Some Questions in the Theory of Pseudoholomorphic Curves

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Abstract

This survey article, in honor of G. Tian's 60th birthday, is inspired by R. Pandharipande's 2002 note highlighting research directions central to Gromov-Witten theory in algebraic geometry and by G. Tian's complex-geometric perspective on pseudoholomorphic curves that lies behind many important developments in symplectic topology since the early 1990s.

Symplectic topology is an area of geometry originating in and closely associated with classical mechanics. While long established, it has been flourishing especially since the introduction of pseudoholomorphic curves techniques in [39]. These techniques have led to an immense wealth of remarkable applications, mutually enriching interplay with algebraic geometry, and striking connections with string theory. They have in particular given rise to counts of such curves in symplectic manifolds, now known as the Gromov-Witten invariants. While many long-standing problems have been spectacularly resolved, new profound questions that could have been hardly imagined in the past have arisen in their place. This article, greatly influenced by G. Tian's perspective on the field, highlights a number of questions concerning pseudoholomorphic curves and their applications in symplectic topology, algebraic geometry, and string theory.

R. Pandharipande's ICM note [73] assembled three conjectures concerning structures in Gromov-Witten theory:

(P1) a Poincare Duality for the tautological cohomology ring of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of stable nodal n-marked genus g curves, known as the Gorenstein property of $R^*(\overline{\mathcal{M}}_{g,n})$;

(P2) integral counts of holomorphic curves in smooth complex projective threefolds, known as the BPS states;

(P3) algebraic restrictions on Gromov-Witten invariants, known as the Virasoro constraints.

Each of these conjectures presented a deep quandary requiring fundamentally new ideas to address.

The Gorenstein property is a triviality for $g=0$, since $\overline{\mathcal{M}}_{0,n}$ is a smooth projective variety and $R^*(\overline{\mathcal{M}}_{0,n})=H^*(\overline{\mathcal{M}}_{0,n})$. It is established for $g=1$ in [80] and shown to fail for $g=2$ whenever $n\geq 20$ in [82, 81]. The Virasoro constraints had been established for the Gromov-Witten invariants of manifolds with only even-dimensional cohomology in genus 0, of a point, of a curve, and of the

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complex projective space $\mathbb{P}^n$ before [73] in [60, 70, 69, 35], respectively, with the last case extended to arbitrary symplectic manifolds with semi-simple quantum cohomology in [90]. However, no geometric rationale behind this conjecture that might confirm it in general has emerged so far, and its testing outside of fairly standard cases in algebraic geometry has been limited by the available computational techniques. Just as (P1), the Virasoro Conjecture of (P3) may yet turn out to fail, at least for non-projective symplectic manifolds.

Unlike (P1) and perhaps (P3), (P2) is most naturally viewed from the symplectic topology perspective in which it splits into three parts. The extensive work on (P2) in algebraic geometry since [73] has not succeeded in confirming this conjecture even in special cases. On the other hand, fundamentally new approaches to the three different parts of (P2) have emerged in symplectic topology which should fully resolve its original formulation in a stronger formulation; see Section 2.

The questions collected in this article fall under four distinct, but related, topics:

(1) the topology of moduli spaces of pseudoholomorphic maps and applications to the mirror symmetry predictions of string theory and to the enumerative geometry of algebraic curves;

(2) integral counts of pseudoholomorphic curves in arbitrary compact symplectic manifolds;

(3) decomposition formulas for counts of pseudoholomorphic curves under "flat" degenerations of symplectic manifolds;

(4) applications of pseudoholomorphic curves techniques in symplectic topology and algebraic geometry.

Each of these topics involves fundamental issues concerning pseudoholomorphic curves and a deep contribution from G. Tian.

G. Tian's perspectives on Gromov-Witten theory had a tremendous influence on the content of the present article in particular and the work of the author in general, and he is very grateful to G. Tian for generously sharing his insights on Gromov-Witten theory over the past two decades. The author would also like to thank J. Li, R. Pandharipande, and R. Vakil for introducing him to the richness of the algebro-geometric side of Gromov-Witten theory indicated by many of the questions in this article and P. Georgieva for acquainting him with the many related mysteries of the real sector of Gromov-Witten theory, as well as E. Brugallé, A. Doan, and C. Wendl for comments on parts of this article.

Contents

1 Topology of moduli spaces

A symplectic form on a $2n$-dimensional manifold X is a closed 2-form on X such that ω^n is a volume form on X. A tame almost complex structure on a symplectic manifold (X,ω) is a bundle endomorphism

$$J\colon TX \longrightarrow TX \qquad \text{s.t.} \qquad J^2=-\mathrm{Id}, \quad \omega(v,Jv)>0 \ \ \forall\, v\in T_xX,\, x\in X,\, v\neq 0.$$

If Σ is a (possibly nodal) Riemann surface with complex structure $\mathfrak{j}$, a smooth map $u\colon\Sigma\longrightarrow X$ is called J-holomorphic if it solves the Cauchy-Riemann equation corresponding to $(J,\mathfrak{j})$:

$$\bar\partial_J u \equiv \frac{1}{2}\big(\mathrm{d}u + J\circ \mathrm{d}u\circ \mathfrak{j}\big)=0.$$

The image of such a map in X is called a J-holomorphic curve. GW-invariants are rational counts of such curves that depend only on (X,ω).

The most fundamental object in GW-theory is the moduli space $\overline{\mathfrak{M}}_{g,k}(A;J)$ of stable k-marked (*geometric*) genus g J-holomorphic maps in the homology class $A\in H_2(X)$. This compact space is generally highly singular. However, as shown in [57], $\overline{\mathfrak{M}}_{g,k}(A;J)$ still determines a rational homology class, called virtual fundamental class (VFC) and denoted by $[\overline{\mathfrak{M}}_{g,k}(A;J)]^{vir}$. This class lives in an arbitrarily small neighborhood of $\overline{\mathfrak{M}}_{g,k}(A;J)$ in the naturally stratified configuration space $\mathfrak{X}_{g,k}(A)$ of smooth stable maps introduced in [57] and is independent of J. Integration of cohomology classes against $[\overline{\mathfrak{M}}_{g,k}(A;J)]^{vir}$ gives rise to GW-invariants; see (2.1). The construction of [57] adapts the deformation-obstruction analysis from the algebro-geometric setting of [56] to symplectic topology via local versions of the inhomogeneous deformations the $\bar\partial_J$-equation introduced in [86, 87] and presents $[\overline{\mathfrak{M}}_{g,k}(A;J)]^{vir}$ as the homology class of a space stratified by even-codimensional orbifolds. This approach is ideally suited for a range of concrete applications, some of which are indicated below, and can be readily extended via [102] beyond the so-called perfect deformation-obstruction settings.

While $\overline{\mathfrak{M}}_{g,k}(A;J)$ is often called a "compactification" of its subspace

$$\mathfrak{M}_{g,k}(A;J)\subset \overline{\mathfrak{M}}_{g,k}(A;J)$$

of maps from smooth domains, $\mathfrak{M}_{g,k}(A;J)$ usually is not a dense subset of $\overline{\mathfrak{M}}_{g,k}(A;J)$. For example,

$$\overline{\mathfrak{M}}_1(\mathbb{P}^n,d)\equiv \overline{\mathfrak{M}}_{1,0}\big(dL;J_{\mathbb{P}^n}\big),$$

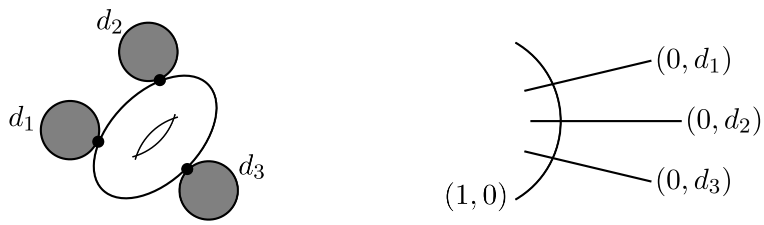
where $L\in H_2(\mathbb{P}^n)$ is the standard generator and $J_{\mathbb{P}^n}$ is the standard complex structure on $\mathbb{P}^n$, is a quasi-projective variety over $\mathbb{C}$ containing $\mathfrak{M}_1(\mathbb{P}^n,d)$ as a Zariski open subspace; see [25]. For $m\in\mathbb{Z}^+$ with $m\le n$, the dimension of the Zariski open subspace $\mathfrak{M}_1^m(\mathbb{P}^n,d)$ of $\overline{\mathfrak{M}}_1(\mathbb{P}^n,d)$ consisting of maps u from a smooth genus 1 curve Σ_P with m copies of $\mathbb{P}^1$ attached directly to Σ_P so that $u(\Sigma_P)\subset\mathbb{P}^n$ is a point is

$$\dim_{\mathbb{C}}\mathfrak{M}_1^m(\mathbb{P}^n,d)=(n+1)d+n-m\ge (n+1)d=\dim_{\mathbb{C}}\mathfrak{M}_1(\mathbb{P}^n,d);$$

see Figure 1. For example,

$$\mathfrak{M}_1^1(\mathbb{P}^n,d)\approx \mathcal{M}_{1,1}\times\mathfrak{M}_{0,1}(\mathbb{P}^n,d)\,.$$

Thus, $\mathfrak{M}_1(\mathbb{P}^n,d)$ is not dense in $\overline{\mathfrak{M}}_1(\mathbb{P}^n,d)$. This motivates the following deep question concerning the convergence of J-holomorphic maps in the sense of [39].



$d_1+d_2+d_3=d$
$d_1,d_2,d_3>0$

Figure 1: The domain of an element of $\mathfrak{M}_1^3(\mathbb{P},d)$ from the points of view of symplectic topology and algebraic geometry, with the first number in each pair in the second diagram denoting the genus of the associated smooth irreducible component and the second number denoting the degree of the restriction of the map to this component.

