mathbb{P}^2_3), M \rangle - \mathcal{C}_{\partial M}(\varphi_3) \\ &= \langle (y + 4a_3)(y^2 + 5ya_3 + 7a_3^2), \tilde{\mathcal{N}}'_2 \times \mathbb{P}^2_3 \rangle - \mathcal{C}_{\partial M}(\varphi_3) \\ &= 27 \langle y, \tilde{\mathcal{N}}'_2 \rangle - \mathcal{C}_{\partial M}(\varphi_3) = 27 \cdot 2|\mathcal{N}_2| - \mathcal{C}_{\partial M}(\varphi_3) = 27 \cdot 450 - \mathcal{C}_{\partial M}(\varphi_3), \end{aligned} \quad (3.23)$$

where $a_3 = \pi_2^* c_1(\gamma_{\mathbb{P}^2_3}^*)$. In order to determine $\mathcal{C}_{\partial M}(\varphi_3)$, we split ∂M into five strata:

$$\begin{aligned} \mathcal{Z}_{1,i} &= \{([\underline{a}], x_1, x_2, x_3) : x_3 = x_i, x_3 \neq x_j, ([\underline{a}], x_i) \in \mathcal{N}''_1 - \mathcal{K}'_1\}, & \{i, j\} &= \{1, 2\}; \\ \mathcal{Z}_{0,i} &= \{([\underline{a}], x_1, x_2, x_3) : x_3 = x_i, x_3 \neq x_j, ([\underline{a}], x_i) \in \mathcal{K}'_1 - \mathcal{T}_1\} \approx \mathcal{K}_2, & \{i, j\} &= \{1, 2\}; \\ \mathcal{Z}_{0,12} &= \{([\underline{a}], x, x, x) : ([\underline{a}], x) \in \mathcal{T}_1\}. \end{aligned}$$

Note that Lemma 3.8 implies that the union of these five spaces is indeed ∂M . Similarly to the proof of Lemma 3.4, we have

$$\begin{aligned} \mathcal{C}_{\mathcal{Z}_{1,1}}(\varphi_3) &= \mathcal{C}_{\mathcal{Z}_{1,2}}(\varphi_3) = N(\alpha), & \text{where} & \\ \alpha \in \Gamma(\tilde{\mathcal{N}}'_2; \text{Hom}(T\mathbb{P}^2_1, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}^2_1)), & \alpha([\underline{a}], x_1, x_2; v) &= (0, H_{\underline{a}, x_1} v), \end{aligned} \quad (3.24)$$

while

$$\mathcal{C}_{\mathcal{Z}_{0,1}}(\varphi_3) = \mathcal{C}_{\mathcal{Z}_{0,2}}(\varphi_3) = 3|\mathcal{K}_2|. \quad (3.25)$$

Finally, suppose that $([\underline{a}], x) \in \mathcal{T}_1$. Let $N_{(\underline{a}, x)}^1$ and $N_{(\underline{a}, x)}^2$ be the normal bundles of \mathcal{T}_1 in \mathcal{K}'_1 and of \mathcal{K}'_1 in \mathcal{N}''_1 , respectively, at $([\underline{a}], x)$. Let $\mathcal{L}_{(\underline{a}, x)}$ and $\mathcal{L}_{(\underline{a}, x)}^\perp$ be as in the proof of Lemma 3.4. Then, with appropriate identifications, for some $\beta_2, \beta_4 \in \mathbb{C}^*$, $C \in \mathbb{R}^+$, and $i = 2, 3$,

$$\begin{aligned} |\varphi_i([\underline{a}], x; u_1, u_2, v_i, w_i) - \alpha_0(u_1, u_2, v_i, w_i)| &\leq C(|v_i|^5 + |w_i|^2) \\ \forall u_1 \in N_{(\underline{a}, x)}^1, u_2 \in N_{(\underline{a}, x)}^2, v_i \in \mathcal{L}_{(\underline{a}, x)}, w_i \in \mathcal{L}_{(\underline{a}, x)}^\perp, \end{aligned} \quad (3.26)$$

$$\text{where } \alpha_0(u_1, u_2, v, w) = \left(\frac{1}{6}u_1v^3 + \frac{1}{2}u_2v^2 + \frac{1}{12}\beta_4v^4, \frac{1}{2}u_1v^2 + u_2v + \frac{1}{3}\beta_4v^3, \beta_2w\right).$$

Since $\tilde{\mathcal{N}}'_{2,0} = \varphi_2^{-1}(0)$, the φ_3 -contribution of $([\underline{a}], x, x, x)$ is the number of small solutions of the system

$$\begin{cases} \varphi_2(u_1, u_2, v_2, w_2) = 0 \\ \varphi_3(u_1, u_2, v_3, w_3) = t\nu(u_1, u_2, v_2, w_2, v_3, w_3) \end{cases} \quad \begin{aligned} (u_1, u_2, v_2, w_2, v_3, w_3) &\in \mathbb{C}^6 \\ (v_2, w_2) &\neq (0, 0), \end{aligned} \quad (3.27)$$

for a generic $\nu \in \mathbb{C}^3$ and $t \in \mathbb{R}^+$ sufficiently small. By (3.26) and a rescaling and cobordism argument as in Subsection 3.1 of [Z1], the number of small solutions of (3.27) is the same as the number of solutions of the system

$$\begin{cases} \frac{1}{6}u_1v_2^3 + \frac{1}{2}u_2v_2^2 + \frac{1}{12}\beta_4v_2^4 = 0 \\ \frac{1}{2}u_1v_2^3 + u_2v_2^2 + \frac{1}{3}\beta_4v_2^4 = 0 \\ \frac{1}{6}u_1v_3^3 + \frac{1}{2}u_2v_3^2 + \frac{1}{12}\beta_4v_3^4 = \nu \\ \frac{1}{2}u_1v_3^3 + u_2v_3^2 + \frac{1}{3}\beta_4v_3^4 = 0 \end{cases} \quad (u_1, u_2, v_2, v_3) \in \mathbb{C}^{*4}, \quad (3.28)$$

for a generic $\nu \in \mathbb{C}$. Dividing the first two equations by v_2^2 and the last equation by v_3^2 and then solving for u_2 and u_1 in terms of v_1 and v_2 , we find that the system (3.28) is equivalent to

$$\begin{cases} u_1 = -\beta_4v_2 \\ u_2 = \frac{1}{6}\beta_4v_2^2 \\ v_2 = v_3 \text{ or } v_2 = 2v_3 \\ -\frac{1}{6}v_2v_3^3 + \frac{1}{12}v_2^2v_3^2 + \frac{1}{12}v_4^4 = \nu \end{cases} \quad (u_1, u_2, v_2, v_3) \in \mathbb{C}^4. \quad (3.29)$$

If $v_2 = v_3$, the last equation has no solutions for $\nu \neq 0$. On the other hand, if $v_2 = 2v_3$, the last equation in (3.29) has four solutions. We conclude that

$$\mathcal{C}_{\mathcal{Z}_{0,12}}(\varphi_3) = 4|\mathcal{T}_1|. \quad (3.30)$$

From (3.24), (3.25), and (3.30), along with Lemmas 3.3 and 3.6, we conclude that

$$\mathcal{C}_{\partial M}(\varphi_3) = 2\mathcal{C}_{\mathcal{Z}_{1,1}}(\varphi_3) + 2\mathcal{C}_{\mathcal{Z}_{0,1}}(\varphi_3) + \mathcal{C}_{\mathcal{Z}_{0,12}}(\varphi_3) = 2 \cdot 1130 + 6 \cdot 840 + 4 \cdot 200 = 8100. \quad (3.31)$$

The lemma follows from (3.23) and (3.31), since $\mathcal{N}_3 = \tilde{\mathcal{N}}_3/S_3$.

Lemma 3.10 *If $\tilde{\mathcal{N}}'_2 \subset \mathcal{D} \times \mathbb{P}_1^2 \times \mathbb{P}_2^2$ is as in Lemma 3.8 and*

$$\alpha \in \Gamma(\tilde{\mathcal{N}}'_2; \text{Hom}(T\mathbb{P}_1^2, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}_1^2)), \quad \alpha([\underline{a}], x_1, x_2; \nu) = (0, H_{\underline{a}, x_1} \nu),$$

then $N(\alpha) = 1130$.

