Basic Estimates of Riemannian Geometry Used in Gluing Pseudoholomorphic Maps

Aleksey Zinger

September 17, 2004

1 Introduction

In this short note, we collect the basic facts of Riemannian geometry necessary for gluing pseudoholomorphic curves in symplectic geometry via the method of [LT]. Proofs of these facts are omitted in [LT], but only an estimate similar to that of Proposition 2.11 seems to have a well-known proof; see [F]. The estimates given here are actually sharper than needed in [LT] or even in [Z]. However, these sharpened estimates may help globalize the construction in [Z].

Section 2 collects various simple facts from Riemannian geometry to obtain a Poincare lemma for vector fields along closed curves in Proposition 2.5 and an expansion for the $\bar{\partial}$-operator in Proposition 2.11. In Section 3, we refine, in the $n=2$ case, the proofs of Sobolev Embedding Theorems given in [M] to obtain a C^0-estimate in Proposition 3.7 and elliptic estimates for the $\bar{\partial}$-operator in Propositions 3.10 and 3.12 for vector fields along smooth maps into a compact manifold. The main reason for collecting these five propositions here is to describe the dependence of the constants involved on the underlying smooth map. This dependence of constants plays an important role in the gluing constructions.

If $u\colon \mathcal{D}\longrightarrow V$ is a smooth map, we write $\Gamma(u)$ and $\Gamma^1(u)$ for $\Gamma(\mathcal{D};u^*TV)$ and $\Gamma(\mathcal{D};T^*\mathcal{D}\otimes u^*TV)$, respectively. We denote the subspace of compactly supported sections in $\Gamma(u)$ by $\Gamma_c(u)$.

2 Riemannian Geometry Estimates

2.1 Parallel Transport

Let (V,g,∇) be a compact Riemannian manifold, where g is a metric on V and ∇ is a connection in TV, which is metric-compatible, but not necessarily torsion-free. Let T_∇ denote the torsion of ∇. If $p\in V$ and $X\in T_pV$, denote by $\exp_p X$ the exponential of X defined with respect to the connection ∇ and by r_V the injectivity radius of V defined with respect to the connection ∇ and metric g; see [C]. If $\alpha\colon (a,b)\longrightarrow V$ is a piecewise smooth curve, let $\Pi_\alpha\colon T_{\alpha(a)}V\longrightarrow T_{\alpha(b)}V$ denote the parallel-transport map along α defined with respect to the connection ∇. If $R=[a,b]\times[c,d]$ is a rectangle in $\mathbb{R}^2$ and $u\colon R\longrightarrow V$ is a smooth map, let

$$\Pi_{\partial u} : T_{u(a,c)}V \longrightarrow T_{u(a,c)}V$$

be the parallel transport along u restricted to the boundary of R traversed in the positive direction. Denote by Π_X the parallel-transport map along the geodesic $\gamma_X\colon s \longrightarrow \exp_p sX$, where $s\in[0,1]$. If α is a smooth curve and ξ is a smooth vector field along α, we write $\frac{D}{dt}\xi$ for the covariant derivative of ξ along α if the variable parameterizing α is t. More generally, if $u\colon \mathcal{D}\longrightarrow V$ is any smooth map, we define $\nabla\xi\in\Gamma^1(u)$ as follows. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow\mathcal{D}$ is a smooth curve, let

$$\{\nabla\xi\}_{\alpha(0)}\big(\alpha'(0)\big) = \frac{D}{dt}(\xi\circ\alpha)\Big|_{t=0}.$$

Lemma 2.1 *There exists a constant $C\in\mathbb{R}^+$ such that for any smooth map $u\colon R\longrightarrow V$,*

$$|\Pi_{\partial u} - I| \le C\int_R |u_s||u_t|dsdt,$$

where the norm of $(\Pi_{\partial u}-I)\in End(T_{u(a,c)}V)$ is computed with respect to the metric g.

Proof: (1) Let $R=[a,b]\times[c,d]$ be as before. Choose an orthonormal frame $\{X_i\}$ for $T_{u(a,c)}V$. Extend each X_i to a vector field along $t\longrightarrow u(a,t)$, where $t\in[c,d]$, by parallel transport. Then extend each X_i to a vector field along $u\colon R\longrightarrow V$ by parallel-transporting the vector $X_i(a,t)$ along the curves $s\longrightarrow u(s,t)$. By definition, $\frac{D}{ds}X_i=0$ on R. Let A be the matrix-valued function on R such that

$$\frac{D}{dt}X_i\Big|_{(s,t)} = A_{ik}(s,t)X_k(s,t);$$

here and below we use the generalized Einstein summation convention. Note that $A_{ij}(a,t)=0$ and

$$\begin{aligned}\big\langle\mathcal{R}(u_s,u_t)X_i,X_j\big\rangle &= \Big\langle\frac{D}{ds}\frac{D}{dt}X_i - \frac{D}{dt}\frac{D}{ds}X_i, X_j\Big\rangle\\ &= \Big\langle\Big(\frac{\partial}{\partial s}A_{ik}\Big)X_k, X_j\Big\rangle = \frac{\partial}{\partial s}A_{ij}.\end{aligned} \tag{2.1}$$

where $\mathcal{R}$ denotes the Riemann curvature tensor of the connection of ∇. Since V is compact, it follows that

$$|A_{ij}(b,t)| \le C\int_a^b |u_s||u_t|ds. \tag{2.2}$$

(2) The parallel transport of X_i along the curves

$$\tau\longrightarrow u(\tau,c),\quad \tau\longrightarrow u(\tau,d),\quad \tau\longrightarrow u(a,\tau)$$

is X_i itself. Thus, it remains to estimate the parallel transport of each X_i along the curve $\tau\longrightarrow u(b,\tau)$. Let h_{ij} be the SO_N-valued function on $[c,d]$ such that

$$h(c)=I \quad\text{and}\quad \frac{D}{dt}h_{ij}X_j\Big|_{(b,t)} = 0 \ \ \forall i,t.$$

The second equation is equivalent to

$$h'_{ij}(t)X_j(b,t) + h_{ij}(t)A_{jk}(b,t)X_k(b,t) = 0 \Longleftrightarrow h' = -hA(b,\cdot). \tag{2.3}$$

Since A_{ij} is always traceless by (2.1), equation (2.3) has a unique solution in SO_n such that $h(0)=I$, where n is the dimension of V. Furthermore, by (2.2)

$$\big|h(d)-I\big| \le \int_c^d |h'(t)|dt \le \int_c^d |h||A|dt \le n^2\int_c^d\int_a^b C|u_s||u_t|dsdt. \tag{2.4}$$

Since $\Pi_{\partial\alpha}X_i = h_{ij}(d)X_j$ by the above, the lemma follows from equation (2.4).