Question 1 ([86, p276]) *Is there a natural Hausdorff space $\overline{\mathfrak{M}}_{g,k}^0(A;J)$ of k marked J-holomorphic maps to X with images of arithmetic genus at least g containing $\mathfrak{M}_{g,k}(A;J)$ as an open subspace so that $\overline{\mathfrak{M}}_{g,k}^0(A;J)$ is compact whenever X is?*

The "natural" requirement in particular includes that

$$\bigsqcup_{\substack{B\in H_2(Y)\\ \iota_*B=A}}\overline{\mathfrak{M}}_{g,k}^0\big(B;J|_Y\big)=\big\{u\in\overline{\mathfrak{M}}_{g,k}^0(A;J)\colon \mathrm{Im}\;u\subset Y\big\}$$

for every inclusion $\iota\colon Y\longrightarrow X$ of an almost complex submanifold and relatedly that $\overline{\mathfrak{M}}_{g,k}^0(A;J)$ determines a fundamental class $[\overline{\mathfrak{M}}_{g,k}^0(A;J)]^{vir}$. For $g=0$, the usual moduli spaces already have the desired properties and so

$$\overline{\mathfrak{M}}_{0,k}^0(A;J)=\overline{\mathfrak{M}}_{0,k}(A;J)\,.$$

We also note that $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is a smooth irreducible quasi-projective variety containing $\mathfrak{M}_{0,k}(\mathbb{P}^n,d)$ as a Zariski dense open subspace and that

$$\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)-\mathfrak{M}_{0,k}(\mathbb{P}^n,d)\subset\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$$

is a normal crossings divisor.

For $g=1$, Question 1 is answered affirmatively in [104, 105] by defining

$$\overline{\mathfrak{M}}_{1,k}^0(A;J)\subset\overline{\mathfrak{M}}_{1,k}(A;J)$$

and showing that $\overline{\mathfrak{M}}_{1,k}^0(A;J)$ determines a fundamental class. In particular, this subspace contains an element u of $\mathfrak{M}_{1,k}^m(A;J)$ if and only if the differentials of the restrictions of u to the m copies of $\mathbb{P}^1$ at the nodes attached to Σ_P span a subspace of $T_{u(\Sigma_P)}X$ of complex dimension less than m. This imposes no condition if $2m>\dim_{\mathbb{R}}X$. If $m\le n$, this imposes a condition of complex codimension $n+1-m$ on $\mathfrak{M}_1^m(\mathbb{P}^n,d)$ and ensures that

$$\dim_{\mathbb{C}}\big(\overline{\mathfrak{M}}_1^0(\mathbb{P}^n,d)\cap\mathfrak{M}_1^m(\mathbb{P}^n,d)\big)=\dim_{\mathbb{C}}\mathfrak{M}_1(\mathbb{P}^n,d)-1\,.$$

We also note that $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is a singular irreducible quasi-projective variety containing $\mathfrak{M}_{1,k}(\mathbb{P}^n,d)$ as a Zariski dense open subspace and that

$$\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) - \mathfrak{M}_{1,k}(\mathbb{P}^n,d) \subset \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

is a divisor. An explicit desingularization $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ of this space is constructed in [95] so that

$$\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) - \mathfrak{M}_{1,k}(\mathbb{P}^n,d) \subset \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

is a normal crossings divisor. The numerical curve-counting invariants obtained by integrating cohomology classes against $[\overline{\mathfrak{M}}_{1,k}^0(A;J)]^{vir}$ as in (2.1) are called reduced genus 1 GW-invariants in [105]. An algebro-geometric approach to these invariants is suggested in [94].

For sufficiently positive symplectic manifolds (X,ω), the *standard* genus 0 and *reduced* genus 1 GW-invariants with insertions pulled back from X only are *integer* counts of J-holomorphic counts of J-holomorphic curves in X for a generic ω-compatible almost complex structure J. The standard complex structure $J_{\mathbb{P}^n}$ on $\mathbb{P}^n$ works for these purposes. As demonstrated in [86, 71], the good properties of $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ indicated above are key to the enumeration of genus 0 curves in $\mathbb{P}^n$ and in particular establish Kontsevich's recursion for counts of such curves. The explicit constructions of $\overline{\mathfrak{M}}_{1,k}^0(A;J)$ and $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ in [104, 95] have opened the door for similar applications to the enumerative geometry of genus 1 curves.

For example, the Eguchi-Hori-Xiong recursion for counts of genus 1 curves in $\mathbb{P}^2$ is established in [72] by lifting Getzler's relation [33] from $\overline{\mathcal{M}}_{1,4}$ to $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^2,d)$ and obtaining a recursion for the genus 1 GW-invariants of $\mathbb{P}^2$; the latter are the same as the corresponding enumerative invariants in this particular case. Getzler's relation can also be lifted to $\overline{\mathfrak{M}}_{1,k}(A;J)$, $\overline{\mathfrak{M}}_{1,k}^0(A;J)$, and $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ to yield relations between the genus 0 GW and standard (resp. reduced) genus 1 GW-invariants from the first (resp. second/third) lift. The reduced genus 1 GW-invariants of $\mathbb{P}^n$ are the same as the corresponding enumerative invariants. As shown in [103], the difference between the standard and reduced genus 1 GW-invariants is a combination of the genus 0 GW-invariants; this combination takes a very simple form in complex dimension 3. This leads to the following, very concrete question.

Question 2 *Can any of the above three lifts be used to obtain a recursion for the genus 1 standard or reduced GW-invariants of $\mathbb{P}^n$ for $n\geq 3$ and thus a $\mathbb{P}^n$ analogue of the Eguchi-Hori-Xiong recursion enumerating genus 1 curves?*

For $g=2$, [67] provides the affirmative answer to the main part of Question 1 by defining

$$\overline{\mathfrak{M}}_{2,k}^0(A;J) \subset \overline{\mathfrak{M}}_{2,k}(A;J)$$

and leaves no fundamental difficulty in constructing a fundamental class for this space. The description of this subspace is significantly more complicated than of its $g=1$ analogue. In addition to the simple "level 1" condition appearing in the $g=1$ case, this description involves a more elaborate "level 2" condition which depends on precisely how the "level 1" condition is satisfied relative to the involution and the Weierstrass points on the principal component Σ_P of the domain. While $\overline{\mathfrak{M}}_{2,k}^0(\mathbb{P}^n,d)$ is still a quasi-projective variety, it is no longer irreducible and $\mathfrak{M}_{2,k}(\mathbb{P}^n,d)$ is not dense in $\overline{\mathfrak{M}}_{2,k}^0(\mathbb{P}^n,d)$. However, this is not material for some applications.

While Question 1 concerns a foundational issue in GW-theory (and thus is of interest in itself), a satisfactory answer to this problem is key to relating GW-invariants of a compact symplectic submanifold Y of a compact symplectic manifold (X,ω) given as the zero set of a transverse bundle section to the GW-invariants of the ambient symplectic manifold X. If $\pi_{\mathcal{L}}: \mathcal{L} \longrightarrow X$ is a holomorphic vector bundle and $\iota_{\mathcal{L}}: X \longrightarrow \mathcal{L}$ is the inclusion as the zero section, there is a natural projection map

$$\widetilde{\pi}_{\mathcal{L}}\colon \mathcal{V}_{g,k}^A(\mathcal{L}) \equiv \overline{\mathfrak{M}}_{g,k}(\iota_{\mathcal{L}*}A;J) \longrightarrow \overline{\mathfrak{M}}_{g,k}(A;J), \quad \big[\widetilde{u}\colon \Sigma \longrightarrow \mathcal{L}\big] \longrightarrow \big[\pi_{\mathcal{L}}\circ\widetilde{u}\colon \Sigma \longrightarrow X\big]. \tag{1.1}$$

The fiber of $\widetilde{\pi}_{\mathcal{L}}$ over an element $[u\colon \Sigma \longrightarrow X]$ is $H^0(\Sigma;u^*\mathcal{L})$, the space of holomorphic sections of the holomorphic bundle $u^*\mathcal{L} \longrightarrow \Sigma$. If X and $\mathcal{L}$ are sufficiently positive (such as $\mathbb{P}^n$ and sum of positive line bundles) and $g=0$, $\widetilde{\pi}_{\mathcal{L}}$ is in fact a vector orbi-bundle and

$$\sum_{\substack{B\in H_2(Y)\\ \iota_*B=A}} \iota_*\big[\overline{\mathfrak{M}}_{0,k}(B;J)\big]^{vir} = e\big(\mathcal{V}_{0,k}^A(\mathcal{L})\big) \cap [\overline{\mathfrak{M}}_{0,k}(A;J)]^{vir}. \tag{1.2}$$

This observation in [49], now known as the Quantum Lefschetz Hyperplane Theorem for genus 0 GW-invariants, was the starting point for the proofs of the genus 0 mirror symmetry prediction of [7] for the quintic threefold $X_5 \subset \mathbb{P}^4$ in [34, 59].

Question 3 *Is there an analogue of the $g=0$ Quantum Lefschetz Hyperplane Theorem (1.2) for $g\geq 1$?*

While $\widetilde{\pi}_{\mathcal{L}}$ is not even a vector bundle for $g\geq 1$ (even for sufficiently positive X and $\mathcal{L}$), it is shown in [101, 58] that the restriction

$$\widetilde{\pi}_{\mathcal{L}}\colon \mathcal{V}_{1,k}^A(\mathcal{L})\big|_{\overline{\mathfrak{M}}_{1,k}^0(A;J)} \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(A;J) \tag{1.3}$$

carries a well-defined Euler class, which in turn relates the reduced genus 1 GW-invariants of the submanifold Y and the ambient manifold X:

$$\sum_{\substack{B\in H_2(Y)\\ \iota_*B=A}} \iota_*[\overline{\mathfrak{M}}_{1,k}^0(B;J)]^{vir} = \mathrm{PD}_{[\overline{\mathfrak{M}}_{1,k}^0(A;J)]} e\big(\mathcal{V}_{1,k}^A(\mathcal{L})\big). \tag{1.4}$$

This Quantum Lefschetz Hyperplane Theorem for the reduced genus 1 GW-invariants introduced in [105] and the comparison of the standard and reduced genus 1 GW-invariants established in [103] provide a Quantum Lefschetz Hyperplane Theorem for the standard genus 1 GW-invariants. The latter is combined in [106] with the desingularization of the relevant special cases of (1.3) constructed in [95] to confirm the genus 1 mirror symmetry prediction of [2] for X_5 and to obtain similar mirror symmetry formulas for Calabi-Yau hypersurfaces in all projective spaces.

The concrete topological construction of virtual fundamental class in [57] is particularly convenient for the purposes of [104, 105, 101]. It readily handles the moduli spaces $\overline{\mathfrak{M}}_{1,k}^0(A;J)$, which are not virtually smooth, but are virtually stratified by smooth orbifolds of even codimensions. The representation of VFC by a geometric object in [57] also fits well with the comparisons carried out in [101, 58]. However, later variations on [57] of topological flavor, such as [24, 75], should also fit with [104, 105, 103, 101, 58].