Proof: We put

$$M = \mathbb{P}T\mathbb{P}_1^2|_{\tilde{\mathcal{N}}_2'}, \quad \partial M = \{([\underline{a}], x_1, x_2) \in M : \tilde{H}_{\underline{a}, x} = 0\},$$

where $\tilde{H}_{\cdot, \cdot}$ is as in the proof of Lemma 3.3. Using Lemma 3.8, we split ∂M into two subsets:

$$\begin{aligned} \mathcal{Z}_{0,1} &= \{([\underline{a}], x_1, x_2) \in M : \tilde{\pi}_1([\underline{a}], x_1) \neq x_2, ([\underline{a}], \underline{x}_1) \in \mathcal{K}'_1 - \mathcal{T}_1\}, \\ \mathcal{Z}_{0,2} &= \{([\underline{a}], x_1, x_2) \in M : \tilde{\pi}_1([\underline{a}], x_1) = x_2, ([\underline{a}], \underline{x}_1) \in \mathcal{T}_1\}, \end{aligned}$$

where $\tilde{\pi}_1$ is as in the proof of Lemma 3.6. Here \mathcal{K}'_1 and \mathcal{T}_1 are viewed as subspaces of $\mathbb{P}T\mathbb{P}_1^2|_{\mathcal{N}''_1}$, as defined in the proof of Lemma 3.3. Let

$$\tilde{\alpha} = (0, \tilde{H}) \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}_1^2))$$

be the section induced by α . Similarly to the proof of Lemma 3.5,

$$\begin{aligned} N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes 4} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes 4} \otimes T^*\mathbb{P}_1^2) c(T\mathbb{P}_1^2)^{-1}, \tilde{\mathcal{N}}_2' \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3y + 6a_1, \tilde{\mathcal{N}}_2' \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (3 \cdot 2|\mathcal{N}_2| + 6|\mathcal{N}_{2,1}|) - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp). \end{aligned} \quad (3.32)$$

As in the proof of Lemma 3.5, we have

$$\mathcal{C}_{\mathcal{Z}_{0,1}}(\tilde{\alpha}^\perp) = \pm |\mathcal{Z}_{0,1}| = |\mathcal{K}_2|. \quad (3.33)$$

On the other hand, suppose $([\underline{a}], x_1, x_2) \in \mathcal{Z}_{0,2}$ and thus $([\underline{a}], x_1) \in \mathcal{T}_1$, while $x_2 = \tilde{\pi}_1([\underline{a}], x_1)$. Then, with identifications similar to the ones used at the end of the proof of Lemma 3.9, the $\tilde{\alpha}^\perp$ -contribution of $([\underline{a}], x_1, x_2)$ is the number of small solutions of the system

$$\begin{cases} \varphi_2(u_1, u_2, v_2, w_2) = 0 & (u_1, u_2, v_2, w_2, w_3) \in \mathbb{C}^5 \\ \tilde{\alpha}^\perp(u_1, u_2, w_3) = t\nu(u_1, u_2, v_2, w_2, w_3) & (v_2, w_2) \neq (0, 0), \end{cases} \quad (3.34)$$

for a generic $\nu \in \mathbb{C}^2$ and $t \in \mathbb{R}^+$ sufficiently small. In this case, $w_3 \in \gamma_{(\underline{a}, x_1)}^* \otimes \gamma_{(\underline{a}, x_1)}^\perp$. For a good choice of identifications

$$\tilde{\alpha}^\perp(u_1, u_2, v_2, w_2, w_3) = (u_2, w_3). \quad (3.35)$$

By the $i = 2$ case of (3.26) and (3.35), the number of small solutions of the system (3.34) is the same as the number of solutions of the system

$$\begin{cases} \frac{1}{6}u_1v_2^3 + \frac{1}{2}u_2v_2^2 + \frac{1}{12}\beta_4v_2^4 = 0 \\ \frac{1}{2}u_1v_2^3 + u_2v_2^2 + \frac{1}{3}\beta_4v_2^4 = 0 \\ u_2 = \nu \end{cases} \quad (u_1, u_2, v_2) \in \mathbb{C}^{*3},$$

for a generic $\nu \in \mathbb{C}$. Thus, each point of $\mathcal{Z}_{0,2}$ contributes two, and

$$\mathcal{C}_{\mathcal{Z}_{0,2}}(\tilde{\alpha}^\perp) = 2|\mathcal{T}_1|. \quad (3.36)$$

The lemma follows from (3.32), (3.33), and (3.36), along with Lemmas 3.3, 3.4, and 3.6.

set	singularities	$d \geq$	co-# pts	cardinality
\mathcal{N}_1	1 node	1	1	$3(d-1)^2$
$\mathcal{N}_{1,1}$	1 node on a fixed line	1	2	$3(d-1)$
\mathcal{K}_1	1 cusp	1	2	$12(d-1)(d-2)$
$\mathcal{K}_{1,1}$	1 cusp on a fixed line	3	3	$4(2d-3)$
\mathcal{T}_1	1 tacnode	3	3	$2(25d^2 - 96d + 84)$
\mathcal{N}_2	2 nodes	1	2	$3(d-1)(d-2)(3d^2 - 3d - 11)/2$
$\mathcal{N}_{2,1}$	2 nodes, one on a fixed line	3	3	$9d^3 - 27d^2 - d + 30$
\mathcal{K}_2	1 node and 1 cusp	3	3	$12(d-3)(3d^3 - 6d^2 - 11d + 18)$
\mathcal{N}_3	3 nodes	3	3	$(9d^6 - 54d^5 + 9d^4 + 423d^3 - 458d^2 - 829d + 1050)/2$

Table 2: Some Characteristic Numbers of Degree- d Plane Curves

3.5 Generalization to Arbitrary-Degree Curves

The computations in the previous subsections generalize to higher-degree curves, as well as to other types of singularities. We list the results of the generalization to arbitrary-degree curves in Table 2. The number in the third column is the lowest value of the degree d for which the formula given in the last column is applicable. Note that in the cases when this number is higher than one, the constraints are -1 points for $d=1$ and two points for $d=2$. So, the corresponding count of curves makes no sense for $d=1$, while for $d=2$ this is a count of structures on the double line through two distinct points in \mathbb{P}^2 . The number in the fourth column is the difference between

$$\dim(d) \equiv \dim \{ \text{deg. } -d \text{ curves} \} = \frac{d(d+3)}{2}$$

and the number of points in general position. Below we state the changes that are needed to be made in the above lemmas to obtain these results.

3.5.1 The Numbers \mathcal{N}_1 and $\mathcal{N}_{1,1}$

In order to compute the number \mathcal{N}_1 , we take $\mathcal{D} \approx \mathbb{P}^1$ to be the subspace of degree- d plane curves that pass through a set of $\dim(d)-1$ points in general position. We define \mathcal{N}_1 as in (1) of the proof of Lemma 3.1, except now

$$\varphi \in \Gamma(\mathcal{D} \times \mathbb{P}^2; \gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^* \mathbb{P}^2).$$

Since φ is transverse to the zero set, we obtain

$$\begin{aligned} |\mathcal{N}_1| &= |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^* \mathbb{P}^2), \mathcal{D} \times \mathbb{P}^2 \rangle \\ &= \langle (y+da)(y^2 + (2d-3)ya + (d^2 - 3d+3)a^2), \mathcal{D} \times \mathbb{P}^2 \rangle = 3(d-1)^2. \end{aligned}$$

With the analogous changes in (2) of the proof of Lemma 3.1, we find that

$$\begin{aligned} |\mathcal{N}_{1,1}| &= |\varphi^{-1}(0)| = \langle e(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^* \mathbb{P}^2), \mathcal{D} \times \mathbb{P}^1 \rangle \\ &= \langle (y+da)(y^2 + (2d-3)ya + (d^2 - 3d+3)a^2), \mathcal{D} \times \mathbb{P}^1 \rangle = 3(d-1). \end{aligned}$$

3.5.2 The Numbers \mathcal{K}_1 and $\mathcal{K}_{1,1}$

We take $\mathcal{D} \approx \mathbb{P}^2$ to be the subspace of degree- d plane curves that pass through a set of $\dim(d)-2$ points in general position. We define \mathcal{N}'_1 and \mathcal{K}_1 as in (1) of the proof of Lemma 3.2, except now

$$H_{\underline{a},x} \in \Gamma(\mathcal{N}'_1; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2)) \quad \text{and} \quad \varphi \in \Gamma(\mathcal{N}'_1; (\gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}).$$

Since φ is transverse to the zero set,

$$\begin{aligned} |\mathcal{K}_1| &= |\varphi^{-1}(0)| = \langle e((\gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}), \mathcal{N}'_1 \rangle \\ &= 2\langle y + (d-3)a, \mathcal{N}'_1 \rangle = 2(|\mathcal{N}'_1| + (d-3)|\mathcal{N}_{1,1}|) \\ &= 2(3(d-1)^2 + (d-3) \cdot 3(d-1)) = 12(d-1)(d-2). \end{aligned}$$

With the analogous changes in (2) of the proof of Lemma 3.2, we find that

$$\begin{aligned} |\mathcal{K}_{1,1}| &= \langle e((\gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}), \mathcal{N}'_{1,1} \rangle \\ &= 2\langle y + (d-3)a, \mathcal{N}'_{1,1} \rangle = 2(|\mathcal{N}'_{1,1}| + (d-3)\langle a, \mathcal{N}'_{1,1} \rangle) \\ &= 2(3(d-1) + (d-3)) = 4(2d-3). \end{aligned}$$

3.5.3 The Number \mathcal{T}_1

In this case, we take $\mathcal{D} \approx \mathbb{P}^3$ to be the subspace of degree- d plane curves that pass through a set of $\dim(d)-3$ points in general position. We define \mathcal{N}''_1 , M , \mathcal{K}'_1 , and \mathcal{T}_1 as in the proof of Lemma 3.3, except now

$$\tilde{H}_{\cdot,\cdot} \in \Gamma(M; \text{Hom}(\gamma, \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2)) \quad \text{and} \quad \varphi \in \Gamma(M; \text{Hom}(\gamma^{\otimes 3}, \gamma_0^* \otimes \gamma_1^{*\otimes d})).$$