Corollary 2.2 *There exists* $C\in\mathbb{R}^+$ *such that for any closed curve* $\alpha\colon[a,b]\longrightarrow V$*,*

$$\big|\Pi_\alpha - I\big| \le C\min\big(\|d\alpha\|_1, (b-a)\|d\alpha\|_2^2\big).$$

Proof: Since the group SO_N is compact and $\|d\alpha\|_1^2\le(b-a)\|d\alpha\|_2^2$ by Holder's inequality, it is enough to assume that $\|d\alpha\|_1\le\frac{1}{2}r_V$. Then we can write

$$\alpha(t)=\exp_{\alpha(0)}\tilde{\alpha}(t), \qquad \text{where} \qquad |\tilde{\alpha}(t)|_{\alpha(0)}<r_V.$$

Define $u\colon[0,1]\times[a,b]\longrightarrow V$ by

$$u(s,t)=\exp_{\alpha(0)}s\tilde{\alpha}(t).$$

Note that

$$\begin{aligned} u_s(s,t) &= d\exp_{\alpha(0)}\big|_{s\tilde{\alpha}(t)}\tilde{\alpha}(t), &&\Longrightarrow\quad |u_s|_{(s,t)}\le C\|d\alpha\|_1\\ u_t(s,t) &= sd\exp_{\alpha(0)}\big|_{s\tilde{\alpha}(t)}(d\exp_{\alpha(0)}|_{\tilde{\alpha}(t)})^{-1}\alpha'(t) &&\Longrightarrow\quad |u_t|_{(s,t)}\le C|d\alpha|_t. \end{aligned}$$

Thus, by Lemma 2.1,

$$\begin{aligned} \big|\Pi_\alpha - I\big| = \big|\Pi_{\partial u}-I\big| &\le C\int_a^b\!\!\int_0^1|u_s||u_t|dsdt\\ &\le C\|d\alpha\|_1^2\le C(b-a)\|d\alpha\|_2^2. \end{aligned}$$

Since $\|d\alpha\|_1\le\frac{1}{2}r_V$, it follows that $|\Pi_\alpha - I|\le C'\|d\alpha\|_1$.

Corollary 2.3 *There exist* $C,C'\in\mathbb{R}^+$ *such that for all smooth maps* $\alpha,\xi\colon(-\epsilon,\epsilon)\longrightarrow T_pV$*,*

$$\left|\frac{D}{dt}\Big(\Pi_{\alpha(t)}\xi(t)\Big)\Big|_{t=0} - \Pi_{\alpha(0)}\xi'(0)\right| \le C|\alpha(0)||\alpha'(0)||\xi(0)|.$$

Proof: Let $R=[0,1]\times\big[0,\frac{1}{2}\epsilon\big]$ and define $u: R\longrightarrow V$ by $u(s,t)=\exp_p s\alpha(t)$. Let $\{X_i\}$ be an orthonormal basis for T_pV. Extend each $\{X_i\}$ to a vector field along u by parallel transport along the geodesics $s\longrightarrow u(s,t)$. Write $\xi(t)=f(t)X_i$; then $\Pi_{\alpha(t)}\xi(t)=f(t)X_i(1,t)$, and thus

$$\begin{aligned} \frac{D}{dt}\Big(\Pi_{\alpha(t)}\xi(t)\Big)\Big|_{t=0} &= f'(0)X_i(1,t)+f(0)\frac{D}{dt}X_i(1,t)\Big|_{t=0}\\ &= \Pi_{\alpha(0)}\xi'(0)+f(0)\frac{D}{dt}X_i(1,t)\Big|_{t=0}. \end{aligned}$$

On the other hand, by the proof of Lemma 2.1,

$$\Big|\frac{D}{dt}X_i(1,t)\Big|_{t=0}\le C\int_0^1|u_s|_{(s,0)}|u_t|_{(s,0)}ds\le C'|\alpha(0)||\alpha'(0)|.$$

2.2 Poincare Lemmas

Lemma 2.4 *If* $f\colon S^1\longrightarrow\mathbb{R}^N$ *is a smooth function such that* $\int_0^{2\pi}f(\theta)d\theta=0$*,*

$$\int_0^{2\pi}|f(\theta)|^2d\theta\le\int_0^{2\pi}|f'(\theta)|^2d\theta.$$

Proof: We can write $f(\theta) = \sum\limits_{n>-\infty}^{n<\infty} a_n e^{in\theta}$. Since $\int_0^{2\pi} f(\theta)d\theta = 0$, $a_0 = 0$. Thus,

$$\int_0^{2\pi} |f(\theta)|^2 d\theta = \sum_{n>-\infty}^{n<\infty} |a_n|^2 \leq \sum_{n>-\infty}^{n<\infty} |na_n|^2 = \int_0^{2\pi} |f'(\theta)|^2 d\theta.$$

Proposition 2.5 *Let (V, g, ∇) be a compact Riemannian manifold. There exists $C \in \mathbb{R}^+$ such that for any curve $\alpha\colon S^1 \longrightarrow V$ and vector fields ξ_1 and ξ_2 along α,*

$$\big|\langle\!\langle \nabla_\theta \xi_1, \xi_2 \rangle\!\rangle\big| \leq \|\nabla_\theta \xi_1\|_2 \|\nabla_\theta \xi_2\|_2 + C \min\big(\|d\alpha\|_1, \|d\alpha\|_2^2\big) \|\xi_1\|_{2,1} \|\xi_2\|_2,$$

where all the norms are computed with respect to the standard metric on S^1.

Proof: (1) Identify $T_{\alpha(0)}V$ with $\mathbb{R}^N$, preserving the metric. Given $v \in T_{\alpha(0)}V$, let $v(\theta)$ denote the parallel transport of v along the curve $t \longrightarrow \alpha(t)$ with $0 \leq t \leq \theta$. By Corollary 2.2, there exists $X \in so(T_{\alpha(0)}V) = so_N$ such that

$$|X| \leq C \min\big(\|d\alpha\|_1, \|d\alpha\|_2^2\big) \quad \text{and} \quad v(2\pi) = \{\exp(X)\} \cdot v(0) = \{\exp(X)\} \cdot v$$

for all $v \in T_{\alpha(0)}V$, where $\exp(X)$ is taken in $SO_N = SO(T_{\alpha(0)}V)$.
(2) For any $v \in T_{\alpha(0)}V$, let $\Psi_\theta v$ denote the parallel transport of $e^{-\theta X/2\pi} v$ along the curve $t \longrightarrow \alpha(t)$ with $0 \leq t \leq \theta$. Then $\Psi\colon S^1 \times \mathbb{R}^N \longrightarrow \alpha^* TV$ is a smooth isometry. Put

$$\bar{\xi}_2 = \frac{1}{2\pi} \int_0^{2\pi} \{\Psi^{-1} \xi_2\}(\theta) d\theta.$$