A satisfactory affirmative answer to Question 1 for each $g \geq 2$, combined with the geometric virtual fundamental class perspective of [57], should readily lead to a Quantum Lefschetz Hyperplane Theorem and to computations of GW-invariants of projective complete intersections in the same genus g. In light of [67], there are no fundamental difficulties left to confirm the genus 2 mirror symmetry predictions of [2] for X_5 and other projective complete intersections by paralleling the genus 1 approach initiated in [104] and completed in [106]. The same approach should also yield confirmations of the mirror symmetry predictions of [97] for the real GW-invariants constructed in [30], after the additional topological subtleties typically arising in the real setting are addressed.

The methods of [104, 67] provide "level 1" and "level 2" obstructions to smoothing J-holomorphic maps from nodal domains and can be used to define natural closed subspaces

$$\overline{\mathfrak{M}}^0_{g,k}(A;J) \subset \overline{\mathfrak{M}}_{g,k}(A;J)$$

for $g \geq 3$, which refine Gromov's Compactness Theorem and determine fundamental classes giving rise to curve-counting invariants of compact symplectic manifolds. However, these sharper compactifications would still not be sufficiently small to exclude all J-holomorphic maps to X with images of arithmetic genus below g, but above 1. The associated reduced GW-invariants would then include lower-genus contributions, even for very positive almost complex structures J. Furthermore, there are indications in [67] that the answer to Question 1 may in fact be negative for an arbitrary almost complex structure J on X if $g > 2$ (or perhaps slightly larger) and the dimension of X is sufficiently large.

On the other hand, an affirmative answer to Question 1 in full generality is not needed for specific applications, including to the enumerative geometry of positive-genus curves in the spirit of [71] and to the mirror symmetry predictions in the spirit of [103, 106]. While the complexity of a complete description of $\overline{\mathfrak{M}}^0_{g,k}(A;J)$, whenever it can be defined, would increase rapidly with the genus g, it is likely not to be needed for specific applications either. In particular, it appears feasible to set up a scheme paralleling the genus 1 approach initiated in [104] and completed in [106] that would compute all GW-invariants of X_5 modulo finitely many inputs in each genus g. This could potentially show that the generating functions F_g for these invariants satisfy the holomorphic anomaly equations as predicted in [2], without determining each specific F_g explicitly.

2 BPS states for arbitrary symplectic manifolds

GW-invariants of a symplectic manifold (X,ω) are in general rational numbers arising from families of J-holomorphic curves in X of possibly lower genus and/or "lower" degree (relative to the symplectic deformation equivalence class of ω). The primary genus 0 GW-invariants of positive symplectic manifolds (such as smooth Fano varieties) and of symplectic fourfolds arise only from J-holomorphic curves of the same genus and degree, for a generic ω-compatible almost complex structure J on X, and are integer counts of such curves. One might hope that the GW-invariants of (X,ω) in general are expressible in terms of some integer invariants of (X,ω) arising from J-holomorphic curves on X, for J generic at least in some non-empty open subset of such J's. The explicit prediction of [38] relating GW-invariants of Calabi-Yau (or CY) sixfolds (X,ω) to certain conjecturally integer counts could be interpreted in such a way; this prediction has since been extended to a number of other special cases.

For a compact symplectic manifold (X,ω), we denote by $\mathcal{J}_\omega$ the space of ω-compatible almost complex structures on X. For $g,k\in\mathbb{Z}^{\ge 0}$, $A\in H_2(X)$, $J\in\mathcal{J}_\omega$, and $i=1,\ldots,k$, let

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{g,k}(A;J)\longrightarrow X, \quad \mathrm{ev}_i\big([u,z_1,\ldots,z_k]\big)=u(z_i),$$

be the evaluation map at the i-th marked point. We denote by

$$\begin{split}\mathrm{GW}_{g,A}^X\colon \mathcal{H}^*(X)\equiv\bigsqcup_{k=1}^{\infty}H^*(X)^{\oplus k}\longrightarrow\mathbb{Q},\\ \mathrm{GW}_{g,A}^X\big(\mu_1,\ldots,\mu_k\big)=\Big\langle\prod_{i=1}^{k}\mathrm{ev}_i^*\mu_i,\,\big[\overline{\mathfrak{M}}_{g,k}(A;J)\big]^{vir}\Big\rangle,\end{split} \tag{2.1}$$

the primary genus g degree A GW-invariants of (X,ω); these multilinear functionals are graded symmetric. The number above vanishes unless

$$\sum_{i=1}^{k}\dim_{\mathbb{R}}\mu_i=\dim\big[\overline{\mathfrak{M}}_{g,k}(A;J)\big]^{vir}=2\big(\langle c_1(X,\omega),A\rangle+k\big)+\dim_{\mathbb{R}}X-6. \tag{2.2}$$

In general, this number arises from the families of genus g' degree A' J-holomorphic curves in X that pass through generic pseudocycle representatives for the Poincare duals of $\mu_1,\ldots,\mu_k$ in the sense of [102].

We denote the symplectic deformation equivalence class of a symplectic form ω on a manifold X by $[\omega]$ and let

$$\mathcal{A}\big([\omega]\big)=\big\{(g,A)\in\mathbb{Z}^{\ge 0}\times(H_2(X)-\{0\})\colon \overline{\mathfrak{M}}_g(A;J)\neq\emptyset\ \forall\,J\in\mathcal{J}_{\omega'},\,\omega'\in[\omega]\big\}.$$

The genus g degree A GW-invariants of a compact symplectic manifold (X,ω) depend only on $[\omega]$ and vanish unless $(g,A)\in\mathcal{A}([\omega])$ or $A=0$. In general, they arise from families of connected J-holomorphic curves in X described by decorated graphs, i.e. tuples of the form

$$\Gamma=\big(\mathrm{V},\mathrm{Edg},\mathfrak{g}\colon\mathrm{V}\longrightarrow\mathbb{Z}^{\ge 0},\mathfrak{d}\colon\mathrm{V}\longrightarrow H_2(X)-\{0\}\big). \tag{2.3}$$

In such a tuple, $\mathrm{V}(\Gamma)\equiv\mathrm{V}$ and $\mathrm{Edg}(\Gamma)\equiv\mathrm{Edg}$ are finite collections of vertices and edges, respectively; the latter are pairs of vertices, but of not necessarily distinct ones, and some pairs may appear multiple times in the collection Edg. The vertices and the edges index the irreducible components $\mathcal{C}_v$ of the curves and the nodes between them, respectively. The values of the maps $\mathfrak{g}$ and $\mathfrak{d}$ at $v\in\mathrm{V}$ specify the geometric genus of $\mathcal{C}_v$ and its degree, respectively. For a tuple as in (2.3), we define

$$g(\Gamma)=1+|\mathrm{Edg}|-|\mathrm{V}|+\sum_{v\in\mathrm{V}}\mathfrak{g}(v), \quad \mathfrak{g}_v(\Gamma)=\mathfrak{g}(v),\ \mathfrak{d}_v(\Gamma)=\mathfrak{d}(v)\ \ \forall\,v\in\mathrm{V}\,.$$

We denote by $\mathcal{P}([\omega])$ the collection of connected decorated graphs Γ as in (2.3) such that $(\mathfrak{g}(v),\mathfrak{d}(v))$ is an element of $\mathcal{A}([\omega])$ for every $v\in\mathrm{V}$.

For $(g,A)\in\mathcal{A}([\omega])$, let $\Gamma_0(g,A)$ be the unique connected edgeless graph with

$$\mathfrak{g}_v\big(\Gamma_0(g,A)\big)=g \qquad \text{and} \qquad \mathfrak{d}_v\big(\Gamma_0(g,A)\big)=A$$

for the unique vertex v. Define

$$\begin{split}\widetilde{\mathcal{P}}_{g,A}\big([\omega]\big)=\Big\{(\Gamma,\mathfrak{m})\colon \Gamma\in\mathcal{P}([\omega]),\ g(\Gamma)\le g,\ |\mathrm{Edg}(\Gamma)|\le(n-3)(g-g(\Gamma)),\\ \mathfrak{m}\in(\mathbb{Z}^+)^{\mathrm{V}(\Gamma)},\ \sum_{v\in\mathrm{V}}\mathfrak{m}_v\mathfrak{d}_v(\Gamma)=A\Big\},\end{split}$$

where $2n\equiv\dim_{\mathbb{R}}X$. By Gromov's Compactness Theorem [39], this collection is finite for every $(g,A)\in\mathcal{A}(\omega)$. Let

$$\widetilde{\mathcal{P}}^{\star}_{g,A}\big([\omega]\big)\subset\widetilde{\mathcal{P}}_{g,A}([\omega])$$

be the complement of the pair $(\Gamma_0(g,A),1)$.

For a graded symmetric multilinear functional

$$\mathrm{E}\colon\mathcal{H}^*(X)\longrightarrow\mathbb{Q}$$

and $\mu\in H^*(X)^{\oplus k_0}$, we denote by $\mathrm{E}(\mu,\cdot)$ the graded symmetric multilinear functional obtained by inserting additional k inputs after the k_0 inputs μ. For $m\in\mathbb{Z}^+$, define

$$\langle\mathrm{E}\rangle_m\colon\mathcal{H}^*(X)\longrightarrow\mathbb{Q},\qquad \langle\mathrm{E}\rangle_m(\mu)=m^k\mathrm{E}(\mu)\ \ \forall\,\mu\in H^k(X),\ k\in\mathbb{Z}^{\ge0}\,.$$

For graded symmetric multilinear functionals $\mathrm{E}_1,\ldots,\mathrm{E}_r$ as above, let

$$\prod(\mathrm{E}_1,\ldots,\mathrm{E}_r)\colon\mathcal{H}^*(X)\longrightarrow\mathbb{Q}$$

be the graded symmetric multilinear functional obtained by distributing the k inputs between the r functionals $\mathrm{E}_1,\ldots,\mathrm{E}_r$, multiplying their outputs, and summing over all possible distributions with the appropriate signs depending on the degrees of the inputs.

For a symplectic form ω on X and $A,A^*\in H_2(X)$, we define $A\le_\omega A^*$ if $\omega'(A)\le\omega'(A^*)$ for some $\omega'\in[\omega]$. For the purposes of the question below, we identify the vertices V of each graph as in (2.3) with the set $\{1,\ldots,|\mathrm{V}|\}$.