Since the sections φ and $\tilde{H}_{\cdot,\cdot}$ are transverse to the zero set, we obtain

$$\begin{aligned} |\mathcal{T}_1| &= |\varphi^{-1}(0)| = \langle e(\gamma^{\otimes 3} \otimes \gamma_0^* \otimes \gamma_1^{*\otimes d})e(\gamma^* \otimes \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2), M \rangle \\ &= \langle 3\lambda^3 + (7y + (7d-9)a)\lambda^2 + (5y^2 + (10d-12)ya + (5d^2 - 12d + 9)a^2)\lambda, M \rangle \\ &= \langle 5y^2 + (10d-33)ya + (5d^2 - 33d + 54)a^2, \mathcal{N}''_1 \rangle \\ &= 5|\mathcal{N}'_1| + (10d-33)|\mathcal{N}_{1,1}| + (5d^2 - 33d + 54)\langle a, \mathcal{N}'_{1,1} \rangle \\ &= 5 \cdot 3(d-1)^2 + (10d-33) \cdot 3(d-1) + (5d^2 - 33d + 54) = 2(25d^2 - 96d + 84). \end{aligned}$$

3.5.4 The Numbers \mathcal{N}_2 and $\mathcal{N}_{2,1}$

In order to compute the number \mathcal{N}_2 , we take $\mathcal{D} \approx \mathbb{P}^2$ to be the subspace of degree- d plane curves that pass through a set of $\dim(d)-2$ points in general position. We define \mathcal{N}'_1 , M , ∂M , $\tilde{\mathcal{N}}_2$, \mathcal{Z}_1 , \mathcal{Z}_0 , and α as in (1) of the proof of Lemma 3.4, except now

$$\varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}_2^2), \quad \alpha \in \Gamma(\mathcal{N}'_1; \text{Hom}(T\mathbb{P}^2, \gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2)).$$

Since $\varphi|_{M^0}$ is transverse to the zero set,

$$\begin{aligned} |\tilde{\mathcal{N}}_2| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}_2^2), M \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= \langle (y + da_2)(y^2 + (2d-3)ya_2 + (d^2 - 3d + 3)a_2^2), \mathcal{N}'_1 \times \mathbb{P}_2^2 \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 3(d-1)^2 \langle y, \mathcal{N}'_1 \rangle - \mathcal{C}_{\partial M}(\varphi) = 3(d-1)^2 |\mathcal{N}'_1| - (\mathcal{C}_{\mathcal{Z}_0}(\varphi) + \mathcal{C}_{\mathcal{Z}_1}(\varphi)). \end{aligned} \tag{3.37}$$

As in (1) of the proof of Lemma 3.4, we have

$$\mathcal{C}_{\mathcal{Z}_1}(\varphi) = N(\alpha) \quad \text{and} \quad \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 3|\mathcal{K}_1|. \quad (3.38)$$

Similarly to (1) of the proof of Lemma 3.5,

$$\begin{aligned} N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2) c(T\mathbb{P}^2)^{-1}, \mathcal{N}'_1 \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3y + 3(d-2)a, \mathcal{N}'_1 \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (3|\mathcal{N}_1| + 3(d-2)|\mathcal{N}_{1,1}|) - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp), \\ &\quad \text{where} \quad \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = |\mathcal{K}_1|. \end{aligned}$$

Combining these observations with (3.37) and (3.38), we obtain

$$|\tilde{\mathcal{N}}_2| = 3d(d-2)|\mathcal{N}_1| - 3(d-2)|\mathcal{N}_{1,1}| - 2|\mathcal{K}_1| = 3(d-1)(d-2)(3d^2 - 3d - 11).$$

With the analogous modifications in (2) of the proof of Lemma 3.4, we obtain

$$\begin{aligned} |\mathcal{N}_{2,1}| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}_2^2), \mathcal{N}'_{1,1} \times \mathbb{P}_2^2 \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 3(d-1)^2 \langle y, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\partial M}(\varphi) = 3(d-1)^2 |\mathcal{N}_{1,1}| - (\mathcal{C}_{\mathcal{Z}_0}(\varphi) + \mathcal{C}_{\mathcal{Z}_1}(\varphi)), \end{aligned} \quad (3.39)$$

$$\text{where} \quad \mathcal{C}_{\mathcal{Z}_1}(\varphi) = N(\alpha) \quad \text{and} \quad \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 3|\mathcal{K}_{1,1}|. \quad (3.40)$$

By the argument in (2) of the proof of Lemma 3.5,

$$\begin{aligned} N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2) c(T\mathbb{P}^2)^{-1}, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\ &= \langle 3y + 3(d-2)a, \mathcal{N}'_{1,1} \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (3|\mathcal{N}_{1,1}| + 3(d-2)) - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp), \\ &\quad \text{where} \quad \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = |\mathcal{K}_{1,1}|. \end{aligned}$$

Combining these identities with (3.39) and (3.40), we obtain

$$|\mathcal{N}_{2,1}| = 3d(d-2)|\mathcal{N}_{1,1}| - 3(d-2) - 2|\mathcal{K}_{1,1}| = 9d^3 - 27d^2 - d + 30.$$

3.5.5 The Number \mathcal{K}_2

We take $\mathcal{D} \approx \mathbb{P}^3$ to be the subspace of degree- d plane curves that pass through a set of $\dim(d) - 3$ points in general position. We define \mathcal{N}''_1 , \mathcal{K}'_1 , M , ∂M , \mathcal{K}_2 , \mathcal{Z}_1 , \mathcal{Z}_0 , and α as in the proof of Lemma 3.6, except now

$$\varphi \in \Gamma(M; \gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}_2^2), \quad \alpha \in \Gamma(\mathcal{K}'_1; \text{Hom}(\gamma^{\otimes 2} \oplus \gamma^\perp, \gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2)).$$

Since $\varphi|_{M^0}$ is transverse to the zero set,

$$\begin{aligned} |\mathcal{K}_2| &= \pm |\varphi^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}_2^2), M \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= \langle (y + da_2)(y^2 + (2d-3)ya_2 + (d^2 - 3d + 3)a_2^2), \mathcal{K}'_1 \times \mathbb{P}_2^2 \rangle - \mathcal{C}_{\partial M}(\varphi) \\ &= 3(d-1)^2 \langle y, \mathcal{K}'_1 \rangle - \mathcal{C}_{\partial M}(\varphi) = 3(d-1)^2 |\mathcal{K}_1| - (\mathcal{C}_{\mathcal{Z}_0}(\varphi) + \mathcal{C}_{\mathcal{Z}_1}(\varphi)). \end{aligned} \quad (3.41)$$

As in (1) of the proof of Lemma 3.6, we have

$$\mathcal{C}_{\mathcal{Z}_1}(\varphi) = 2N(\alpha) \quad \text{and} \quad \mathcal{C}_{\mathcal{Z}_0}(\varphi) = 4|\mathcal{T}_1|. \quad (3.42)$$

Similarly to (1) of the proof of Lemma 3.7,

$$\begin{aligned}
N(\tilde{\alpha}) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes \gamma^*) c(\gamma^{\otimes 2})^{-1}, \mathcal{K}'_1 \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\
&= \langle (3\lambda + 2y + 2da)(\lambda^2 + (2y + (2d-3)a)\lambda + (2d-3)ya + (d^2 - 3d + 3)a^2), \mathbb{P}T\mathbb{P}^2|_{\mathcal{N}''_1} \rangle - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp) \\
&= \langle 7y^2 + (14d-39)ya + (7d^2 - 39d + 54)a^2, \mathcal{N}''_1 \rangle - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp) \\
&= (7|\mathcal{N}_1| + (14d-39)|\mathcal{N}_{1,1}| + (7d^2 - 39d + 54)) - \mathcal{C}_{\mathcal{T}_1}(\tilde{\alpha}^\perp), \\
&\quad \text{where } \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = |\mathcal{T}_1|.
\end{aligned}$$

Combining these observations with (3.41) and (3.42), we obtain

$$\begin{aligned}
|\mathcal{K}_2| &= 3(d-1)^2|\mathcal{K}_1| - 2(7|\mathcal{N}_1| + (14d-39)|\mathcal{N}_{1,1}| + (7d^2 - 39d + 54)) - 2|\mathcal{T}_1| \\
&= 12(d-3)(3d^3 - 6d^2 - 11d + 18).
\end{aligned}$$

3.5.6 The Number \mathcal{N}_3

We take $\mathcal{D} \approx \mathbb{P}^3$ as above and define $\mathcal{N}''_1, \tilde{\mathcal{N}}'_{2,0}, \tilde{\mathcal{N}}'_2, M, M^0, \tilde{\mathcal{N}}_3, \mathcal{Z}_{k,i}$ for $k=0, 1$ and $i=1, 2$, $\mathcal{Z}_{0,12}$, and α as in Lemmas 3.8 and Lemma 3.9, except now

$$\begin{aligned}
\varphi_2 &\in \Gamma(\mathcal{N}''_1 \times \mathbb{P}^2_2; \gamma_0^* \otimes \gamma_2^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_2^{*\otimes d} \otimes T^*\mathbb{P}^2_2), & \varphi_3 &\in \Gamma(M; \gamma_0^* \otimes \gamma_3^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_3^{*\otimes d} \otimes T^*\mathbb{P}^2_3), \\
&\text{and } \alpha &\in \Gamma(\tilde{\mathcal{N}}'_2; \text{Hom}(T\mathbb{P}^2_1, \gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2_1)).
\end{aligned}$$