By Holder's inequality and Lemma 2.4,

$$\begin{aligned} \big|\langle\!\langle \nabla_\theta \xi_1, \xi_2 - \Psi\bar{\xi}_2 \rangle\!\rangle\big| &\leq \|\nabla_\theta \xi_1\|_2 \|\xi_2 - \Psi\bar{\xi}_2\|_2 \\ &= \|\nabla_\theta \xi_1\|_2 \|\Psi^{-1}\xi_2 - \bar{\xi}_2\|_2 \leq \|\nabla_\theta \xi_1\|_2 \|d\Psi^{-1}\xi_2\|_2. \end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned} \|d\Psi^{-1}\xi_2\|_2 &\leq \|\nabla_\theta \xi_2\|_2 + |X/2\pi| \|\xi_2\|_2 \\ &\leq \|\nabla_\theta \xi_2\|_2 + C \min\big(\|d\alpha\|_1, \|d\alpha\|_2^2\big) \|\xi_2\|_2. \end{aligned} \tag{2.6}$$

On the other hand, by integration by parts, we obtain

$$\langle\!\langle \nabla_\theta \xi_1, \xi_2 - \Psi\bar{\xi}_2 \rangle\!\rangle = \langle\!\langle \nabla_\theta \xi_1, \xi_2 \rangle\!\rangle + \langle\!\langle \xi_1, \nabla_\theta \Psi\bar{\xi}_2 \rangle\!\rangle. \tag{2.7}$$

Since $\Psi\bar{\xi}_2$ is the parallel transport of $e^{\theta X/2\pi}\bar{\xi}_2$,

$$\begin{aligned} \big|\langle\!\langle \xi_1, \nabla_\theta \Psi\bar{\xi}_2 \rangle\!\rangle\big| &\leq \|\xi_1\|_2 \|\nabla_\theta \Psi\bar{\xi}_2\|_2 = \|\xi_1\|_2 |X/2\pi| \|\Psi\bar{\xi}_2\|_2 \\ &\leq C \min\big(\|d\alpha\|_1, \|d\alpha\|_2^2\big) \|\xi_1\|_2 \|\xi_2\|_2. \end{aligned} \tag{2.8}$$

The corollary follows from equations (2.5)-(2.8).

If $R > r \geq 0$, let $B_{R,r} = \big\{x \in \mathbb{R}^2 \colon r < |x| < R\big\}$ and $\tilde{B}_{R,r} = \big\{x \in \mathbb{R}^2 \colon r \leq |x| < R\big\}$. The following lemma is used in the next section as well as in [Z].

Lemma 2.6 *For all $p\ge 1$, there exists $C_p\in\mathbb{R}^+$ such that for all $R\in\mathbb{R}^+$, $\epsilon\in[0,\frac{1}{2})$, and $f\in C^\infty\big(B_{R,\epsilon R};\mathbb{R}^n\big)$,*

$$\int_{B_{R,\epsilon R}} f=0 \quad\Longrightarrow\quad \|f\|_{C^0}\le C_pR^{\frac{p-2}{p}}\|df\|_p.$$

Proof: (1) We first assume that $R=1$. Suppose $\epsilon_k\longrightarrow\epsilon\in\big[0,\frac{1}{2}\big]$, $f_k\in C^\infty(B_{1,\epsilon_k};\mathbb{R}^n)$,

$$\int_{B_{1,\epsilon_k}} f_k=0,\quad \|f_k\|_{C^0}=1,\quad\text{and}\quad \|df_k\|_p\longrightarrow 0.$$

By the Sobolev Embedding Theorem, f_k converges to some $f\in L^p(B_{1,\epsilon};\mathbb{R}^n)$ with respect to the L^p-norm on compact subsets of $B_{1,\epsilon}$. Since $\|df_k\|_p\longrightarrow 0$, $df=0$ and $f\in L^p_1(B_{1,\epsilon};\mathbb{R}^n)$. Since d is elliptic, it follows that f is smooth and constant, and thus zero. However, this contradicts the fact that $\|f\|_{C^0}=1$.
(2) Suppose $R>0$, $f\in C^\infty(B_{R,\epsilon R};\mathbb{R}^n)$, and $\int f=0$. Let $\tilde f\in C^\infty(B_{1,\epsilon};\mathbb{R}^n)$ be given by $\tilde f(z)=f(Rz)$. Then $\int_{B_{1,\epsilon}}\tilde f=0$, and thus by (1),

$$\|f\|_{C^0}=\|\tilde f\|_{C^0}\le C_p\|d\tilde f\|_p\le C_pR^{\frac{p-2}{p}}\|df\|_p.$$

2.3 The Exponential Map and Differentiation

With notation as in Subsection 2.1, let

$$\tilde TV=\bigcup_{p\in V}\tilde T_pV,\quad\text{where}\quad \tilde T_pV=\big\{X\in T_pV: |X|\le\frac{1}{2}r_V\big\}.$$

If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow V$ is a smooth curve and ξ is a vector fields along α such that $\xi(0)\in\tilde TV$, put

$$\Phi_{\alpha(0)}\Big(\xi(0);\alpha'(0),\frac{D}{ds}\xi\Big|_{s=0}\Big)=\Pi^{-1}_{\xi(0)}\Big(\frac{d}{ds}\exp_{\alpha(s)}\xi(s)\Big|_{s=0}\Big).$$

Note that Φ is well-defined, i.e. its definition is dependent only on the vectors $\alpha'(0)$, $\xi(0)$, and $\frac{D}{ds}\xi(0)\big|_{s=0}$, since Φ involves only first-order differentiation.