Question 4 *Let (X,ω) be a compact symplectic manifold. Are there a collection*

$$C^{(g)}_{\mathfrak{g},\mathfrak{m}}\in\mathbb{Q}\qquad g\in\mathbb{Z}^{\ge0},\ (\mathfrak{g},\mathfrak{m})\in(\mathbb{Z}^{\ge0})^r\times(\mathbb{Z}^+)^r,\ r\in\mathbb{Z}^+,$$

of rational numbers and collections

$$\mathrm{E}^X_{\Gamma,\mathfrak{m}}\colon\mathcal{H}^*(X)\longrightarrow\mathbb{Z},\ \Gamma\in\mathcal{P}\big([\omega]\big),\,\mathfrak{m}\in(\mathbb{Z}^+)^{\mathrm{V}(\Gamma)},\quad \mathrm{E}^X_{g,A}\colon\mathcal{H}^*(X)\longrightarrow\mathbb{Z},\ (g,A)\in\mathcal{A}([\omega]),$$

of graded symmetric multilinear functionals that depend only on $[\omega]$ and satisfy the following properties?

(E1) for every $(g,A)\in\mathcal{A}([\omega])$,

$$\mathrm{GW}^X_{g,A}=\mathrm{E}^X_{g,A}+\sum_{(\Gamma,\mathfrak{m})\in\widetilde{\mathcal{P}}^{\star}_{g,A}([\omega])}\mathrm{E}^X_{\Gamma,\mathfrak{m}}\,;\tag{2.4}$$

(E2) for every $\Gamma\in\mathcal{P}([\omega])$ as in (2.3), there exist $N(\Gamma)\in\mathbb{Z}^{\ge 0}$ and $\mu_{r;v}\in\mathcal{H}^*(X)$ with $r=1,\ldots,N(\Gamma)$ and $v\in\mathrm{V}$ such that

$$\mathrm{E}^X_{\Gamma,\mathfrak{m}} = C^{(g)}_{\mathfrak{g}(\Gamma),\mathfrak{m}}\sum_{r=1}^{N(\Gamma)}\prod\Big(\big(\big\langle \mathrm{E}^X_{\mathfrak{g}(v),\mathfrak{d}(v)}\big\rangle_{\mathfrak{m}_v}(\mu_{r;v},\cdot)\big)_{v\in\mathrm{V}}\Big) \quad \forall\,\mathfrak{m}\in(\mathbb{Z}^+)^{\mathrm{V}(\Gamma)}; \tag{2.5}$$

(E3) for every $A\in H_2(X)$ with $\omega'(A)>0$ for all $\omega'\in[\omega]$,

$$\sup\big\{g\in\mathbb{Z}^{\ge 0}\colon \mathrm{E}^X_{g,A}\neq 0\big\} < \infty; \tag{2.6}$$

(E4) for all $g^*\in\mathbb{Z}^{\ge 0}$ and $A^*\in H_2(X)$ there exists a subset $\mathcal{J}^{reg}_\omega\subset\mathcal{J}_\omega$ of second category in a nonempty open subset of $\mathcal{J}_\omega$ so that for all $(g,A)\in\mathcal{A}([\omega])$ with $g\le g^*$ and $A\le_\omega A^*$, $J\in\mathcal{J}^{reg}_\omega$, and $\mu_1,\ldots,\mu_k\in H^*(X)$ satisfying (2.2), there exist pseudocycle representatives f_i for the Poincare duals of μ_i such that

- *the set of genus g degree A J-holomorphic curves meeting the pseudocycles $f_1,\ldots,f_k$ is cut out transversely and thus is finite,*
- *the number of such curves counted with the associated signs is $\mathrm{E}^X_{g,A}(\mu_1,\ldots,\mu_k)$.*

For all $n\in\mathbb{Z}^{\ge 0}$ and $A\in H_2(\mathbb{P}^n)$, there exists $g_A\in\mathbb{Z}^+$ so that every degree A $J_{\mathbb{P}^n}$-holomorphic map $u\colon\Sigma\longrightarrow\mathbb{P}^n$ from a smooth closed connected genus $g\ge g_A$ Riemann surface is a branched cover of a line $\mathbb{P}^1\subset\mathbb{P}^n$; this is a special of the classical Castelnuovo bound [36, p252]. In light of (E4), (E3) is an analogue of this bound for J-holomorphic curves in arbitrary symplectic manifolds.

For symplectic fourfolds, i.e. $n=2$ in the definition of the collection $\widetilde{\mathcal{P}}_{g,A}([\omega])$, (2.4) and (2.5) reduce to $\mathrm{GW}^X_{g,A}=\mathrm{E}^X_{g,A}$; (E4) is well-known to hold in this case. For symplectic sixfolds, i.e. $n=3$, $(g,A)\notin\mathcal{A}([\omega])$ unless

$$\big\langle c_1(X,\omega),A\big\rangle=0 \qquad\text{or}\qquad \big\langle c_1(X,\omega),A\big\rangle>0\,. \tag{2.7}$$

In both cases, only edgeless connected graphs appear in (2.4). Precise predictions for the structure of (2.4) and (2.5) for symplectic sixfolds involve the coefficients $C_{h,A}(g)\in\mathbb{Q}$ specified by

$$\sum_{g=0}^{\infty}C_{h,A}(g)t^{2g} = \bigg(\frac{\sin(t/2)}{t/2}\bigg)^{2h-2+\langle c_1(X,\omega),A\rangle}. \tag{2.8}$$

In the second, Fano, case of (2.7), (2.4) and (2.5) were predicted in [73] to reduce to

$$\mathrm{GW}^X_{g,A}(\mu)=\sum_{h=0}^{g}C_{h,A}(g-h)\mathrm{E}^X_{h,A}(\mu) \qquad \forall\,\mu\in\mathcal{H}^*(X). \tag{2.9}$$

In the $g=0,1$ cases, this becomes

$$\mathrm{GW}^X_{0,A}(\mu)=\mathrm{E}^X_{0,A}(\mu), \quad \mathrm{GW}^X_{1,A}(\mu)=\mathrm{E}^X_{1,A}(\mu)+\frac{2-\langle c_1(X,\omega),A\rangle}{24}\mathrm{E}^X_{0,A}(\mu), \tag{2.10}$$

respectively.

The first equation in (2.10) with $\mathrm{E}^X_{0,A}(\mu)$ described by (E4) is the original *definition* of $\mathrm{GW}^X_{0,A}(\mu)$ for Fano classes A in the basic case of the semi-positive symplectic manifolds (which include all symplectic sixfolds). The second equation in (2.10) holds with $\mathrm{E}^X_{1,A}(\mu)$ replaced by the reduced genus 1 GW-invariants $\mathrm{GW}^{X;0}_{1,A}(\mu)$ constructed in [105], which satisfy the first bullet in (E4) whenever (X,ω) is semi-positive; see [105, Theorem 1.1] and [104, Section 1.3], respectively. The existence of a subspace $\mathcal{J}^{reg}_\omega \subset \mathcal{J}_\omega$ of second category satisfying (E4) for the Fano classes A on symplectic sixfolds is established in [107]. Since the system of equations (2.9) with all such classes A is invertible and the GW-invariants depend only on $[\omega]$, this implies that the resulting counts $\mathrm{E}^X_{h,A}(\mu)$ depend only on $[\omega]$ and thus affirmatively answers Question 4 with the exception of (E3) in the Fano case of (2.7).

The first, CY, case of (2.7) is much harder because degree $m\geq 2$ covers of genus h degree A/m J-holomorphic curves $\mathcal{C}\subset X$ contribute to the genus g degree A GW-invariants of (X,ω). For $d\in\mathbb{Z}^+$, we denote by $\mathcal{P}(d)$ the set of partitions of d into positive integers $d_1\geq\ldots\geq d_k$. Each such partition ρ corresponds to a Ferrers diagram, i.e. a collection of boxes indexed by the set

$$S(\rho)=\big\{(i,j)\colon i\in 1,\ldots,k,\ j\in 1,\ldots,d_i\big\},$$

and to a dual partition $\rho'\equiv(d_1'\geq\ldots\geq d_{k'}')$ of d specified by

$$k'=d_1, \qquad d_j'=\max\big\{i=1,\ldots,k\colon d_i\geq j\big\}.$$

The hooklength of a box $(i,j)\in S(\rho)$ is defined to be

$$\ell_{ij}(\rho)=d_i+d_j-i-j+1\in\mathbb{Z}^+\,.$$

The degree d contribution $n^{(h)}_{h',d}\in\mathbb{Z}^+$ of a genus h curve to the genus h' curve count was predicted in [6] to be given by

$$\begin{split}\exp\Bigg(\sum_{d=1}^{\infty}\sum_{h'=h}^{\infty}n^{(h)}_{h',d}\bigg(\sum_{m=1}^{\infty}\frac{q^{md}}{m}\big(2\sin(mt/2)\big)^{2h'-2}\bigg)\Bigg)&\\ =1+\sum_{d=1}^{\infty}q^d\bigg(\sum_{\rho\in\mathcal{P}(d)}\prod_{(i,j)\in S(\rho)}&\Big(2\sin\big(\ell_{ij}(\rho)t/2\big)\Big)^{2h-2}\bigg).\end{split}\tag{2.11}$$

We note that

$$\exp\bigg(\sum_{m=1}^{\infty}\frac{q^m}{m}\big(2\sin(mt/2)\big)^{-2}\bigg)=1+\sum_{d=1}^{\infty}q^d\bigg(\sum_{\rho\in\mathcal{P}(d)}\prod_{(i,j)\in S(\rho)}\Big(2\sin\big(\ell_{ij}(\rho)t/2\big)\Big)^{-2}\bigg),$$

$$\sum_{d=1}^{\infty}\bigg(\sum_{m=1}^{\infty}\frac{q^{md}}{m}\bigg)=-\sum_{d=1}^{\infty}\ln\big(1-q^d\big)=\ln\bigg(\prod_{d=1}^{\infty}\big(1-q^d\big)^{-1}\bigg)=\ln\bigg(1+\sum_{d=1}^{\infty}q^d\big|\mathcal{P}(d)\big|\bigg);$$

the first identity above is the $t_1=t$, $t_2=t^{-1}$ case of [66, (4.5)]. Combining these two identities with the $h=0,1$ cases of (2.11), we obtain

$$n^{(0)}_{h',d}=\begin{cases}1, & \text{if } (h',d)=(0,1);\\ 0, & \text{otherwise;}\end{cases}\qquad n^{(1)}_{h',d}=\begin{cases}1, & \text{if } h'=1;\\ 0, & \text{otherwise.}\end{cases}\tag{2.12}$$

However, $n^{(h)}_{h',d}$ is generally nonzero for $h\geq 2$, $d\in\mathbb{Z}^+$, and some $h'>h$.