Since $\varphi_3|_{M^0}$ is transverse to the zero set,

$$\begin{aligned}
|\tilde{\mathcal{N}}_3| &= \pm |\varphi_3^{-1}(0) \cap M^0| = \langle e(\gamma_0^* \otimes \gamma_3^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_3^{*\otimes d} \otimes T^*\mathbb{P}^2_3), M \rangle - \mathcal{C}_{\partial M}(\varphi_3) \\
&= \langle (y + da_3)(y^2 + (2d-3)ya_3 + (d^2 - 3d + 3)a_3^2), \tilde{\mathcal{N}}'_2 \times \mathbb{P}^2_3 \rangle - \mathcal{C}_{\partial M}(\varphi_3) \\
&= 3(d-1)^2 \langle y, \tilde{\mathcal{N}}'_2 \rangle - \mathcal{C}_{\partial M}(\varphi_3) = 6(d-1)^2|\mathcal{N}_2| - 2\mathcal{C}_{\mathcal{Z}_{1,1}}(\varphi_3) - 2\mathcal{C}_{\mathcal{Z}_{0,1}}(\varphi_3) - \mathcal{C}_{\mathcal{Z}_{0,12}}(\varphi_3).
\end{aligned} \tag{3.43}$$

Similarly to the proof of Lemma 3.9, we have

$$\mathcal{C}_{\mathcal{Z}_{1,1}}(\varphi) = N(\alpha), \quad \mathcal{C}_{\mathcal{Z}_{0,1}}(\varphi) = 3|\mathcal{K}_2|, \quad \text{and} \quad \mathcal{C}_{\mathcal{Z}_{0,12}}(\varphi) = 4|\mathcal{T}_1|. \tag{3.44}$$

In order to compute $N(\alpha)$, we define $M, \mathcal{Z}_{0,1}$, and $\mathcal{Z}_{0,2}$ as in the proof of Lemma 3.10. By the same argument as before, we find that

$$\begin{aligned}
N(\alpha) &= \langle c(\gamma_0^* \otimes \gamma_1^{*\otimes d} \oplus \gamma_0^* \otimes \gamma_1^{*\otimes d} \otimes T^*\mathbb{P}^2_1) c(T\mathbb{P}^2_1)^{-1}, \tilde{\mathcal{N}}'_2 \rangle - \mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^\perp) \\
&= \langle 3y + (3d-2)a_1, \tilde{\mathcal{N}}'_2 \rangle - \mathcal{C}_{\partial M}(\tilde{\alpha}^\perp) = (6|\mathcal{N}_2| + 3(d-2)|\mathcal{N}_{2,1}|) - \mathcal{C}_{\mathcal{Z}_{0,1}}(\tilde{\alpha}^\perp) - \mathcal{C}_{\mathcal{Z}_{0,2}}(\tilde{\alpha}^\perp), \\
&\quad \text{where } \mathcal{C}_{\mathcal{Z}_{0,1}}(\tilde{\alpha}^\perp) = |\mathcal{K}_2| \quad \text{and} \quad \mathcal{C}_{\mathcal{Z}_{0,2}}(\tilde{\alpha}^\perp) = 2|\mathcal{T}_1|.
\end{aligned}$$

Combining this result with (3.43) and (3.44), we conclude that

$$\begin{aligned}
|\tilde{\mathcal{N}}_3| &= 6((d-1)^2 - 2)|\mathcal{N}_2| - 6(d-2)|\mathcal{N}_{2,1}| - 4|\mathcal{K}_2| \\
&= 3(9d^6 - 54d^5 + 9d^4 + 423d^3 - 458d^2 - 829d + 1050).
\end{aligned}$$

Remark: For $d \geq 5$, the middle component of the polynomial α_0 in the proof of Lemma 3.9 should be increased by $\frac{1}{4}\beta_5 v^5$. However, this term vanishes as we proceed from (3.27) to (3.28).

4 Stable Maps and Recursive Formula

4.1 The Moduli Space of Four Marked Points on a Sphere

In this section, we derive recursion (1.1), following the argument in [RuT]. We start by defining an invariant that counts holomorphic maps into \mathbb{P}^n . A priori, the number we describe depends on the cross ratio of the chosen four points on a sphere. However, it turns out that this number is well-defined. We use its independence to express this invariant in terms of the numbers n_d in two different ways. By comparing the two expressions, we obtain (1.1).

Let x_0, x_1, x_2 and x_3 be the four points in \mathbb{P}^2 given by

$$x_0 = [1, 0, 0], \quad x_1 = [0, 1, 0], \quad x_2 = [0, 0, 1], \quad x_3 = [1, 1, 1].$$

We denote by $H^0(\mathbb{P}^2; \gamma^{*\otimes 2})$ the space of holomorphic sections of the holomorphic line bundle $\gamma^{*\otimes 2} \rightarrow \mathbb{P}^2$, or equivalently of the degree-two homogeneous polynomials in three variables; see Lemma A.3. Let

$$\begin{aligned} \mathcal{U} &= \{([s], x) \in \mathbb{P}H^0(\mathbb{P}^2; \gamma^{*\otimes 2}) \times \mathbb{P}^2 : s(x_i) = 0 \ \forall i = 0, 1, 2, 3; \ s(x) = 0\} \\ &\approx \{([A, B]; [z_0, z_1, z_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 : (A - B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0\}. \end{aligned}$$

The space \mathcal{U} is a compact complex two-manifold.

Let $\pi : \mathcal{U} \rightarrow \overline{\mathfrak{M}}_{0,4} \equiv \mathbb{P}^1$ denote the projection onto the first component. If $[A, B] \in \overline{\mathfrak{M}}_{0,4}$, the fiber $\pi^{-1}([A, B])$ is the conic

$$\mathcal{C}_{A,B} = \{[z_0, z_1, z_2] \in \mathbb{P}^2 : (A - B)z_0z_1 - Az_1z_2 + Bz_0z_2 = 0\}.$$

If $[A, B] \neq [1, 0], [0, 1], [1, 1]$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero. In other words, $\mathcal{C}_{A,B}$ is a sphere with four distinct marked points by Lemma 2.1. If $[A, B] = [1, 0], [0, 1], [1, 1]$, $\mathcal{C}_{A,B}$ is a union of two lines. One of the lines contains two of the four points x_0, \dots, x_3 , and the other line passes through the remaining two points. The two lines intersect in a single point. Figure 2 shows the three singular fibers of the projection map $\pi : \mathcal{U} \rightarrow \overline{\mathfrak{M}}_{0,4}$. The other fibers are smooth conics. The fibers should be viewed as lying in planes orthogonal to the horizontal line in the figure.

We conclude this subsection with a few remarks concerning the family $\mathcal{U} \rightarrow \overline{\mathfrak{M}}_{0,4}$. These remarks are irrelevant for the purposes of the next subsection and can be omitted.

If $[A, B] \in \overline{\mathfrak{M}}_{0,4} - \{[1, 0], [0, 1], [1, 1]\}$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero, i.e. it is a sphere holomorphically embedded in \mathbb{P}^2 . Thus, there exists a one-to-one holomorphic map $f : \mathbb{P}^1 \rightarrow \mathcal{C}_{A,B}$. Using Lemma A.1, it can be shown directly that if $[u_i, v_i] = f^{-1}(x_i)$,

$$\frac{v_0/u_0 - v_2/u_2}{v_0/u_0 - v_3/u_3} : \frac{v_1/u_1 - v_2/u_2}{v_1/u_1 - v_3/u_3} = \frac{B}{A}.$$

The cross-ratio is the only invariant of four distinct points on \mathbb{P}^1 ; see [A], for example. Thus,

$$\begin{aligned} \mathbb{P}^1 - \{[1, 0], [0, 1], [1, 1]\} &= \mathfrak{M}_{0,4} \equiv \{(x_0, x_1, x_2, x_3) \in (\mathbb{P}^1)^4 : x_i \neq x_j \text{ if } i \neq j\} / \sim, \\ \text{where } (x_0, x_1, x_2, x_3) &\sim (\tau(x_0), \tau(x_1), \tau(x_2), \tau(x_3)) \quad \text{if } \tau \in \text{PSL}_2 \equiv \text{Aut}(\mathbb{P}^1). \end{aligned}$$

Furthermore, the restriction of the projection map $\pi : \mathcal{U}|_{\mathfrak{M}_{0,4}} \rightarrow \mathfrak{M}_{0,4}$ to each fiber $\mathcal{C}_{[A,B]}$ is the cross ratio of the points x_0, \dots, x_3 on $\mathcal{C}_{[A,B]}$, viewed as an element of $\mathbb{P}^1 \supset \mathbb{C}$.