Lemma 2.7 *There exists $C\in\mathbb{R}^+$ such that for all $p\in V$, $X\in\tilde T_pV$, and $Y,Z\in T_pV$,*

$$\Big|\Phi_p(X;Y,Z)-\big(Y+Z+T_\nabla(Y,Z)\big)\Big|\le C\big(|X|^2|Y|+|X||Z|\big).$$

Proof: Let $\alpha\colon(-\epsilon,\epsilon)\longrightarrow V$ be a smooth curve and ξ a smooth vector field along α such that

$$\alpha(0)=p,\quad \alpha'(0)=Y,\quad \xi(0)=X,\quad \frac{D}{ds}\xi(s)\Big|_{s=0}=Z.$$

Put

$$F_{X,Y,Z}(t)=\frac{d}{ds}\exp_{\alpha(s)}t\xi(s)\Big|_{s=0},\qquad H_{X,Y,Z}(t)=\Pi_{tX}\big(Y+tZ+tT_\nabla\big(Y,X\big)\big).$$

Then,

$$F_{X,Y,Z}(0)=\frac{d}{ds}\alpha(s)\Big|_{s=0}=H_{X,Y,Z}(0);$$
$$\frac{D}{dt}F_{X,Y,Z}(t)\Big|_{t=0}=\frac{D}{ds}\frac{d}{dt}\exp_{\alpha(s)}t\xi(s)\Big|_{t=0}\Big|_{s=0}+T_\nabla\big(\alpha'(0),\xi(0)\big)=\frac{D}{dt}H_{X,Y,Z}(t)\Big|_{t=0};$$

see Corollary 2.3. Combining the last two equations, we obtain

$$\big|F_{X,Y,Z}(t)-H_{X,Y,Z}(t)\big|\le C_p(X,Y,Z)t^2 \quad \forall\, t\in[-1,1],~ X,Y,Z\in TV,$$

where C is a smooth function on $TV\oplus TV\oplus TV$. Since

$$F_{X,Y,Z}(t)=F_{tX,Y,tZ}(1), \qquad H_{X,Y,Z}(t)=H_{tX,Y,tZ}(1),$$

the space $\{X\in TV\colon |X|=1\}$ is compact, and $F_{X,Y,Z}$ and $H_{X,Y,Z}$ are linear in (Y,Z), we conclude that there exists $C\in\mathbb{R}^+$ such that

$$\big|F_{X,Y,Z}(1)-H_{X,Y,Z}(1)\big|\le C\big(|X|^2|Y|+|X||Z|\big) \quad \forall\, X\in\tilde{T}_pV,~ Y,Z\in T_pV, \tag{2.9}$$

as claimed.

For any $X,Y,Z\in T_pV$, let $\tilde{\Phi}_p(X;Y,Z)=\Phi_p(X;Y,Z)-\big(Y+Z+T_\nabla(Y,Z)\big)$.

Corollary 2.8 *There exists $C\in\mathbb{R}^+$ such that for all $p\in V$, $X_1,X_2\in\tilde{T}_pV$, and $Y,Z_1,Z_2\in T_pV$,*

$$\begin{aligned}&\Big|\tilde{\Phi}_p(X_1;Y,Z_1)-\tilde{\Phi}_p(X_2;Y,Z_2)\Big|\\ &\qquad\le C\Big(\big((|X_1|+|X_2|)|Y|+|Z_1|+|Z_2|\big)|X_1-X_2|+\big(|X_1|+|X_2|\big)|Z_1-Z_2|\Big).\end{aligned}$$

Proof: By the proof of Lemma 2.7, $\tilde{\Phi}=\tilde{\Phi}_1+\tilde{\Phi}_2$, where

$$\tilde{\Phi}_1,\tilde{\Phi}_2\colon \tilde{T}V\longrightarrow \pi_{TV}^*\big(T^*V\otimes TV\big)$$

are smooth sections such that

$$\big|\tilde{\Phi}_1(X;\cdot)\big|\le C_1|X|^2 \quad\text{and}\quad \big|\tilde{\Phi}_2(X;\cdot)\big|\le C_2|X| \qquad \forall\, X\in\tilde{T}V.$$

Since the space $\tilde{T}V$ is compact,

$$\left.\begin{aligned}\big|\tilde{\Phi}_1(X_1;\cdot)-\tilde{\Phi}_1(X_2;\cdot)\big|&\le C_1'\big(|X_1|+|X_2|\big)|X_1-X_2|,\\ \big|\tilde{\Phi}_2(X_1;\cdot)-\tilde{\Phi}_2(X_2;\cdot)\big|&\le C_2'|X_1-X_2|,\end{aligned}\right. \qquad \forall\, X_1,X_2\in\tilde{T}_pV.$$

From the linearity of $\tilde{\Phi}_1(X:\cdot)$ and $\tilde{\Phi}_2(X:\cdot)$, we conclude that

$$\begin{aligned}\big|\tilde{\Phi}_1(X_1;Y)-\tilde{\Phi}_1(X_2;Y)\big|&\le C_1'\big(|X_1|+|X_2|\big)|X_1-X_2||Y|,\\ \Big|\tilde{\Phi}_2(X_1;Z_1)-\tilde{\Phi}_2(X_2;Z_2)\Big|&\le C_2'|X_1-X_2||Z_1|+C_2|X_2||Z_1-Z_2|.\end{aligned}$$

2.4 Expansion of the $\bar{\partial}$-Operator

Let (V,g,J) be a compact almost-complex manifold. Here J is a complex structure in TV, which is orthogonal with respect to g. Then there exists a canonical connection ∇ in TV that commutes with J. Explicitly, if ∇^{LC} is the Levi-Civita connection of the metric g, for any $X\in T_pV$ and $\xi\in\Gamma(V;TV)$, let

$$\nabla_X\xi=\frac{1}{2}\Big(\nabla_X^{LC}\xi-J\nabla_X^{LC}(JX)\Big).$$

We will always associate this canonical connection with any given triple (V,g,J). This connection agrees with the Levi-Civita connection if and only if (V,g,J) is Kahler.

Let $(\mathcal{D},j)$ be any one-dimensional complex manifold. If $u\colon \mathcal{D}\longrightarrow V$ is a smooth map, we denote by $\Lambda^{0,1}_{J,j}T^*\mathcal{D}\otimes u^*TV\longrightarrow\mathcal{D}$ the bundle of (J,j)-antilinear homomorphisms from $T\mathcal{D}$ to u^*TV. Let

$$\bar{\partial}u=\frac{1}{2}\big(du+J\circ du\circ j\big)\in\Gamma^{0,1}_{J,j}(u)\equiv\Gamma(\mathcal{D};\Lambda^{0,1}_{J,j}T^*\mathcal{D}\otimes u^*TV).$$

If $\xi\in\Gamma(u)$, we define $\bar{\partial}_u\xi, D_u\xi, L_u\xi, \tilde{D}_u\xi, N_u\xi\in\Gamma^{0,1}_{J,j}(u)$ by

$$\big\{\bar{\partial}_u\xi\big\}_z(v)=\Pi^{-1}_{\xi(z)}\big(\bar{\partial}\{\exp_u\xi\}\big|_z v\big),\qquad 2\big\{L_u\xi\big\}_z(v)=T_\nabla\big(du|_z v,\xi(z)\big)+JT_\nabla\big(du|_z jv,\xi(z)\big),$$
$$2D_u\xi=\nabla\xi+J\circ\nabla\xi\circ j\qquad \tilde{D}_u\xi=D_u\xi+L_u\xi,\qquad \bar{\partial}_u\xi=\bar{\partial}u+\tilde{D}_u\xi+N_u\xi.$$

The operator $\tilde{D}_u\colon\Gamma(u)\longrightarrow\Gamma^{0,1}_{J,j}(u)$ is the linearization of the $\bar{\partial}$-operator at u; see [MS]. What this means is described in detail by Lemma 2.9 and Proposition 2.11 below.