The primary GW-invariants (2.1) in the CY classes A are encoded by the rational numbers $N^X_{g,A}\equiv \mathrm{GW}_{g,A}()$, i.e. the GW-invariants with no insertions. In this case, (2.4) and (2.5) were predicted in [38] to reduce to

$$N^X_{g,A}=\sum_{\substack{m\in\mathbb{Z}^+\\ A/m\in\mathcal{A}([\omega])}} m^{2g-3}\sum_{h=0}^{g}\Bigg(\sum_{\substack{d\in\mathbb{Z}^+\\ m/d\in\mathbb{Z}}} d^{3-2g}\sum_{h'=h}^{g} C_{h',0}(g-h')n^{(h)}_{h',d}\Bigg)n^X_{h,A/m}\,, \tag{2.13}$$

where $n^X_{g,A}\equiv \mathrm{E}_{g,A}()$. For $m\in\mathbb{Z}^+$, we denote by $\langle m\rangle$ the sum of the positive divisors of m. By (2.12), the $g=0,1$ cases of (2.13) become

$$N^X_{0,A}=\sum_{\substack{m\in\mathbb{Z}^+\\ A/m\in\mathcal{A}([\omega])}} m^{-3}n^X_{0,A/m}, \quad N^X_{1,A}=\sum_{\substack{m\in\mathbb{Z}^+\\ A/m\in\mathcal{A}([\omega])}} m^{-1}\bigg(\langle m\rangle n^X_{1,A/m}+\frac{1}{12}n^X_{0,A/m}\bigg), \tag{2.14}$$

respectively.

The system of equations (2.13) with all CY classes A on a symplectic sixfold (X,ω) is also invertible. Thus, it determines the numbers $n^X_{g,A}\in\mathbb{Q}$ from the number $N^X_{g,A}$. The original version of Question 4, known as the Gopakumar-Vafa Conjecture for projective CY threefolds, in fact predicted *only* the integrality of the numbers $n^X_{g,A}$ obtained in this way and the existence of a Castelnuovo-type bound for them. However, (E4) has been generally believed to be the underlying reason for the validity of this conjecture since its appearance in the late 1990s. Until [46], (E4) had also been central to every claim, including by the authors of [46] in the early 2000s, to establish the integrality part of this conjecture; all of these claims had quickly turned out to be erroneous.

A fundamentally new perspective on the integrality part of the Gopakumar-Vafa Conjecture for symplectic sixfolds is introduced in [46]. It completely bypasses the analytic step (E4) and appears to succeed in establishing the integrality of the numbers $n^X_{g,A}$ arising from (2.13) via local arguments that are generally topological in spirit. The existence of a subset $\mathcal{J}^{reg}_\omega\subset\mathcal{J}_\omega$ of second category satisfying the first bullet in (E4) for symplectic CY sixfolds is treated in [99] following the general approach to this transversality issue in [11], but with additional technical input. However, it still remains to establish that the resulting counts of J-holomorphic curves satisfy the second bullet in (E4). Taking a geometric analysis perspective previously unexplored in GW-theory, [10] uses [84], which established an analogue of Gromov's Convergence Theorem for J-holomorphic maps without an a priori genus bound, to reduce the Castelnuovo-type bound (E3) for symplectic CY sixfolds to the existence of $J\in\mathcal{J}_\omega$ satisfying (E4).

Precise predictions for the structure of (2.4) and (2.5) have also been made in some cases for symplectic manifolds of real dimensions $2n\geq 8$. The genus 0 prediction for symplectic CY manifolds is a direct generalization of the first equation in (2.14) and is given by

$$\mathrm{GW}^X_{0,A}(\mu_1,\ldots,\mu_k)=\sum_{\substack{m\in\mathbb{Z}^+\\ A/m\in\mathcal{A}([\omega])}} m^{k-3}\mathrm{E}^X_{0,A/m}(\mu_1,\ldots,\mu_k)\quad\forall\,\mu_1,\ldots,\mu_k\in H^*(X); \tag{2.15}$$

see [47, (2)]. The genus 1 predictions for symplectic CY manifolds of real dimensions 8 and 10 appear in [47] and [74], respectively. In contrast to the arbitrary genus GW-invariants of symplectic sixfolds in (2.9) and (2.13) and to the genus 0 GW-invariants of symplectic CY manifolds in (2.15), the genus 1 GW-invariants of symplectic CY manifolds (X,ω) of real dimensions $2n\geq 8$ include contributions from families of J-holomorphic curves in (X,ω) of positive dimensions ($2(n-3)$-dimensional families of genus 0 curves). This makes the analogues of (2.9), (2.13), and (2.15) in the last case significantly more complicated. All curves appearing in the relevant families of J-holomorphic curves are reduced in the sense of algebraic geometric geometry and have simple nodes if $n=4,5$. As noted in the last paragraphs of [74, Sections 1.2,2.2], non-reduced curves and curves with non-simple nodes appear in such families if $n\geq 6$. In order to obtain a precise prediction for the structure of (2.4) and (2.5) for the genus 1 GW-invariants of symplectic CY manifolds of real dimensions $2n\geq 12$, contributions from such curves to the genus 0 and genus 1 GW-invariants still need to be determined.

Question 4 readily extends to the real GW-invariants $\mathrm{GW}^{\phi}_{g,A}$ of compact real symplectic manifolds (X,ω,ϕ), whenever these invariants are defined. For example, the genus 0 real GW-invariants of real symplectic fourfolds constructed in [98] are just signed counts of J-holomorphic curves. So, the real analogues of (2.4) and (2.5) in this case also reduce to $\mathrm{GW}^{\phi}_{0,A}=\mathrm{E}^{\phi}_{0,A}$. Arbitrary genus real GW-invariants are constructed in [30] for many real symplectic manifolds, including the odd-dimensional projective spaces $\mathbb{P}^{2n-1}$ and quintic threefolds $X_5\subset\mathbb{P}^4$ cut out by real equations. It is established in [68] that the analogue of (2.9) for the Fano classes A on a real symplectic sixfold (X,ω,ϕ) is

$$\mathrm{GW}^{\phi}_{g,A}(\mu)=\sum_{\substack{0\leq h\leq g\\ g-h\in 2\mathbb{Z}}}\widetilde{C}_{h,A}\big(\tfrac{g-h}{2}\big)\mathrm{E}^{\phi}_{h,A}(\mu)\qquad\forall\ \mu\in\mathcal{H}^*(X),\tag{2.16}$$

with the coefficients $\widetilde{C}_{h,A}(g)\in\mathbb{Q}$ defined by

$$\sum_{g=0}^{\infty}\widetilde{C}_{h,A}(g)t^{2g}=\bigg(\frac{\sinh(t/2)}{t/2}\bigg)^{h-1+\langle c_1(X,\omega),A\rangle/2}.\tag{2.17}$$

The invariants $\mathrm{E}^{\phi}_{h,A}(\mu)$ appearing in (2.16) are signed counts of real genus g degree A J-holomorphic curves $\mathcal{C}\subset X$.

The real Fano threefold case treated in [68] and [97, (5.41)] suggest that the real analogue of (2.13) should be

$$N^{\phi}_{g,A}=\sum_{\substack{m\in\mathbb{Z}^+-2\mathbb{Z}\\ A/m\in\mathcal{A}([\omega])}}m^{g-2}\sum_{\substack{0\leq h\leq g\\ g-h\in 2\mathbb{Z}}}\bigg(\sum_{\substack{d\in\mathbb{Z}^+\\ m/d\in\mathbb{Z}}}d^{2-g}\sum_{\substack{h\leq h'\leq g\\ g-h'\in 2\mathbb{Z}}}\widetilde{C}_{h',0}\big(\tfrac{g-h'}{2}\big)\widetilde{n}^{(h)}_{h',d}\bigg)n^{\phi}_{h,A/m},\tag{2.18}$$

for some $\widetilde{n}^{(h)}_{h',d}\in\mathbb{Z}$ (only the d odd cases matter). The right-hand sides of [97, (5.10),(5.28)] suggest that

$$\widetilde{n}^{(0)}_{h',d}=\begin{cases}1, & \text{if } (h',d)=(0,1);\\ 0, & \text{otherwise;}\end{cases}\qquad \widetilde{n}^{(1)}_{h',d}=\begin{cases}1, & \text{if } h'=1;\\ 0, & \text{otherwise.}\end{cases}$$

This would reduce the $g\!=\!0,1$ cases of (2.18) to

$$N^{\phi}_{0,A}=\sum_{\substack{m\in\mathbb{Z}^+-2\mathbb{Z}\\ A/m\in\mathcal{A}([\omega])}} m^{-2}n^{\phi}_{0,A/m}, \qquad N^{\phi}_{1,A}=\sum_{\substack{m\in\mathbb{Z}^+-2\mathbb{Z}\\ A/m\in\mathcal{A}([\omega])}} m^{-1}\langle m\rangle n^{\phi}_{1,A/m}.$$

The numbers $\widetilde{n}^{(h)}_{h',d}$ should arise from a real analogue of (2.11), with the exponent on the left-hand side combining the real curve counts $\widetilde{n}^{(h)}_{h',d}$ and the complex curve counts $n^{(h)}_{h',d}$ to account for the real doublets of [31, Theorem 1.3]. The three theorems of [31, Section 1] should provide the necessary geometric input to adapt the approach of [6] for (2.13) to the real setting; related equivariant localization data is provided by [32, Section 4.2]. Some analogue of (2.11) for the real setting has been apparently obtained by [27].