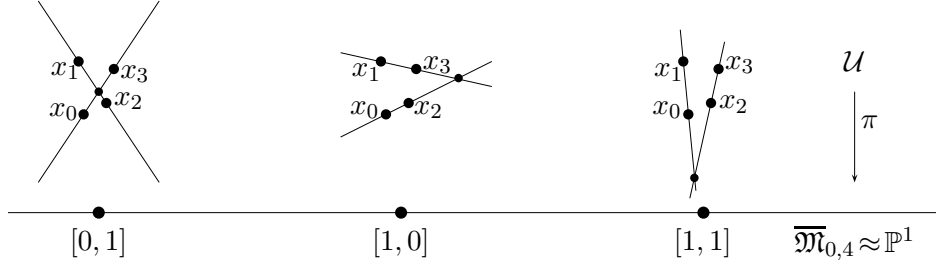


Figure 2: The Family $\mathcal{U} \longrightarrow \overline{\mathfrak{M}}_{0,4}$

4.2 Counts of Holomorphic Maps

If d is an integer and \mathcal{C} is a complex curve, which may be a wedge of spheres, let

$$\mathcal{H}_d(\mathcal{C}) = \{f \in C^\infty(\mathcal{C}; \mathbb{P}^2) : f \text{ is holomorphic, } f_*[\mathcal{C}] = d[L]\}, \quad (4.1)$$

where $[L] \in H_2(\mathbb{P}^2; \mathbb{Z})$ is the homology class of a line in \mathbb{P}^2 . We give a more explicit description of the space $\mathcal{H}_d(\mathcal{C})$ in the relevant cases below.

Suppose ℓ_0, ℓ_1 and p_2, \dots, p_{3d-1} are two lines and $3d-2$ points in general position in \mathbb{P}^2 . If $\sigma \in \overline{\mathfrak{M}}_{0,4}$, let $N_d^\sigma(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ denote the cardinality of the set

$$\{f \in \mathcal{H}_d(\mathcal{C}_\sigma) : f(x_0) \in \ell_0, f(x_1) \in \ell_1, f(x_2) = p_2, f(x_3) = p_3, p_i \in \text{Im } f \forall i\}. \quad (4.2)$$

Here \mathcal{C}_σ denotes the rational curve with four marked points, x_0, x_1, x_2 , and x_3 , whose cross ratio is σ ; see Subsection 4.1. If $\sigma \neq [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a sphere with four, distinct, marked points. In this case, the condition $f \in \mathcal{H}_d(\mathcal{C}_\sigma)$ means that f has the form

$$f([u, v]) = [P_0(u, v), P_1(u, v), P_2(u, v)] \quad \forall [u, v] \in \mathbb{P}^1,$$

for some degree- d homogeneous polynomials P_0, P_1, P_2 that have no common factor; see Lemma A.1. If $\sigma = [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a wedge of two spheres, $\mathcal{C}_{\sigma,1}$ and $\mathcal{C}_{\sigma,2}$, with two marked points each. In this case, the first condition in (4.1) means that f is continuous and $f|_{\mathcal{C}_{\sigma,1}}$ and $f|_{\mathcal{C}_{\sigma,2}}$ are holomorphic. The second condition in (4.1) means that $d = d_1 + d_2$ if $f_*[\mathcal{C}_{\sigma,1}] = d_1[L]$ and $f_*[\mathcal{C}_{\sigma,2}] = d_2[L]$.

The requirement that the two lines, ℓ_0 and ℓ_1 , and the $3d-2$ points, p_2, \dots, p_{3d-1} , are in general position means that they lie in a dense open subset \mathcal{U}_σ of the space of all possible tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$:

$$\mathfrak{X} \equiv \text{Gr}_2\mathbb{C}^3 \times \text{Gr}_2\mathbb{C}^3 \times (\mathbb{P}^2)^{3d-2}.$$

Here $\text{Gr}_2\mathbb{C}^3$ denotes the Grassmanian manifold of two-planes through the origin in \mathbb{C}^3 , or equivalently of lines in \mathbb{P}^2 . The dense open subset \mathcal{U}_σ of \mathfrak{X} consists of tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ that satisfy a number of geometric conditions. In particular, $\ell_0 \neq \ell_1$, none of the points p_2, \dots, p_{3d-1} lies on either ℓ_0 or ℓ_1 , the $3d-1$ points $\ell_0 \cap \ell_1, p_2, \dots, p_{3d-1}$ are distinct, no three of them lie on the same line, and so on. In addition, we need to impose certain cross-ratio conditions on the

rational curves that pass through ℓ_0, ℓ_1, p_2, p_3 , and a subset of the remaining $3d-4$ points. These conditions can be stated more formally. Define

$$\text{ev}_\sigma: \mathcal{H}_d(\mathcal{C}_\sigma) \times (\mathcal{C}_\sigma)^{3d-4} \longrightarrow (\mathbb{P}^2)^{3d} \quad \text{by} \quad \text{ev}_\sigma(f; x_4, \dots, x_{3d-1}) = (f(x_0), f(x_1), \dots, f(x_{3d-1})).$$

Lemma A.1 implies that $\mathcal{H}_d(\mathcal{C}_\sigma)$ is a dense open subset of \mathbb{P}^{3d+2} and the evaluation map ev_σ is holomorphic. The space $\mathcal{H}_d(\mathcal{C}_\sigma)$ has a natural compactification $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$, which is the union of spaces of holomorphic maps from various wedges of spheres into \mathbb{P}^2 . The complex dimension of each such boundary stratum is less than that of $\mathcal{H}_d(\mathcal{C}_\sigma)$. The evaluation map ev_σ admits a continuous extension over $\partial\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$, whose restriction to each stratum is holomorphic. The elements $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ of the subspace \mathcal{U}_σ of \mathfrak{X} are characterized by the condition that the restriction of the evaluation map to each stratum of $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$ is transversal to the submanifold

$$\ell_0 \times \ell_1 \times p_2 \times \dots \times p_{3d-1} \subset (\mathbb{P}^2)^{3d}.$$

This condition implies that

$$\text{ev}_\sigma^{-1}(\ell_0 \times \ell_1 \times p_2 \times \dots \times p_{3d-1}) \cap \partial\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d) = \emptyset$$

and the set in (4.2) is a finite subset of $\mathcal{H}_d(\mathcal{C}_\sigma)$.

The set \mathcal{U}_σ of “general” tuples $(\ell_0, \ell_1, p_2, \dots, p_{3d-1})$ is path-connected. Indeed, it is the complement of a finite number of proper complex submanifolds in \mathfrak{X} . It follows that the number in (4.2) is independent of the choice of two lines and $3d-2$ points in general position in \mathbb{P}^2 . We thus may simply denote it by N_d^σ . If $\sigma \neq [1, 0], [0, 1], [1, 1]$, \mathcal{C}_σ is a sphere with four distinct points. In such a case, it is fairly easy to show that the number N_d^σ does not change under small variations of σ , or equivalently of the four points on the sphere. Thus, N_d^σ is independent of

$$\sigma \in \mathfrak{M}_{0,4} = \mathbb{P}^1 - \{[1, 0], [0, 1], [1, 1]\} = \overline{\mathfrak{M}}_{0,4} - \{[1, 0], [0, 1], [1, 1]\}.$$

It is far harder to prove

Proposition 4.1 *The function $\sigma \longrightarrow N_d^\sigma$ is constant on $\overline{\mathfrak{M}}_{0,4}$.*

This proposition is a special case of the gluing theorems first proved in [McSa] and [RuT]. A more straightforward proof can be obtained via the approach of [LT].

4.3 Holomorphic Maps vs. Complex Curves

In this subsection, we express the numbers $N_d^{[0,1]}$ and $N_d^{[1,1]}$ of Subsections 4.2 in terms of the numbers $n_{d'}$, with $d' \leq d$, of Question 1.1. By Proposition 4.1, $N_d^{[0,1]} = N_d^{[1,1]}$. We obtain a recursion for the numbers of Question 1.1 by comparing the expressions for $N_d^{[0,1]}$ and $N_d^{[1,1]}$.