Lemma 2.9 *There exists a constant $C\in\mathbb{R}^+$ dependent only on (V,g,J) such that for any smooth map $u\colon(\mathcal{D},j)\longrightarrow V$, $z\in\mathcal{D}$, $v\in T_z\mathcal{D}$, and $\xi_1,\xi_2\in\Gamma(u)$, with $|\xi_1|_z,|\xi_2|_z\le\frac{1}{2}r_V$,*

$$\Big|\big\{N_u\xi_1\big\}_z v-\big\{N_u\xi_2\big\}_z v\Big|\le C\Big(\big(\big|du|_z v\big|(|\xi_1|_z+|\xi_2|_z)+|\nabla\xi_1(v)+\nabla\xi_2(v)|\big)\big|\xi_1-\xi_2\big|$$
$$+\big(|\xi_1|_z+|\xi_2|_z\big)\big|\nabla\xi_1(v)-\nabla\xi_2(v)\big|\Big).$$

Furthermore, $N_u0=0$.

Proof: Since the connection ∇ commutes with J, so does the parallel transport Π. Thus, with notation as in the previous subsection,

$$\big\{N\xi\big\}_z v=\tilde{\Phi}\big(\xi(z);du|_z v,\nabla\xi|_z v\big)+J\big(u(z)\big)\tilde{\Phi}\big(\xi(z);du|_z jv,\nabla\xi|_z jv\big).$$

The claim now follows from Corollary 2.8.

Definition 2.10 *If $C_0\in\mathbb{R}^+$ and $u\colon\mathcal{D}\longrightarrow V$ is a smooth map, norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ on $\Gamma(u)$ and $\Gamma^1(u)$, respectively, are <u>C_0-admissible</u> if for all $\xi\in\Gamma(u)$, $\eta\in\Gamma^1(u)$, and any continuous function $f\colon\mathcal{D}\longrightarrow\mathbb{R}$,*

$$\|\nabla\xi\|_p\le\|\xi\|_{p,1},\quad \|f\eta\|_p\le\|f\|_{C^0}\|\eta\|_p,\quad \|\xi\|_{C^0}\le C_0\|\xi\|_{p,1}.$$

Proposition 2.11 *If (V,g,J) is a compact almost-complex manifold, there exists $C_V\in C^\infty(\mathbb{R};\mathbb{R})$ with the following property. Suppose $(\mathcal{D},j)$ is a one-dimensional complex manifold, $u\colon(\mathcal{D},j)\longrightarrow V$ is a smooth map, and $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ are C_0-admissible norms on $\Gamma(u)$ and $\Gamma^1(u)$, respectively. Then for all $\xi_1,\xi_2\in\Gamma(u)$ such that $\|\xi_1\|_{p,1},\|\xi_2\|_{p,1}<\frac{r_V}{2C_0}$,*

$$\big\|N_u\xi_1-N_u\xi_2\big\|_p\le C_V\big(C_0+\|du\|_p\big)\big(\|\xi_1\|_{p,1}+\|\xi_2\|_{p,1}\big)\|\xi_1-\xi_2\|_{p,1}.$$

Furthermore, $N_u0=0$. If the geodesic ball of radius δ about $u(z)$ in (V,g,J) is isomorphic to an open subset of $\mathbb{C}^n$ and $|\xi|_z<\delta$, then $\{N_u\xi\}_z=0$.

Proof: The first two statements follow from Lemma 2.9 and Definition 2.10. The last claim is clear from the construction of N_u.

Remark: As the notation suggests, one possibility for the norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ is the usual Sobolev L^p_1 and L^p-norms with respect to some Riemannian metric on $\mathcal{D}$, where $p>2$. Another natural possibility is the modified Sobolev norms of [LT], which are particularly suited for gluing pseudoholomorphic curves. In Proposition 3.7 below, we will see that the constant C_0 itself in either of these two cases is a function of $\|du\|_p$ only.

3 Sobolev and Elliptic Inequalities

3.1 Eucledian Case

Proofs of the next four statements are slight refinements of the proofs of similar statements in [M] in the $n=2$ case.

Lemma 3.1 *For any bounded convex domain $\mathcal{D}\subset\mathbb{R}^2$, $f\in C^\infty(\mathcal{D})$, and $x\in\mathcal{D}$,*

$$\big|f_{\mathcal{D}}-f(x)\big|\le\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|df|_y|y-x|^{-1}dy,$$

where $2r_0$ is the diameter of $\mathcal{D}$, $|\mathcal{D}|$ is the area of $\mathcal{D}$, and

$$f_{\mathcal{D}}=\frac{1}{|\mathcal{D}|}\Big(\int_{\mathcal{D}}f(y)dy\Big)$$

is the average value of f on $\mathcal{D}$.

Proof: For any $y\in\mathcal{D}$,

$$f(y)-f(x)=\int_0^1\frac{d}{dt}f\big(x+t(y-x)\big)dt=\int_0^1 df|_{x+t(y-x)}(y-x)dt.$$

Put $g(z)=|df|_z$ if $z\in\mathcal{D}$ and $g(z)=0$ otherwise. Then,

$$\big|f_{\mathcal{D}}-f(x)\big|\le\frac{1}{|\mathcal{D}|}\int_{y\in\mathcal{D}}|f(y)-f(x)|dy\le\frac{1}{|\mathcal{D}|}\int_{y\in D}\int_0^\infty g\big(x+t(y-x)\big)|y-x|dtdy.$$

Rewriting the last integral in polar coordinates (r,θ) centered at x, we obtain

$$\begin{aligned}\big|f_{\mathcal{D}}-f(x)\big|&\le\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(tr,\theta)r^2dtdrd\theta\\&=\frac{1}{|\mathcal{D}|}\int_0^{2\pi}\int_0^{2r_0}\int_0^\infty g(t,\theta)rdtdrd\theta=\frac{2r_0^2}{|\mathcal{D}|}\int_0^{2\pi}\int_0^\infty g(t,\theta)dtd\theta\\&=\frac{2r_0^2}{|\mathcal{D}|}\int_{\mathcal{D}}|df|_y|y-x|^{-1}dy.\end{aligned}$$