The approach of [46] to the integrality of the numbers $n^X_{g,A}$ determined by (2.13) should be adaptable to other situations when the GW-invariants in question are expected to arise entirely from isolated J-holomorphic curves. These situations include the real genus 0 GW-invariants of many real symplectic manifolds and the real arbitrary genus GW-invariants of real symplectic CY sixfolds constructed in [26] and [30], respectively. In fact, the integrality of the numbers $\mathrm{E}^X_{0,A}(\mu)$ determined by (2.15) is already a (secondary) subject of [46]. On the other hand, the approach of [46] does not appear readily adaptable to situations when positive-dimensional families of J-holomorphic curves in X are expected to contribute to the GW-invariants in question. These situations include the genus 1 GW-invariants of symplectic CY manifolds of real dimensions 8 and 10 studied in [47] and [74], respectively. The approaches of [99] and [10] to the existence of a subset $\mathcal{J}^{reg}_\omega\!\subset\!\mathcal{J}_\omega$ satisfying (E4) and to the Castelnuovo-type bound for the associated counts of J-holomorphic curves, respectively, appear more flexible in this regard.

Enumerative geometry of curves in projective varieties is a classical subject originating in the middle of the nineteenth century. However, the developments in this field had been limited to very low degrees until the emergence of GW-theory and its applications to enumerative geometry in the early 1990s. As the moduli spaces $\overline{\mathfrak{M}}_{g,k}(A;J)$ have fairly nice deformation-obstruction theory, the GW-invariants arising from these spaces are often amendable to computations. Whenever these invariants can be related to enumerative curve counts as in Question 4, computations of GW-invariants translate into direct applications to enumerative geometry. The most famous such application is perhaps Kontsevich's recursion for counts of genus 0 curves in $\mathbb{CP}^2$, stated in [50] and proved in [86]. Analogues of this recursion for counts of real genus 0 curves in $\mathbb{P}^2$ defined in [98] and in $\mathbb{P}^{2n-1}$ defined in [26] appear in [89] and [28, 29], respectively. The counts of genus g degree d curves arising from the proofs of the mirror symmetry predictions for the projective CY complete intersections in genus 0 in [34, 59] and in genus 1 in [106, 83] via (2.14) have been shown to match the classical enumerative counts for $g\!=\!0$, $d\!\le\!3$ and for $g\!=\!1$, $d\!\le\!4$; see [12]. The genus 0 real GW-invariants of real symplectic fourfolds defined in [98] and of many higher-dimensional real symplectic manifolds defined in [26] directly provide lower bounds for counts of genus 0 real curves; the arbitrary genus real GW-invariants defined in [30] provide such bounds in arbitrary genera via the relation (2.16) proved in [68]. For local CY manifolds, Question 4 points to intriguing number-theoretic properties of GW-invariants; see G. Martin's conjecture in [74, Section 3.2].

The coefficients $\widetilde{C}_{0,0}(g)$ in (2.17) are the coefficients of the renown A-series central to the Index Theorem [52, Theorem 3.13]; they in particular determine the index of the Dirac operator on a

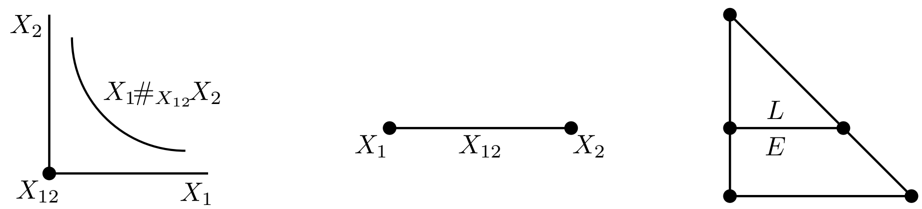


Figure 2: A 2-fold simple normal crossings variety $\mathcal{Z}_0 = X_1 \cup_{X_{12}} X_2$ with its smoothing $\mathcal{Z}_\lambda = X_1\#_{X_{12}}X_2$, its dual intersection complex, and a toric 2-fold decomposition of $\mathbb{P}^2$ into $\mathbb{P}^2$ and its one-point blowup $\widehat{\mathbb{P}}^2$ along a line $L \subset \mathbb{P}^2$ and the exceptional divisor $E \subset \widehat{\mathbb{P}}^2$.

Spin bundle. The coefficients in (2.8) are closely related to the A-series as well. It is tempting to wonder if there is some connection between the multiply covered contributions encoded by (2.8) and by (2.17) and Dirac operators.

3 Symplectic degenerations and Gromov-Witten invariants

It is natural and essential to study the behavior of GW-invariants under reasonable degenerations and decompositions of symplectic manifolds, as pointed out in [91]. The standard example of such a decomposition is provided by the symplectic sum construction of [37]; it joins two symplectic manifolds X_1 and X_2 along a common *smooth* symplectic divisor X_{12} (i.e. a closed symplectic submanifold of real codimension 2) with dual normal bundles in the two manifolds into a symplectic manifold $X_1\#_{X_{12}}X_2$. In fact, the symplectic sum construction of [37] provides a symplectic fibration $\pi\colon \mathcal{Z} \longrightarrow \Delta$ over the unit disk $\Delta \subset \mathbb{C}$, whose central fiber $\mathcal{Z}_0$ is $X_1 \cup_{X_{12}} X_2$ and the remaining fibers are smooth symplectic manifolds which are symplectically deformation equivalent to each other; see the first diagram in Figure 2. While the behavior of GW-invariants under the basic degenerations and decompositions associated with the construction of [37] was understood long ago and has since been followed up by numerous applications throughout GW-theory, the progress beyond these cases has been slow. The interest in finding usable decomposition formulas for GW-invariants in more general situations has grown considerably since the advent of the Gross-Siebert program [41] for a (fairly) direct approach to the mirror symmetry predictions of string theory.

A sequence of J-holomorphic curves in the smooth fibers $\mathcal{Z}_\lambda = X_1\#_{X_{12}}X_2$ of a symplectic fibration $\pi\colon \mathcal{Z} \longrightarrow \Delta$ associated with the construction of [37] with $\lambda \longrightarrow 0$ converges to curves in the singular fiber $\mathcal{Z}_0 = X_1 \cup_{X_{12}} X_2$. Each of the irreducible components of a limiting curve either lies entirely in X_{12} or meets X_{12} in finitely many points (possibly none) and lies entirely in either X_1 or X_2. A key prediction in [91] concerning the behavior of the GW-invariants of $\mathcal{Z}_\lambda$ as $\lambda \longrightarrow 0$ is that they should arise only from J-holomorphic curves in $\mathcal{Z}_0$ with no irreducible components contained in X_{12} and with the irreducible components mapped into X_1 and X_2 having the same contacts with X_{12}; see Figure 3. In particular, there should be *no* direct contribution from the GW-invariants of X_{12}. The multiplicity with which such a limiting curve should contribute to the GW-invariants of $\mathcal{Z}_\lambda$ is determined in [9] based on a straightforward algebraic reason.

Notions of stable J-holomorphic maps to simple normal crossings (or SC) projective varieties of the form $X_1 \cup_{X_{12}} X_2$ and of stable maps to X_i relative to a smooth projective divisor X_{12} are intro-

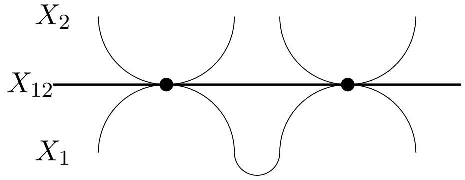


Figure 3: A connected curve in $\mathcal{Z}_0$ possibly contributing to the GW-invariants of $\mathcal{Z}_\lambda$.

duced in [54]. A degeneration formula relating the virtual cycles of the moduli spaces $\overline{\mathfrak{M}}_{g,k}(A_\lambda;J)$ with $A_\lambda\in H_2(\mathcal{Z}_\lambda)$ to the virtual cycles of the moduli spaces $\overline{\mathfrak{M}}_{g,k}(A_0;J)$ with $A_0\in H_2(\mathcal{Z}_0)$ appears in [55]. A splitting formula decomposing the latter into the virtual cycles of the moduli spaces $\overline{\mathfrak{M}}_{g_i,k_i;\mathbf{s}_i}(X_{12},A_i;J)$ of stable relative maps to (X_i,X_{12}) via a Kunneth decomposition of the diagonal

$$\Delta_{X_{12}}=\big\{(x,x)\colon x\in X_{12}\big\}$$

in X_{12}^2 is also established in [55]. The relative GW-invariants of (X_i,X_{12}) are in turn shown to reduce to the (absolute) GW-invariants of X_i and X_{12} in [61]. Thus, [54, 55, 61] fully establish the prediction of [91] in the projective category in the case of basic degenerations of the target as in Figure 2. An expository account of the symplectic topology perspective on the numerical reduction of the decomposition formula of [55] appears in [21].

The standard symplectic sum construction of [37] readily extends to the setting where the disjoint union $X_1\sqcup X_2$ is replaced by a single symplectic manifold $(\widetilde{X},\widetilde{\omega})$ and the two copies of the divisor X_{12} are replaced by a single smooth symplectic divisor $\widetilde{X}_{12}\subset\widetilde{X}$ with a symplectic involution ψ. The NC symplectic variety $\mathcal{Z}_{\psi;0}\equiv X_\psi$ is then obtained from $\widetilde{X}$ by identifying the points on $\widetilde{X}_{12}$ via ψ. This setting is discussed in Example 6.10 in the first two versions of [17]; a construction smoothing $\mathcal{Z}_{\psi;0}$ into symplectic manifolds $\mathcal{Z}_{\psi;\lambda}$ is a special case of the construction outlined in Section 7 of the first version of [18] and detailed in [20]. The reasoning behind the decomposition formulas for GW-invariants in the basic setting of the previous paragraph readily extends to provide a relation between the GW-invariants of a smoothing $\mathcal{Z}_{\psi;\lambda}$ of the NC symplectic variety X_ψ and the relative GW-invariants of $(\widetilde{X},\widetilde{X}_{12})$. The only difference in the resulting formula is that a Kunneth decomposition of the diagonal $\Delta_{X_{12}}\subset X_{12}^2$ is replaced by a Kunneth decomposition of the ψ-diagonal

$$\widetilde{\Delta}_\psi=\big\{\big(\widetilde{x},\psi(\widetilde{x})\big)\colon\widetilde{x}\in\widetilde{X}_{12}\big\};$$

the resulting sum of pairwise products of the GW-invariants of $(\widetilde{X},\widetilde{X}_{12})$ should then be divided by 2.