Let \mathcal{C}_1 denote the component of $\mathcal{C}_{[0,1]}$ containing the marked points x_0 and x_3 ; see Figure 2. We denote by \mathcal{C}_2 the other component of $\mathcal{C}_{[0,1]}$. By definition,

$$N_d^{[0,1]} = \sum_{d_1+d_2=d} N_{d_1, d_2}^{[0,1]} \quad \text{where}$$

$$N_{d_1, d_2}^{[0,1]} = |\{f \in \mathcal{H}_d(\mathcal{C}_{[0,1]}; \mathbb{P}^2) : f_*[\mathcal{C}_1] = d_1[L], f_*[\mathcal{C}_2] = d_2[L]; p_i \in \text{Im } f \ \forall i; f(x_0) \in \ell_0, f(x_1) \in \ell_1, f(x_2) = p_2, f(x_3) = p_3\}|.$$

Since the group PSL_2 of holomorphic automorphisms acts transitively on triples of distinct points on the sphere,

$$N_{d_1, d_2}^{[0,1]} = \left| \left\{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : \begin{aligned} & f_1(\infty) = f_2(\infty), \quad p_i \in f_1(S^2) \cup f_2(S^2) \quad \forall i; \\ & f_1(0) \in \ell_0, \quad f_1(1) = p_3, \quad f_2(0) \in \ell_1, \quad f_2(1) = p_2 \end{aligned} \right\} \right|.$$

Since the maps f_1 and f_2 above are holomorphic, $d_1, d_2 \geq 0$ if $N_{d_1, d_2}^{[0,1]} \neq 0$. Since every degree-zero holomorphic map is constant and $p_3 \notin \ell_0$, $N_{0, d}^{[0,1]} = 0$. Similarly, $N_{d, 0}^{[0,1]} = 0$. Thus, we assume that $d_1, d_2 > 0$. Since the points p_3, \dots, p_{3d-1} are in general position, $f_1(S^2)$ contains at most $3d_1 - 2$ of the points p_4, \dots, p_{3d-1} . Similarly, the curve $f_2(S^2)$ passes through at most $3d_2 - 2$ of the points p_4, \dots, p_{3d-1} . Thus, if $I = \{4, \dots, 3d-1\}$,

$$N_{d_1, d_2}^{[0,1]} = \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1, d_2}^{[0,1]}(I_1, I_2),$$

where $N_{d_1, d_2}^{[0,1]}(I_1, I_2)$ is the cardinality of the set

$$\mathcal{S}_{d_1, d_2}^{[0,1]}(I_1, I_2) = \left\{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : \begin{aligned} & p_i \in f_1(S^2) \quad \forall i \in I_1, \quad p_i \in f_2(S^2) \quad \forall i \in I_2; \\ & f_1(\infty) = f_2(\infty), \quad f_1(0) \in \ell_0, \quad f_1(1) = p_3, \quad f_2(0) \in \ell_1, \quad f_2(1) = p_2 \end{aligned} \right\}.$$

If $(f_1, f_2) \in \mathcal{S}_{d_1, d_2}^{[0,1]}(I_1, I_2)$, $f_1(S^2)$ is one of the n_{d_1} curves passing through the points $\{p_i : i \in \{3\} \sqcup I_1\}$. Similarly, $f_2(S^2)$ is one of the n_{d_2} curves passing through the points $\{p_i : i \in \{2\} \sqcup I_2\}$. The point $f_1(\infty) = f_2(\infty)$ must be one of the $d_1 d_2$ points of $f_1(S^2) \cap f_2(S^2)$; see Lemma A.5. Finally, $f_1(0)$ must be one of the d_1 points of $f_1(S^2) \cap \ell_0$, while $f_2(0)$ must be one of the d_2 points of $f_2(S^2) \cap \ell_1$. Thus, we conclude that

$$\begin{aligned} N_d^{[0,1]} &= \sum_{d_1+d_2=d} N_{d_1, d_2}^{[0,1]} = \sum_{d_1+d_2=d} \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-2} N_{d_1, d_2}^{[0,1]}(I_1, I_2) \\ &= \sum_{d_1+d_2=d} \sum_{I_1 \subset I, |I_1|=3d_1-2} (d_1 d_2) (d_1 n_{d_1}) (d_2 n_{d_2}) \\ &= \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-2} d_1^2 d_2^2 n_{d_1} n_{d_2}; \end{aligned} \tag{4.3}$$

where $I = \{4, \dots, 3d-1\}$.

We compute the number $N_d^{[1,1]}$ similarly. We denote by \mathcal{C}_1 the component of $\mathcal{C}_{[1,1]}$ containing the points x_0 and x_1 and by \mathcal{C}_2 the other component of $\mathcal{C}_{[1,1]}$. By definition,

$$\begin{aligned} N_d^{[1,1]} &= \sum_{d_1+d_2=d} N_{d_1, d_2}^{[1,1]}, \quad \text{where} \\ N_{d_1, d_2}^{[1,1]} &= \left| \left\{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : \begin{aligned} & f_1(\infty) = f_2(\infty), \quad p_i \in f_1(S^2) \cup f_2(S^2) \quad \forall i; \\ & f_1(0) \in \ell_0, \quad f_1(1) \in \ell_1, \quad f_2(0) = p_2, \quad f_2(1) = p_3 \end{aligned} \right\} \right|. \end{aligned}$$

Since every degree-zero holomorphic map is constant, $N_{d,0}^{[1,1]} = 0$ as before. However,

$$N_{0,d}^{[1,1]} = \left| \left\{ f_2 \in \mathcal{H}_d(S^2) : f_2(\infty) \in \ell_0 \cap \ell_1, f_2(0) = p_2, f_2(1) = p_3; \right. \right. \\ \left. \left. p_i \in f_2(S^2) \forall i = 4, \dots, 3d-1 \right\} \right|.$$

Thus, $N_{0,d}^{[1,1]} = n_d$. If $d_1, d_2 > 0$,

$$N_{d_1,d_2}^{[1,1]} = \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-1} N_{d_1,d_2}^{[1,1]}(I_1, I_2),$$

where $N_{d_1,d_2}^{[1,1]}(I_1, I_2)$ is the cardinality of the set

$$\mathcal{S}_{d_1,d_2}^{[1,1]}(I_1, I_2) = \left\{ (f_1, f_2) \in \mathcal{H}_{d_1}(S^2) \times \mathcal{H}_{d_2}(S^2) : p_i \in f_1(S^2) \forall i \in I_1, p_i \in f_2(S^2) \forall i \in I_2; \right. \\ \left. f_1(\infty) = f_2(\infty), f_1(0) \in \ell_0, f_1(1) \in \ell_1, f_2(0) = p_2, f_2(1) = p_3 \right\}.$$

Proceeding as in the previous paragraph, we conclude that

$$\begin{aligned} N_d^{[1,1]} &= \sum_{d_1+d_2=d} N_{d_1,d_2}^{[1,1]} = n_d + \sum_{d_1+d_2=d} \sum_{I=I_1 \sqcup I_2, |I_1|=3d_1-1} N_{d_1,d_2}^{[1,1]}(I_1, I_2) \\ &= n_d + \sum_{d_1+d_2=d} \sum_{I_1 \subset I, |I_1|=3d_1-1} (d_1 d_2) (d_1^2 n_{d_1}) (n_{d_2}) \\ &= n_d + \sum_{d_1+d_2=d} \binom{3d-4}{3d_1-1} d_1^3 d_2 n_{d_1} n_{d_2}; \end{aligned} \tag{4.4}$$

Comparing equations (4.3) and (4.4), we obtain

$$n_d = \sum_{d_1+d_2=d} \left(\binom{3d-4}{3d_1-2} d_1 d_2 - \binom{3d-4}{3d_1-1} d_1^2 \right) d_1 d_2 n_{d_1} n_{d_2}. \tag{4.5}$$

The recursive formula (1.1) is the symmetrized version of (4.5).

A The Basics

A.1 Complex Projective Spaces

The *complex projective space* \mathbb{P}^n is the space of (complex) lines through the origin in \mathbb{C}^{n+1} . Equivalently,

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \mathbb{C}^*, \quad \text{where } (z_0, \dots, z_n) \sim (tz_1, \dots, tz_n) \text{ if } t \in \mathbb{C}^*.$$

This space is a smooth $2n$ -manifold. For $i=0, \dots, n$, let

$$U_i = \{[z_0, \dots, z_n] \in \mathbb{P}^n : z_i \neq 0\}, \\ \phi_i : \mathbb{C}^n \longrightarrow U_i, \quad \phi_i(w_1, \dots, w_n) = [w_1, \dots, w_i, 1, w_{i+1}, \dots, w_n].$$

The set $\{(U_i, \phi_i, \mathbb{C}^n)\}$ is the *standard atlas* for \mathbb{P}^n . If $i < j$, the corresponding overlap map is given by

$$\begin{aligned} \phi_{ij} &\equiv \phi_i^{-1} \circ \phi_j|_{\phi_j^{-1}(U_i)}: \{(w_1, \dots, w_n) \in \mathbb{C}^n: w_{i+1} \neq 0\} \longrightarrow \{(w_1, \dots, w_n) \in \mathbb{C}^n: w_j \neq 0\} \\ (w_1, \dots, w_n) &\longrightarrow \left(\frac{w_1}{w_{i+1}}, \dots, \frac{w_i}{w_{i+1}}, \frac{w_{i+2}}{w_{i+1}}, \dots, \frac{w_j}{w_{i+1}}, w_{i+1}^{-1}, \frac{w_{j+1}}{w_{i+1}}, \dots, \frac{w_n}{w_{i+1}} \right). \end{aligned}$$

Each map ϕ_{ij} is a diffeomorphism. In fact, this map is holomorphic, and so is its inverse ϕ_{ij}^{-1} . In other words, \mathbb{P}^n is naturally a *complex n -manifold*.