Corollary 3.2 *For any $R>0$ and $p>2$, there exists $C_p\in C^\infty(\mathbb{R};\mathbb{R})$ such that*

$$r\in\big[0,\frac{1}{2}R\big],\quad f\in C^\infty(B_{R,r})\quad\Longrightarrow\quad\|f\|_{C^0}\le C_p(R)\|f\|_{p,1}.$$

Proof: For any $x^*\in B_{R,r}$, put

$$\mathcal{D}_{x^*}=\big\{x\in B_{R,r}\colon \langle x^*,|x^*|x-rx^*\rangle\ge 0\big\}.$$

If $x^*\neq 0$, $\mathcal{D}_{x^*}$ is the part of the annulus on the same side of the line $\langle x^*,x-rx^*/|x^*|\rangle=0$ as x^*. In particular,

$$\mathrm{diam}(\mathcal{D}_{x^*})\le 2R \quad\text{and}\quad |\mathcal{D}_{x^*}|\ge\Big(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\Big)R^2.$$

Thus, by the above lemma and Holder's inequality,

$$\begin{aligned}|f(x^*)|&\le\frac{1}{|\mathcal{D}_{x^*}|}\|f\|_1+8\int_{y\in\mathcal{D}_{x^*}}|df|_y|y-x|^{-1}dy\\&\le|\mathcal{D}_{x^*}|^{-\frac{1}{p}}\|f\|_p+8\Big(\int_{y\in B_{2R}(x)}|y-x|^{-\frac{p}{p-1}}\Big)^{\frac{p-1}{p}}\|df\|_p\le C_p(R)\|f\|_{p,1},\end{aligned}\tag{3.1}$$

since $\frac{p}{p-1}<2$.

Lemma 3.3 *For any $R>0$ and $r\in[0,R)$,*

$$f\in C^\infty(B_{R,r}),\quad supp(f)\subset\tilde{B}_{R,r}\quad\Longrightarrow\quad\|f\|_2\le\|df\|_1.$$

Proof: Such a function f can be viewed as a function on the complement of the ball $\bar{B}(0,r)$ in $\mathbb{R}^2$. Since f vanishes at infinity, for any $(x,y)\in B_{R,r}$

$$f(x,y)=\begin{cases}\int_{-\infty}^x f_s(s,y)ds, & \text{if } x\le 0;\\ -\int_x^\infty f_s(s,y)ds, & \text{if } x\ge 0;\end{cases}\quad\text{and}\quad f(x,y)=\begin{cases}\int_{-\infty}^y f_t(x,t)dt, & \text{if } y\le 0;\\ -\int_y^\infty f_t(x,t)dt, & \text{if } y\ge 0.\end{cases}$$

Taking the absolute value in these equations, we obtain

$$|f|_{(x,y)}\le\int_{-\infty}^\infty|df|_{(s,y)}ds\quad\text{and}\quad|f|_{(x,y)}\le\int_{-\infty}^\infty|df|_{(x,t)}dt,\tag{3.2}$$

where we formally set f and df to be zero on the smaller disk. Multiplying the two inequalities in (3.2) and integrating with respect to x and y, we conclude

$$\int_{-\infty}^\infty\int_{-\infty}^\infty|f|^2_{(x,y)}dxdy\le\Big(\int_{-\infty}^\infty\int_{-\infty}^\infty|df|_{(x,y)}dxdy\Big)^2,$$

as claimed.

Corollary 3.4 *For any $R>0$, $p,q\ge1$ with $1-\frac{2}{p}\ge-\frac{2}{q}$, there exists $C_{p,q}\in C^\infty(\mathbb{R};\mathbb{R})$ such that*

$$r\in[0,R),\quad f\in C^\infty(B_{R,r}),\quad supp(f)\subset\tilde{B}_{R,r}\quad\Longrightarrow\quad\|f\|_q\le C_{p,q}(R)\|df\|_p.$$

Proof: For $\epsilon>0$, let $h_\epsilon=(f^2+\epsilon)^{\frac{q}{4}}-\epsilon^{\frac{q}{4}}$. By the above lemma and Holder's inequality,

$$\begin{aligned}\|f\|_q^q&\le\big\|h_\epsilon+\epsilon^{\frac{q}{4}}\big\|_2^2\le 2\|dh_\epsilon\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2=2\Big\|\frac{q}{2}(f^2+\epsilon)^{\frac{q}{4}-1}fdf\Big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\\&\le q^2\big\|(f^2+\epsilon)^{\frac{q}{4}-\frac{1}{2}}df\big\|_1^2+2\epsilon^{\frac{q}{2}}\pi R^2\le q^2\|df\|_p^2\big\|(f^2+\epsilon)^{\frac{q-2}{4}}\big\|^2_{\frac{p}{p-1}}+2\epsilon^{\frac{q}{2}}\pi R^2.\end{aligned}\tag{3.3}$$

Note that

$$1-\frac{2}{p}=-\frac{2}{q}\quad\Longrightarrow\quad\frac{q-2}{4}\frac{p}{p-1}=\frac{q-2}{4}\frac{2q}{q-2}=\frac{q}{2}.$$

Thus, letting ϵ go to zero in (3.3), we obtain

$$\|f\|_q^q\le q^2\|df\|_p^2\|f\|_p^{q-2}\quad\Longrightarrow\quad\|f\|_q\le q\|df\|_p.$$

The case $1-\frac{2}{p}>-\frac{2}{q}$ follows by Holder's inequality.

3.2 Vector Fields along Smooth Maps into Compact Manifolds

Let (V, g, ∇) be a compact Riemannian manifold.

Lemma 3.5 *For any $R>0$ and $p, q\geq 1$ with $1-\frac{2}{p} \geq -\frac{2}{q}$, there exists $C_{p,q}\in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $r\in[0,R)$, $u\in C^\infty(\tilde{B}_{R,r};V)$, and $\xi\in\Gamma_c(u)$,*

$$\|\xi\|_q \leq C_{p,q}(R)\big(\|\xi\|_{p,1} + \|\xi du\|_p\big).$$

Proof: (1) Let $\{U_i : i\in[N]\}$ be a finite open cover of V such that the diameter of each set U_i is at most $\frac{1}{2}r_V$. Let $\{W_i : i\in[N]\}$ be an open cover of V such that $\bar{W}_i\subset U_i$. Choose smooth functions $\eta_i : V \longrightarrow [0,1]$ such that $\eta_i=1$ on W_i and $\eta_i=0$ outside of U_i. For each $i\in[N]$, pick $x_i\in W_i$. If $z\in\tilde{B}_{R,r}$ and $u(z)\in U_i$, define $\tilde{u}_i(z), \xi_i(z)\in T_{x_i}V$ by

$$\exp_{x_i}\tilde{u}_i(z) = u(z), \quad |\tilde{u}_i(z)|<r_V; \quad \Pi_{\tilde{u}_i(z)}\xi_i(z) = \xi(z).$$