The decomposition formulas of [55] do not completely determine the GW-invariants of a smooth fiber $\mathcal{Z}_\lambda=X_1\#_{X_{12}}X_2$ in terms of the GW-invariants of (X_i,X_{12}) in many cases because of the so-called vanishing cycles: second homology classes in $\mathcal{Z}_\lambda$ which vanish under the projection to $\mathcal{Z}_0=X_1\cup_{X_{12}}X_2$. A refinement to the usual relative GW-invariants of (X,V) of [54] is suggested in [44] with the aim of resolving this unfortunate deficiency of the decompositions formulas of [55] in [45]. This refinement is constructed in [22] via a lifting

$$\widetilde{\mathrm{ev}}_X^V\colon\overline{\mathfrak{M}}_{g,k;\mathbf{s}}(V,A;J)\longrightarrow\widehat{V}_{X;\mathbf{s}}$$

of the relative evaluation map to a covering of $V_{\mathbf{s}} \equiv V^{\ell}$, where $\ell \in \mathbb{Z}^{\ge 0}$ is the length of the relative contact vector $\mathbf{s}$. This refinement sharpens the decomposition formulas of [55] by pulling back closed submanifolds

$$\widehat{V}^{A}_{X_1,X_2;\mathbf{s}} \subset \big(\widehat{V}_{X_1;\mathbf{s}} \times \widehat{V}_{X_2;\mathbf{s}}\big)\big|_{\Delta_V^{\ell}}\,, \tag{3.1}$$

with $V = X_{12}$ and $A \in H_2(X_1 \#_{X_{12}} X_2)$, by $\widetilde{\mathrm{ev}}^{V}_{X_1} \times \widetilde{\mathrm{ev}}^{V}_{X_2}$; see [23, Section 1.2]. However, this does not necessarily lead to a decomposition of the GW-invariants of $X_1 \cup_{X_{12}} X_2$ into the GW-invariants of (X_i, X_{12}) that completely describes the former in terms of the latter. The same approach provides a sharper version of the relation between the GW-invariants of a smoothing $\mathcal{Z}_{\psi;\lambda}$ of X_ψ and the GW-invariants of $(\widetilde{X}, \widetilde{X}_{12})$ indicated in the previous paragraph. The submanifolds (3.1) in this case are replaced by certain submanifolds

$$\widehat{V}^{A}_{\widetilde{X};\mathbf{ss}} \subset \widehat{V}_{\widetilde{X};\mathbf{ss}}\big|_{\Delta_\psi^{\ell}}\,,$$

with $V = \widetilde{X}_{12}$; the resulting relative invariants of $(\widetilde{X}, \widetilde{X}_{12})$ should then be divided by 2.

Qualitative applications of the above refinements to relative GW-invariants and to the decomposition formula of [55] are described in [22, 23]. These refinements in principle distinguish between the GW-invariants of $\mathcal{Z}_\lambda$ in degrees A_λ differing by torsion. Torsion classes can also arise from the one-parameter families of smoothings $\mathcal{Z}_{\psi;\lambda}$ of X_ψ as above. *Quantitative* computation of GW-invariants in degrees differing by torsion has been a long-standing problem.

Question 5 *Is it possible to compute GW-invariants in degrees differing by torsion in some cases via the sharper version of the decomposition formula described in [23] and/or its analogue for the degenerations of the form $\mathcal{Z}_{\psi;\lambda}$ above?*

The Enriques surface X forms an elliptic fibration over $\mathbb{P}^1$ with 12 nodal fibers and 2 double fibers; see [62, Section 1.3]. The difference $F_1 - F_2$ between the two double fibers is a 2-torsion class. A smooth genus 1 curve E has a fixed-point-free holomorphic involution ψ. The quotient

$$X_2 \equiv \big(\mathbb{P}^1 \times E\big)/\sim, \qquad (z,p) \longrightarrow \big(-z, \psi(p)\big),$$

forms an elliptic fibration over $\mathbb{P}^1$ with 2 double fibers. The blowup $\widetilde{X}$ of $\mathbb{P}^2$ at the 9-point base locus of a generic pencil of cubics is an elliptic fibration over $\mathbb{P}^1$ with 12 nodal fibers. The NC variety $\mathcal{Z}_0 \equiv X_2 \cup_E \widetilde{X}$ can be smoothed out to an Enriques surface $\mathcal{Z}_\lambda \equiv X$. The genus 1 GW-invariants of X are determined in [62] by applying the decomposition formula of [55] in this setting and using the Virasoro constraints. However, the computation in [62] does not distinguish between the map degrees differing by the torsion $F_1 - F_2$; this torsion arises from the vanishing cycles and thus is not detected by the decomposition formula of [55]. On the other hand, it may be possible to fully compute the genus 1 GW-invariants of X by refining the computation in [62] via the sharper version of this formula described in [23].

Another potential approach to a complete computation of the GW-invariants of the Enriques surface X is provided by the extension of the standard symplectic sum construction of [37] indicated above Question 5. Let ψ be a fixed-point-free holomorphic involution on a smooth fiber $F \approx E$ of $\widetilde{X} \longrightarrow \mathbb{P}^1$. The NC variety

$$\mathcal{Z}_{\psi;0} \equiv X_\psi \equiv \widetilde{X}/\sim, \qquad p \sim \psi(p) \quad \forall\, p \in \widetilde{X}_{12} \equiv F,$$

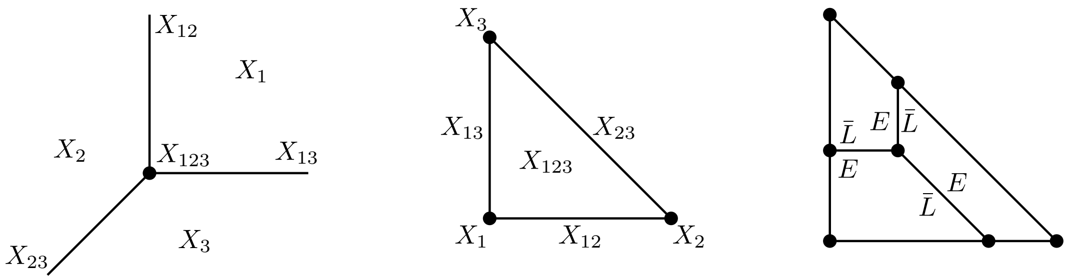


Figure 4: A 3-fold simple normal crossings variety $\mathcal{Z}_0$, its dual intersection complex, and a toric 3-fold decomposition of $\mathbb{P}^2$ into three copies of its one-point blowup $\widehat{\mathbb{P}}^2$ along the exceptional divisor $E \subset \widehat{\mathbb{P}}^2$ and the proper transform $\overline{L} \subset \widehat{\mathbb{P}}^2$ of a line $L \subset \mathbb{P}^2$.

has a $\mathbb{Z}^2$-collection of one-parameter families of smoothings $\mathcal{Z}_{\psi;\lambda}$. The total spaces of these families are $\mathbb{Z}_2$-quotients of the total families of the smoothings of $\widetilde{X} \cup_F \widetilde{X}$. The fibers $\mathcal{Z}_\lambda$ of one of the latter families are $K3$ surfaces. Thus, the fibers $\mathcal{Z}_{\psi;\lambda}$ in one of the families of smoothings of X_ψ should be Enriques surfaces (at least up to symplectic deformation equivalence). The extension of the standard degeneration formula of [55] indicated above applies to these families of smoothings and again distinguishes between the GW-invariants in degrees differing by the torsion $F_1 - F_2$.

The Gross-Siebert program [41] for a direct proof of mirror symmetry requires degeneration and splitting formulas for GW-invariants under degenerations $\pi \colon \mathcal{Z} \longrightarrow \Delta$ of algebraic varieties that are locally of the form

$$\pi \colon \big\{(\lambda, z_1, \ldots, z_k, p) \in \mathbb{C}^{k+2} \times \mathbb{C}^{n-k} \colon z_1 \ldots z_k = \lambda\big\} \longrightarrow \mathbb{C}, \quad \pi\big(\lambda, z_1, \ldots, z_k, p\big) = \lambda, \tag{3.2}$$

around the central fiber $\mathcal{Z}_0 \equiv \pi^{-1}(0)$. The degenerations discussed above, i.e. the standard one associated with the symplectic sum construction of [37] and its extension indicated in [17, 18], correspond to $k=2$ in (3.2). The central fiber of π for $k \geq 3$ in the algebro-geometric category is a more general NC variety; see Figure 4. Degeneration and splitting formulas for GW-invariants in this more general setting require notions of GW-invariants for (smoothable) NC varieties and for smooth varieties relative to NC divisors. A degeneration formula in the projective category extending that of [55] has finally appeared in the setting of the logarithmic GW-theory of [42] in [1]; the latter includes GW-invariants of smoothable NC varieties and of smooth varieties relative to NC divisors. However, a splitting formula for the GW-invariants of NC varieties in the projective category remains to be established.

The logarithmic GW-invariants of [42] are special cases of the GW-invariants of exploded manifolds introduced in [76]. Degeneration *and* splitting formulas for these invariants are studied in [77]. Based on the $k=2$ case established in [55], one might expect that all curves in $\mathcal{Z}_0$ contributing to the GW-invariants of $\mathcal{Z}_\lambda$ either

- have no irreducible components lying in the singular locus $\mathcal{Z}_0'$ of $\mathcal{Z}_0$ and meet at the smooth points of $\mathcal{Z}_0$ or at least
- have no irreducible components in $\mathcal{Z}_0'$.

As demonstrated in [77], even the weaker alternative does not hold in general. This makes any general splitting formula necessarily complicated; its $k=3$ case is described in [78]. A more geometric perspective on the GW-invariants of [76] appears in [43], without analogues of the crucial degeneration and splitting formulas.

The GW-invariants of exploded manifolds of [76] and their interpretation in some cases in [43] are essentially invariants of deformation equivalence classes of almost Kähler structures on manifolds. While these classes are much larger than the deformation equivalence classes of the algebro-geometric structures in [42, 1], GW-invariants are fundamentally invariants of the still larger deformation equivalence classes of symplectic structures. Purely topological notions of NC symplectic divisors and varieties are introduced in [17, 19], addressing a fundamental quandary of [40, p343] in the case of NC singularities. Crucial to the introduction of these long desired notions is the new perspective proposed in [17]:

> *A symplectic variety/subvariety should be viewed as a deformation equivalence class of objects with the same topology, not as a single object.*

It is then shown in [17, 19] that the spaces of NC symplectic divisors and varieties are weakly homotopy equivalent to the spaces of almost Kähler structures, as needed for geometric applications.