Suppose X and Y are complex manifolds, of complex dimensions m and n , and with (holomorphic) atlases $\{(U_i, \phi_i, U'_i)\}_{i \in I}$ and $\{(V_j, \varphi_j, V'_j)\}_{j \in J}$, respectively. A smooth map $f: X \longrightarrow Y$ is called *holomorphic* if for all $i \in I$ and $j \in J$, the map

$$\varphi_j^{-1} \circ f \circ \phi_i: \phi_i^{-1}(f^{-1}(V_j)) \longrightarrow \mathbb{C}^n$$

is holomorphic as a \mathbb{C}^n -valued function on an open subset of \mathbb{C}^m . In the case of interest to us, i.e. $X = \mathbb{P}^1$ and $Y = \mathbb{P}^n$, the holomorphic maps have a much simpler description, see Lemma A.1 below. This lemma can be checked directly. The simpler characterization of Lemma A.1 can be taken as the definition of what it means to be a holomorphic map between \mathbb{P}^1 and \mathbb{P}^n .

Lemma A.1 *If $f: \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ is a holomorphic map, there exist homogeneous polynomials p_0, \dots, p_n in two variables such that p_0, \dots, p_n are of the same degree, have no common factor, and*

$$f([z_0, z_1]) = [p_0(z_0, z_1), \dots, p_n(z_0, z_1)] \quad \forall [z_0, z_1] \in \mathbb{P}^1. \quad (\text{A.1})$$

Conversely, if p_0, \dots, p_n are homogeneous polynomials in two variables that are of the same degree and have no common factor, the map $f: \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ given by (A.1) is well-defined and holomorphic.

A.2 Almost Complex and Symplectic Structures

This subsection is not relevant for understanding Sections 2-4. However, it puts the last section in perspective.

Let X be a smooth manifold. An *almost complex structure* on X is a smooth section J of the bundle $\text{End}(TX) \longrightarrow X$ such that $J^2 = -I$. In other words, an almost complex structure is a smooth family of linear maps $J_p: T_p X \longrightarrow T_p X$ such that $J_p J_p v = -v$ for all $v \in T_p X$ and $p \in X$. For example, if $X = \mathbb{C}^n$, $T_p \mathbb{C}^n = \mathbb{C}^n$ and the desired endomorphism on $T_p \mathbb{C}^n$ is simply the multiplication by $i \equiv \sqrt{-1}$.

Every complex n -manifold X carries a natural almost complex structure J , defined as follows. Let $\{(U_i, \phi_i, U'_i)\}_{i \in I}$ be the (holomorphic) atlas for X . If $p \in U_i$, we set

$$J_p = d\phi_i|_{\phi_i^{-1}(p)} \circ i \circ d\phi_i^{-1}|_p.$$

Since all overlap maps $\phi_i^{-1} \circ \phi_j$ are holomorphic, the endomorphism J_p is independent of the choice of $i \in I$ such that $p \in U_i$. An almost complex structure arising in such a way is called *complex or integrable*.

A typical almost complex structure is not integrable, unless the real dimension of the manifold is two. In fact, there is a criterion that characterizes integrable almost complex structures. If (X, J) is an almost complex manifold, $p \in X$, and V and W are vector fields on X , let

$$N_p^J(V_p, W_p) = \frac{1}{4}([V, W]_p + J_p[JV, W]_p + J_p[V, JW]_p - [JV, JW]_p).$$

The vector $N_p^J(V_p, W_p) \in T_p X$ depends only on the values V_p and W_p of the vector fields V and W at the point p . In addition, N_p^J is linear in each of the two inputs. Thus,

$$N^J \in \Gamma(X; \text{Hom}(TX \otimes TX, TX)),$$

i.e. N^J is a $(2, 1)$ -tensor field on X . This tensor field is called the *Nijenhuis torsion* of J . It is easy to see that $N^J \equiv 0$ if J is an integrable almost complex structure. The converse is proved in [NeNi]. Since $N^J \equiv 0$ if (X, J) is an almost complex manifold of real dimension two, it follows every almost complex structure on a smooth two-manifold is integrable. Such a manifold is called a *Riemann surface*.

Suppose (X, j) and (Y, J) are almost complex manifolds and $f: X \rightarrow Y$ is a smooth map. If $z \in X$, we set

$$\bar{\partial}_{J,j} f|_z = df|_z + J_{f(z)} \circ df|_z \circ j_z \in \text{Hom}(T_z X, T_{f(z)} Y).$$

Note that $\bar{\partial}_{J,j} f|_z \circ j_z = -J_{f(z)} \circ \bar{\partial}_{J,j} f|_z$, i.e. the linear map $\bar{\partial}_{J,j} f|_z$ is (J, j) -antilinear. Thus,

$$\bar{\partial}_{J,j} f \in \Gamma(X, \Lambda_{J,j}^{0,1} T^* X \otimes f^* T Y),$$

where $\Lambda_{J,j}^{0,1} T^* X \otimes f^* T Y \rightarrow X$ is the bundle of $(f^* J, j)$ -antilinear homomorphisms from (TX, j) to $f^*(TY, J)$. The smooth map $f: X \rightarrow Y$ is called (J, j) -*holomorphic*, or *pseudoholomorphic*, if $\bar{\partial}_{J,j} f \equiv 0$. If (X, j) and (Y, J) are complex manifolds, this definition agrees with the one given in the previous subsection. More generally, if (X, j) is a wedge of finitely many almost complex manifolds (X_l, j_l) , we will call a continuous map $f: X \rightarrow Y$ (J, j) -*holomorphic* if $f|_{X_l}$ is (J, j_l) -holomorphic for all l .

If (X, J) is an almost complex manifold, $A \in H_2(X; \mathbb{Z})$, and g and n are nonnegative integers, let

$$\mathfrak{M}_{g,n}(X, A; J) = \left\{ (\Sigma, j, x_1, \dots, x_n; f) : \begin{array}{l} (\Sigma, j) = \text{Riemann surface of genus } g; \\ x_i \in \Sigma, x_i \neq x_j \text{ if } i \neq j; f \in C^\infty(\Sigma; X), f_*[\Sigma] = A, \bar{\partial}_{J,j} f = 0 \end{array} \right\} / ,$$

$$\text{where } (\Sigma, j, z_1, \dots, z_n; f) \sim (\Sigma', j', \tau(z_1), \dots, \tau(z_n), f \circ \tau^{-1}) \text{ if } \tau \in C^\infty(\Sigma; \Sigma'), \bar{\partial}_{j',j'} \tau = 0.$$

This moduli space has a natural topology, as well as n evaluation maps

$$\text{ev}_i: \mathfrak{M}_{g,n}(X, A; J) \rightarrow X, \quad [\Sigma, j, z_1, \dots, z_n; f] \rightarrow f(z_i).$$

In general, $\mathfrak{M}_{g,n}(X, A; J)$ is not a compact topological space. However, under certain conditions on (X, J) , $\mathfrak{M}_{g,n}(X, A; J)$ admits a natural compactification and in fact carries a (virtual) fundamental class.

Let X be a smooth manifold. A *symplectic form* on X is a closed two-form ω on X which is nondegenerate at every point of X . In other words, $d\omega = 0$, and for every point p in X and nonzero tangent vector $v \in T_p X$, there exists $w \in T_p X$ such that $\omega_p(v, w) \neq 0$. For example, if $(x_1, y_1, \dots, x_n, y_n)$ are the standard coordinates on \mathbb{C}^n ,

$$\omega \equiv dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$$

is a symplectic form on \mathbb{C}^n . More generally, if X admits a symplectic form, the (real) dimension of X is even.

If (X, ω) is a symplectic manifold, the almost complex structure J on X is ω -tame if for every point p in X and nonzero tangent vector $v \in T_p X$, $\omega_p(v, J_p v) > 0$. The ω -tame almost complex structure J is ω -compatible if

$$\omega_p(Jv, Jw) = \omega_p(v, w) \quad \forall p \in X, v, w \in T_p X.$$

For example, if ω is the standard symplectic form on \mathbb{C}^n , defined in the previous paragraph, the standard complex structure i , defined in the second paragraph of this subsection, is ω -compatible. For a general symplectic manifold (X, ω) , the spaces of ω -tame and ω -compatible almost complex structures on X are non-empty and contractible. The most fundamental result in the theory of pseudoholomorphic curves is Gromov's Compactness Theorem, stated roughly below.

Theorem A.2 [Gro] *Suppose (X, ω) is a compact symplectic manifold and J is an almost complex ω -tame structure on X . If $A \in H_2(X; \mathbb{Z})$ and g and n are nonnegative integers, the moduli space $\mathfrak{M}_{g,n}(X, A; J)$ admits a natural compactification $\overline{\mathfrak{M}}_{g,n}(X, A; J)$. In particular, the evaluation maps ev_i extend continuously over $\overline{\mathfrak{M}}_{g,n}(X, A; J)$.*

The compactification $\overline{\mathfrak{M}}_{g,n}(X, A; J)$ consists of equivalence classes of tuples $(\Sigma, j, x_1, \dots, x_n, f)$, where (Σ, j) is a possibly singular genus- g Riemann surface, i.e. a wedge of smooth Riemann surfaces, x_1, \dots, x_n are distinct points on Σ , and $f: \Sigma \rightarrow X$ is a (J, j) -holomorphic map such that $f_*[\Sigma] = A$. Notice that the space $\overline{\mathfrak{M}}_{g,n}(X, A; J)$ is described by the almost complex structure J , and not the symplectic form ω . However, this space may not be compact if J is not ω -tame for some symplectic form ω on X .