For any $z\in B_{R,r}$, put $\tilde{\xi}_i(z)=\eta_i(u(z))\xi_i(z)$. Then $\tilde{\xi}_i\in C_c^\infty(\tilde{B}_{R,r};T_{x_i}V)$.
(2) By Corollary 3.4, there exists $C_{p,q}(R)>0$ such that

$$\|\xi\|_{L^q(u^{-1}(W_i))} \leq \|\tilde{\xi}_i\|_q \leq C_{p,q}(R)\|\tilde{\xi}_i\|_{p,1} \leq C_{p,q}(R)(\|\xi\|_p + \|d\tilde{\xi}_i\|_p). \tag{3.4}$$

Since $d\tilde{\xi}_i = (d\eta_i\circ du)\xi_i + (\eta\circ u)d\xi_i$ on $u^{-1}(U_i)$ and vanishes outside of $u^{-1}(U_i)$,

$$\|d\tilde{\xi}_i\|_p \leq \|d\xi_i\|_{L^p(u^{-1}(U_i))} + C\|\xi_i du\|_p. \tag{3.5}$$

On the other hand, by Corollary 2.3, if $u(z)\in U_i$

$$\Big|\nabla\xi|_z - \Pi_{\tilde{u}(z)}\circ d\xi_i|_z\Big|_z \leq C|du|_z|\xi|_z. \tag{3.6}$$

Combining equations (3.4)-(3.6), we obtain

$$\|\xi\|_{L^q(u^{-1}(W_i))} \leq C_{p,q}(R)\big(\|\xi\|_{p,1} + \|\xi du\|_p\big).$$

The claim follows by summing the last inequality over all i.

Lemma 3.6 *For any $R>0$ and $p>2$, there exists $C_p\in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $r\in[0,\frac{1}{2}R]$, $u\in C^\infty(B_{R,r};V)$, and $\xi\in\Gamma(u)$,*

$$\|\xi\|_{C^0} \leq C_p(R)\big(\|\xi\|_{p,1} + \|\xi du\|_p\big).$$

Proof: With notation as above, by Corollary 3.2, there exists $C_p(R)$ such that

$$\|\xi\|_{C^0(u^{-1}(W_i))} \leq \|\tilde{\xi}_i\|_{C^0} \leq C_p(R)\|\tilde{\xi}_i\|_{p,1} \leq C_p(R)\big(\|\xi\|_{L^p(u^{-1}(U_i))} + \|d\tilde{\xi}_i\|_p\big).$$

As above, we obtain

$$\|d\tilde{\xi}_i\|_p \leq C\big(\|\xi\|_{p,1} + \|\xi du\|_p\big),$$

and the claim follows.

Proposition 3.7 *If (V,g,∇) is a compact Riemannian manifold and $p>2$, there exists $C_p\in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$ such that for any $R>0$, $r\in[0,\frac{1}{2}R]$, $u\in C^\infty(B_{R,r};V)$ and $\xi\in\Gamma_c(u)$,*

$$\|\xi\|_{C^0}\le C_p\big(R,\|du\|_p\big)\|\xi\|_{p,1}.$$

The same statement holds if $B_{R,r}$ is replaced by a fixed compact Riemann surface (Σ,g_Σ).

Proof: By Lemma 3.6 applied with $\tilde{p}=\frac{p+2}{2}$ and Holder's inequality,

$$\|\xi\|_{C^0}\le C_{\tilde{p}}(R)\big(\|\xi\|_{\tilde{p},1}+\|\xi du\|_{\tilde{p}}\big)\le C'_{\tilde{p}}(R)\big(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_1}\big),\tag{3.7}$$

where $q_1=\frac{2p}{p-2}$. If $q_1\le p$, then the proof is complete. Otherwise, apply Lemma 3.5 with $p_1=\frac{2q_1}{q_1+2}$ and Holder's inequality:

$$\|\xi\|_{q_1}\le C_{p_1,q_1}(R)\big(\|\xi\|_{p_1,1}+\|\xi du\|_{p_1}\big)\le C'_{p_1,q_1}(R)\big(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_2}\big),\tag{3.8}$$

where $q_2=\frac{pp_1}{p-p_1}$. If $q_2\le p$, then the claim follows from equations (3.7) and (3.8). Otherwise, we can continue and construct sequences $\{p_i\},\{q_i\},\{C_i\}$ such that

$$p_i=\frac{2q_i}{q_i+2},\quad q_{i+1}=\frac{pp_i}{p-p_i};\tag{3.9}$$

$$\|\xi\|_{q_i}\le C_i(R)\left(\|\xi\|_{p,1}+\|du\|_p\|\xi\|_{q_{i+1}}\right).\tag{3.10}$$

Equation (3.9) implies that

$$q_{i+1}=\frac{2pq_i}{pq_i-2(q_i-p)}\qquad\Longrightarrow\qquad\text{if } q_i>0,\ \text{then } q_{i+1}<q_i.$$

Thus, if $q_i>2$ for all i, then the sequence $\{q_i\}$ must have a limit $q\ge 2$. The limiting value must satisfy

$$q=\frac{2pq}{pq-2(q-p)}\Longrightarrow(p-2)q=0\Longrightarrow q=0,$$

since $p>2$ by assumption. Thus, for N sufficiently large $q_N\le p$ and the first claim follows from (3.7) and equations (3.10) with i running from 1 to $N+1$, where N is the smallest integer such that $q_N\le p$. The second claim is easily obtained from the first.

3.3 Elliptic Estimates

If $A_1=B_{R_1,r_1}$ and $A_2=\bar{B}_{R_2,r_2}$ are two annuli in $\mathbb{R}^2$, we write $A_2\Subset_\delta A_1$ if $R_1-R_2>\delta$ and $r_2-r_1\ge\delta$.