The equivalence between the topological and geometric notions of NC symplectic variety established in [17, 19] immediately implies that any invariants arising from [77, 43] in fact depend only on the deformation equivalence classes of symplectic structures. These equivalences are also used in [18, 20] to establish a smoothability criterion for NC symplectic varieties. Direct approaches to constructing GW-invariants of symplectic manifolds relative to NC symplectic divisors in the perspective of [17, 19] and to obtaining degeneration and splitting formulas for the degenerations appearing in [18, 20] are discussed in [14] and [15], respectively.

The decomposition and splitting formulas for GW-invariants in [77] involve exploded de Rham cohomology of [79], which makes these formulas very hard to apply. The purpose of this elaborate modification of the ordinary de Rham cohomology is to correct the standard Kunneth decompositions of the diagonals of the strata of the singular locus $\mathcal{Z}_0'$ of the central fiber $\mathcal{Z}_0$ for the presence of lower-dimensional strata. This removes certain degenerate contributions to the Kunneth decompositions of the diagonals of the strata of $\mathcal{Z}_0'$. A local, completely topological approach to computing degenerate contributions in terms of the ordinary cohomology of the strata is presented in [100].

Question 6 *Is there a reasonably usable formula for general NC degenerations $\pi\colon \mathcal{Z} \longrightarrow \Delta$ of symplectic manifolds which splits the GW-invariants of a smooth fiber $\mathcal{Z}_\lambda$ into the GW-invariants of the strata of the central fiber $\mathcal{Z}_0$ that involves only the ordinary cohomology of the strata?*

The introduction of symplectic topology notions of NC divisors, varieties, and degenerations in [17, 18, 19, 20] has made it feasible to study Question 6 entirely in the symplectic topology category, which is far more flexible than the algebraic geometry category of [42, 1] and the almost Kähler category of [76, 43]. A symplectic approach to this question should fit well with the topological approach of [100] to degenerate contributions. A splitting formula for GW-invariants of $\mathcal{Z}_\lambda$ resulting from such an approach should involve sums over finite trees with the edges labeled by integer weights and the vertices labeled by paths in the dual intersection complex of $\mathcal{Z}_0$ with additional de Rham cohomology data; these paths would correspond to the tropical curves of [77]. While such a formula would still be more complicated than in the standard case of [55], it should be more readily applicable than the presently available splitting formula of [77] that involves exploded de Rham cohomology.

Degeneration and splitting formulas for real GW-invariants under real degenerations of real symplectic manifolds have been obtained only in a small number of special cases. A fundamental difficulty for obtaining such formulas is that the standard notions of relative invariants of the complex GW-theory do not have direct analogues in the real GW-theory in most settings. Real GW-invariants of a real symplectic manifold (X,ω,ϕ) with simple contacts with a real symplectic divisor $V\subset X$ can be readily defined whenever the real GW-invariants of (X,ω,ϕ) are defined and V is disjoint from the fixed locus X^{ϕ} of ϕ. This observation lies behind the splitting formula and related vanishing result for some genus 0 real GW-invariants under special real degenerations of real symplectic manifolds obtained in [13].

The reduction of the complex relative GW-invariants of (X,V) to the complex GW-invariants of X and V in [61] suggests the possibility of expressing the real GW-invariants of a real symplectic sum $X_1\#_{X_{12}}X_2$ in terms of the real GW-invariants of X_1, X_2, X_{12}, whenever these are defined. If X_1 and X_2 are of real dimension 4, then X is a real surface and the real GW-invariants of $X_1\#_{X_{12}}X_2$ should reduce to the real GW-invariants of X_1 and X_2. By [4, Theorem 7] and [3, Theorem 1.1], this is indeed the case for the genus 0 real GW-invariants if $X_{12}\approx\mathbb{P}^1$ is a real symplectic submanifold of self-intersection 2 in $X_2=\mathbb{P}^1\times\mathbb{P}^1$ and in some other settings with $X_{12}\approx\mathbb{P}^1$. Genus 0 real GW-invariants have been defined for many real symplectic sixfolds and for all real symplectic fourfolds. This leads to the following question.

Question 7 *Is it possible to express the genus 0 real GW-invariants of a real symplectic sum $X_1\#_{X_{12}}X_2$ of real symplectic sixfolds (X_i,ω_i,ϕ_i) along a common real symplectic divisor X_{12} in terms of the genus 0 real GW-invariants X_1, X_2, X_{12}, whenever the genus 0 real GW-invariants of the sixfolds are defined?*

4 Geometric applications

Pseudoholomorphic curves were originally introduced in [39] with the aim of applications in symplectic topology. These applications have included the Symplectic Non-Squeezing Theorem [39], classification of symplectic 4-manifolds [63, 51], distinguishing diffeomorphic symplectic manifolds [85], symplectic isotopy problem [92, 88], and applications in birational algebraic geometry [48, 93]. However, many deep related problems remain open.

Rational curves, i.e. images of J-holomorphic maps from chains of spheres, play a particularly important role in algebraic geometry. A smooth algebraic manifold X is called uniruled (resp. rationally connected or RC) if there is a rational curve through every point (resp. every pair of points) in X. According to [48], a uniruled algebraic variety admits a nonzero genus 0 GW-invariant with a point insertion (i.e. a count of stable maps in a fixed homology class which pass through a point and some other constraints). This implies that the uniruled property is invariant under symplectic deformations. The RC property is known to be invariant under integrable deformations of the complex structure [48]. It is a long-standing conjecture of J. Kollár that the RC property is invariant under symplectic deformations as well. It is unknown if every RC algebraic manifold admits a nonzero genus 0 GW-invariant with two point insertions; this would immediately imply Kollár's conjecture. The dimension 3 case of this conjecture is established in [93] by combining the special cases treated in [96] with the minimal model program.

As GW-invariants are symplectic invariants, it is natural to consider the parallel situation in symplectic topology. Given the flexibility of the symplectic category, this may also provide a different approach to Kollár's conjecture. A symplectic manifold (X,ω) is called uniruled (resp. RC) if for some ω-compatible almost complex structure J there is a genus 0 connected rational J-holomorphic curve through every point (resp. every pair of points) in X. This leads to the following two pairs of questions.

Question 8 *Let J be any almost complex structure on a uniruled (resp. RC) compact symplectic manifold (X,ω). Is there a connected rational J-holomorphic curve through every point (resp. every pair of points) in X?*

Question 9 *Does every uniruled (resp. RC) compact symplectic manifold (X,ω) admit a nonzero genus 0 GW-invariant with a point insertion (resp. two point insertions)?*

The affirmative answer to each case of Question 9 would immediately imply the affirmative answer to the corresponding case of Question 8. The uniruled case of Question 9 is known only under the rigidity assumptions that X is either Kähler [48] or admits a Hamiltonian S^1-action [64]. It is not difficult to construct J-holomorphic curves in a symplectic manifold that disappear as the almost complex structure J deforms. On the other hand, regular J-holomorphic curves do not disappear under small deformations of J, while J-holomorphic curves contributing to nonzero GW-invariants survive all deformations of J. Thus, the above four questions concern the fundamental issue of the extent of flexibility in the symplectic category with implications to birational algebraic geometry.

If $u:\mathbb{P}^1\longrightarrow X$ is a J-holomorphic map into a Kahler manifold and for *some* $z\in\mathbb{P}^1$ the evaluation map

$$H^0(\mathbb{P}^1;u^*TX)\longrightarrow T_{u(z)}X, \qquad \xi\longrightarrow\xi(z), \tag{4.1}$$

is onto, then $H^1(\mathbb{P}^1;u^*TX)=0$, i.e. u is regular. This statement is key to the arguments of [48] in the algebraic setting. It in particular implies that if the rational J-holomorphic curves cover a nonempty open subset of a connected Kähler manifold, then they cover all of X. As shown in [65], the last implication can fail in the almost Kähler category. The first implication need not hold either, even if the evaluation homomorphism (4.1) is surjective for *every* $z\in\mathbb{P}^1$. However, the main results of [48] may still extend to the almost Kähler category. In particular, for the interplay between openness and closedness of various properties of complex structures exhibited in the proof of deformation invariance of the RC property for integrable complex structures in [48] to extend to a non-integrable complex structure, the vanishing of the obstruction space needs to hold only generically in a family of J-holomorphic maps covering X. This leads to a potentially even more fundamental problem in this spirit.

Question 10 *Let $\{u_\alpha:\mathbb{P}^1\longrightarrow X\}$ be a family of J-holomorphic curves on a compact symplectic manifold (X,ω) that covers X. Is a generic member of this family a regular map?*

There are still many open questions concerning the geography and topology of symplectic manifolds The multifold smoothing constructions of [18, 20] may shed light on some of these questions. Just as the (2-fold) symplectic sum construction of [37], the multifold constructions could be used to build vast classes of non-Kähler symplectic manifolds with various topological properties. They might also be useful for studying properties of symplectic manifolds of algebro-geometric flavor, in the spirit of the perspective on symplectic topology initiated in [39].

Question 11 ([16, Question 14]) *Is every compact almost Kähler manifold with a rational J-holomorphic curve of a fixed homology class through every pair of points simply connected?*

By [8, Theorem 3.5], a compact RC Kähler manifold is simply connected. As noted by J. Starr, the fundamental group of a compact almost Kähler manifold (X,ω,J) as in Question 11 is finite. The multifold sum/smoothing constructions of [18, 20] can be used to obtain symplectic manifolds that are not simply connected from simply connected ones and thus may be useful in answering Question 11 negatively. The constructions of [18, 20] may also be useful in studying this question under the stronger assumption of the existence of a nonzero GW-invariant of (X,ω) with two point insertions.

As in the complex case, it is natural to expect that a real symplectic manifold (X,ω,ϕ) which has well-defined genus 0 real GW-invariants and is covered by real rational curves admits a nonzero genus 0 real GW-invariant with a real point insertion. However, the reasoning neither in [48], which relies on the positivity of intersections in complex geometry, nor in [64], which makes use of quantum cohomology, is readily adaptable to the real setting. Thus, there is not apparent approach to this problem at the present.

Another important question in real algebraic geometry is the existence of real rational curves on real even-degree complete intersections $X\subset\mathbb{P}^n$; this would be implied by the existence of a well-defined nonzero genus 0 real GW-invariant of X. However, the real analogue of the Quantum Lefschetz Hyperplane Principle (1.2) suggests that all such invariants should vanish. On the other hand, one may hope for some real analogue of the reduced/family GW-invariants of [5, 53], which effectively remove a trivial line bundle from the obstruction cone for deformations of J-holomorphic maps to X. The resulting reduced/family real invariants could well be nonzero.

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