Since the space of ω -tame almost complex structures on X is contractible, up to an appropriate equivalence, the space $\overline{\mathfrak{M}}_{g,n}(X, A; J)$ is independent of the choice of J . In particular, the "equivalence class" of $\overline{\mathfrak{M}}_{g,n}(X, A; J)$ is determined by (X, ω) and thus is a symplectic invariant. This is essentially the *Gromov-Witten invariant* of (X, ω) .

A.3 Tautological Line Bundle

We continue with the notation of Subsection A.1. Let

$$\gamma = \{(\ell; z_0, \dots, z_n) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : (z_0, \dots, z_n) \in \ell\}.$$

We denote by $\pi: \gamma \rightarrow \mathbb{P}^n$ the projection map. For each $\ell \in \mathbb{P}^n$, the fiber $\gamma_\ell \equiv \pi^{-1}(\ell)$ over a point $\ell \in \mathbb{P}^n$ is the line ℓ through the origin in \mathbb{C}^n . For each $i=0, \dots, n$, let

$$\begin{aligned} \tilde{U}_i &= \pi^{-1}(U_i) = \{(\ell; z_0, \dots, z_n) \in \gamma : z_i \neq 0\}, \\ \tilde{\phi}_i: \mathbb{C}^n \times \mathbb{C} &\rightarrow \tilde{U}_i, \quad \tilde{\phi}_i(w_1, \dots, w_n; \lambda) = (\phi_i(w_1, \dots, w_n); \lambda w_1, \dots, \lambda w_i, \lambda, \lambda w_{i+1}, \dots, \lambda w_n). \end{aligned}$$

The set $\{(\tilde{U}_i, \tilde{\phi}_i, \mathbb{C}^n) \times \mathbb{C}\}$ is the *standard atlas* for γ . If $i < j$, the corresponding overlap map is given by

$$\tilde{\phi}_{ij} \equiv \tilde{\phi}_i^{-1} \circ \tilde{\phi}_j|_{\tilde{\phi}_j^{-1}(\tilde{U}_i)}: \phi_j^{-1}(U_i) \times \mathbb{C} \longrightarrow \phi_i^{-1}(U_j) \times \mathbb{C}, \quad (w_1, \dots, w_n; \lambda) \longrightarrow (\phi_{ij}(w_1, \dots, w_n); w_{i+1}\lambda).$$

Each map $\tilde{\phi}_{ij}$ is holomorphic, and so is its inverse $\tilde{\phi}_{ij}^{-1}$. Thus, γ is a complex $(n+1)$ -manifold. Furthermore, if $p: \mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}^n$ is the projection map,

$$\pi \circ \tilde{\phi}_i = \phi_i \circ p \quad \forall i = 0, \dots, n,$$

and $\tilde{\phi}_i: p^{-1}(\underline{w}) \longrightarrow \pi^{-1}(\phi_i(\underline{w}))$ is a \mathbb{C} -linear map for all $\underline{w} \in \mathbb{C}^n$. Thus, $\gamma \longrightarrow \mathbb{P}^n$ is a *holomorphic rank-one vector bundle*, i.e. a *holomorphic line bundle*.

Each homogeneous polynomial,

$$p = \sum_{i_0 + \dots + i_n = d} a_{i_0 \dots i_n} z_0^{i_0} \dots z_n^{i_n},$$

of degree d in $n+1$ variables determines a section s_p of the bundle $\gamma^{*\otimes d} \longrightarrow \mathbb{P}^n$, described as follows. At each point $\ell \in \mathbb{P}^n$, $s_p(\ell)$ is to be a map from γ_p to \mathbb{C} such that

$$\{s_p(\ell)\}(t\underline{z}) = t^d \{s_p(\ell)\}(\underline{z}) \quad \forall \underline{z} \in \gamma_p = \ell.$$

Thus, we define s_p by

$$\{s_p(\ell)\}(\ell; z_0, \dots, z_n) = p(z_0, \dots, z_n).$$

Lemma A.3 below can be checked directly from the relevant definitions.

Lemma A.3 *If p is a homogeneous polynomial of degree d in $n+1$ variables, s_p is a holomorphic section of the holomorphic line bundle $\gamma^{\otimes *d}$. Conversely, if s is a holomorphic section of $\gamma^{\otimes *d}$, $s = s_p$ for some homogeneous polynomial p of degree d in $n+1$ variables.*

If s is a section of a vector bundle V over a smooth manifold X and $x \in s^{-1}(0)$, the differential of s at x is a well-defined linear map:

$$ds|_x: T_x X \longrightarrow V_x.$$

It can be constructed using either a chart for V or a connection in V . If $ds|_x$ is surjective, s is said to be *transversal to the zero set at x* . If $ds|_x$ is surjective for all $x \in s^{-1}(0)$, s is to be *transverse to the zero set*. If V is a complex vector bundle of rank n , X is a complex n -manifold, and s is transversal to the zero set at $x \in s^{-1}(0)$, x is an isolated point of $s^{-1}(0)$ and $ds|_x: T_x X \longrightarrow V_x$ is an \mathbb{R} -linear map between complex (and thus, oriented) vector spaces. The point x is assigned the plus sign if this map is orientation-preserving and the minus sign otherwise. Note that if s is a holomorphic section, $ds|_x$ is \mathbb{C} -linear and thus orientation-preserving.

We conclude this subsection by proving Lemma 2.1. With notation as before,

$$g(s^{-1}(0)) = \frac{2 - \chi(s^{-1}(0))}{2}, \tag{A.2}$$

where $\chi(s^{-1}(0))$ is the euler characteristic of the surface $s^{-1}(0)$. On the other hand, by Corollary 11.12 in [MiSt] and by Lemma 2.2,

$$\begin{aligned} \chi(s^{-1}(0)) &= \langle e(Ts^{-1}(0)), s^{-1}(0) \rangle = \langle c_1(T\mathbb{P}^2) - c_1(\gamma^{*\otimes d}), s^{-1}(0) \rangle \\ &= \langle (3a - da) \cdot da, \mathbb{P}^2 \rangle = 3d - d^2. \end{aligned} \tag{A.3}$$

Lemma 2.1 follows immediately from (A.2) and (A.3).

A.4 Plane Curves

A (*reduced, complex*) curve \mathcal{C} in \mathbb{P}^2 is a subset of \mathbb{P}^2 of the form

$$\mathcal{C} = \mathcal{C}_{\underline{a}} \equiv \{[X, Y, Z] \in \mathbb{P}^2 : \sum_{j+k+l=d} a_{jkl} X^j Y^k Z^l = 0\},$$

for some positive integer d and some tuple $\underline{a} = (a_{jkl})_{j+k+l=d}$ of complex numbers, not all zero. In other words, a curve in $\mathbb{P}^2 \equiv (\mathbb{C}^3 - \{0\})/\mathbb{C}^*$ is the quotient of the zero set of a nonzero homogeneous polynomial on $\mathbb{C}^3 - \{0\}$ by the \mathbb{C}^* -action. The *degree* $d(\mathcal{C})$ of the curve \mathcal{C} in \mathbb{P}^2 is the minimal degree of a homogeneous polynomial giving rise to \mathcal{C} . Alternatively, $d(\mathcal{C})$ is the positive number such that

$$[\mathcal{C}] = d(\mathcal{C}) \cdot \ell \in H_2(\mathbb{P}^2; \mathbb{Z}),$$

where ℓ is the homology class of a line in \mathbb{P}^2 .

If $\mathcal{C} \subset \mathbb{P}^2$ is a curve, there exists a smooth Riemann surface Σ , possibly not connected, and a holomorphic map $f: \Sigma \rightarrow \mathbb{P}^2$ such that $\mathcal{C} = f(\Sigma)$. The degree of such a map f is the number $d(f)$ such that

$$f_*[\Sigma] = d(f) \cdot \ell \in H_2(\mathbb{P}^2; \mathbb{Z}).$$

If $\mathcal{C} = f(\Sigma)$, $d(\mathcal{C}) \leq d(f)$. If $d(\mathcal{C}) = d(f)$, $f: \Sigma \rightarrow \mathcal{C}$ is a *normalization* of \mathcal{C} . If $f: \Sigma \rightarrow \mathcal{C}$ is a normalization of \mathcal{C} , the (*geometric*) *genus*, $g(\mathcal{C})$, of the curve \mathcal{C} is the genus of Riemann surface Σ .

The following two lemmas can be proved using basic facts from complex analysis and algebraic topology.

Lemma A.4 *Every complex curve $\mathcal{C} \subset \mathbb{P}^2$ admits a normalization $f: \Sigma \rightarrow \mathcal{C}$. If $f_1: \Sigma_1 \rightarrow \mathcal{C}$ and $f_2: \Sigma_2 \rightarrow \mathcal{C}$ are normalizations of \mathcal{C} , there exists a biholomorphism $\tau: \Sigma_1 \rightarrow \Sigma_2$ such that $f_1 = f_2 \circ \tau$.*

Lemma A.5 *If \mathcal{C}_1 and \mathcal{C}_2 are complex plane curves that intersect at a finite number points, then the number of intersection points counted with appropriate positive multiplicities is $d(\mathcal{C}_1) \cdot d(\mathcal{C}_2)$.*

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