Lemma 3.8 *For any $\delta>0$, $p\ge1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$ and $f\in C^\infty(A_1;\mathbb{C}^n)$,*

$$\|f\|_{L^p_1(A_2)}\le C_p(A_1)\big(\|\bar\partial f\|_p+\|df\|_2+\|f\|_1\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: We can assume that A_2 is the maximal annulus such that $A_2 \Subset_\delta A_1$. Let $\eta\colon A_1 \longrightarrow [0,1]$ be a compactly supported smooth function such that $\eta|_{A_2}=1$. By the usual elliptic inequalities for S^2,

$$\begin{aligned} \|df\|_{L^p(A_2)} &\le \|d(\eta f)\|_p \le C_p(A_1)\big(\|\bar\partial(\eta f)\|_p+\|\eta f\|_p\big)\\ &\le C_p(A_1)\big(\|\bar\partial f\|_p+\|(d\eta)f\|_p+\|\eta f\|_p\big). \end{aligned} \tag{3.11}$$

By Corollary 3.4,

$$\begin{aligned} \|\eta f\|_p &\le C_p(A_1)\big(\|df\|_2 + \|(d\eta)f\|_2 + \|\eta f\|_2\big)\\ &\le C'_p(A_1)\big(\|df\|_2 + \|df\|_1 + \|(\nabla^2\eta)f\|_1 + \|(d\eta)f\|_1 + \|\eta f\|_1\big)\\ &\le C_{\delta,p}(A_1)\big(\|df\|_2 + \|f\|_1\big). \end{aligned} \tag{3.12}$$

Similarly,

$$\|(d\eta)f\|_p \le C_{\delta,p}(A_1)\big(\|df\|_2 + \|f\|_1\big). \tag{3.13}$$

The claim follows by plugging (3.12) and (3.13) into (3.11).

Corollary 3.9 *For any $\delta>0$, $p\ge1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)>0$ such that for any annulus $A_2\Subset_\delta A_1$, and $f\in C^\infty(A_1;\mathbb{C}^n)$,*

$$\|df\|_{L^p(A_2)} \le C_p(A_1)\big(\|\bar\partial f\|_p + \|df\|_2\big).$$

Proof: Let $\bar f = \frac{1}{|A_1|}\int_{A_1} f$, where $|A_1|$ is the area of A_1. By Lemma 3.8,

$$\begin{aligned} \|df\|_{L^p(A_2)} &= \|d(f-\bar f)\|_{L^p(A_2)} \le C_p(A_1)\big(\|\bar\partial(f-\bar f)\|_p + \|d(f-\bar f)\|_2 + \|f-\bar f\|_1\big)\\ &= C_p(A_1)\big(\|\bar\partial f\|_p + \|df\|_2 + \|f-\bar f\|_1\big). \end{aligned} \tag{3.14}$$

The claim follows by applying Lemma 2.6 or its proof, depending on A_1, to A_1 and $f-\bar f$ with $p=2$ to the last term in (3.14).

Proposition 3.10 *If (V,g,J) is a compact almost complex manifold, for any $\delta>0$, $p\ge1$, and open annulus A_1, there exists $C_{\delta,p}(A_1)$ such that for any annulus $A_2 \Subset_\delta A_1$, smooth function $u\colon A_1 \longrightarrow V$ and $\xi\in\Gamma(u)$,*

$$\|\nabla\xi\|_{L^p(A_2)} \le C_{\delta,p}(A_1)\big(\|D_u\xi\|_p + \|\nabla\xi\|_2 + \|\xi du\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: We continue with the notation of the proof of Lemma 3.8. By Corollary 3.9,

$$\begin{aligned} \|d\tilde\xi_i\|_{L^p(A_2)} &\le C_{\delta,p}(A_1)\big(\|\bar\partial\tilde\xi_i\|_p + \|d\tilde\xi_i\|_2\big)\\ &\le C'_{\delta,p}(A_1)\big(\|\bar\partial\xi_i\|_{L^p(u_i^{-1}(U_i))} + \|d\xi_i\|_{L^2(u_i^{-1}(U_i))} + \|\xi du\|_p\big) \end{aligned} \tag{3.15}$$

If $u(z)\in U_i$, by Corollary 2.3,

$$\big|\nabla\xi - \Pi_{\tilde u_i(z)}d\xi_i\big|_z \le C|du|_z|\xi|_z. \tag{3.16}$$

Since $\nabla J=0$ and $\bar\partial_j\tilde\xi_i = (\bar\partial_j(\eta_i\circ u))\xi_i + \eta_i\bar\partial_j\xi_i$ on $u^{-1}(U_i)$, it follows from (3.15) and (3.16) that

$$\begin{aligned} \|\nabla\xi\|_{L^p(A_2\cap u_i^{-1}(W_i))} &\le \|d\tilde\xi_i\|_p + C\|\xi du\|_p\\ &\le C_{p,j}(A_1)\big(\|D_{u,j}\xi\|_p + \|\nabla\xi\|_2 + \|\xi du\|_p\big). \end{aligned} \tag{3.17}$$

The claim is obtained by summing the last equation over all i.

Lemma 3.11 *For any $p \geq 1$ and open ball $B \subset \mathbb{R}^2$, there exists $C_{B,p} \in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $u \in C^\infty(B;V)$ and $\xi \in \Gamma_c(u)$,*

$$\|\xi\|_{p,1} \leq C_{B,p}(\|du\|_p)\big(\|D_u\xi\|_p + \|\xi\|_p\big),$$

where the norms are taken with respect to the standard metric on $\mathbb{R}^2$.

Proof: By an argument nearly identical to the proof of Proposition 3.10, for all $p \geq 1$,

$$\|\xi\|_{p,1} \leq C_{B,p}\big(\|D_u\xi\|_p + \|\xi\|_p + \|\xi du\|_p\big). \tag{3.18}$$

On the other hand, by Proposition 3.7, for all $p > 2$,

$$\|\xi\|_{C^0} \leq C_{B,p}(\|du\|_p)\|\xi\|_p. \tag{3.19}$$

The claim is obtained from (3.18) and (3.19) by taking a sequence (p_i, q_i) as in the proof of Proposition 3.7.

Proposition 3.12 *If (V,g,J) is a compact almost complex manifold, for any $p > 2$ and compact Riemann surface (Σ, g_Σ), there exists $C_p \in C^\infty(\mathbb{R};\mathbb{R})$ such that for any $u \in C^\infty(B_{R,r};V)$ and $\xi \in \Gamma(u)$,*

$$\|\xi\|_{p,1} \leq C_p\big(\|du\|_p\big)(\|D_u\xi\|_p + \|\xi\|_p\big).$$

Proof: This statement is immediate from Lemma 3.11.

References

[C] I. Chavel, *Riemannian Geometry: A Modern Introduction*, Cambridge University Press, 1996.

[F] A. Floer, *The Unregularized Gradient Flow of the Symplectic Action*, Comm. Pure Appl. Math XLI (1988), 775-813.

[LT] J. Li and G. Tian, *Virtual Moduli Cycles and Gromov-Witten Invariants of General Symplectic Manifolds*, Topics in Symplectic 4-Manifolds, 47-83, First Int. Press Lect. Ser., I, Internat. Press, 1998.

[M] T. Mrowka, *18.966 Lecture Notes*, Spring 1998.

[MS] D. McDuff and D. Salamon, *Introduction to J-Holomorphic Curves*, American Mathematical Society, 1994.

[Z] A. Zinger, *Enumerative vs. Symplectic Invariants and Obstruction Bundles*, math.SG/0201255.