Real Ruan-Tian Perturbations

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Abstract

Ruan-Tian deformations of the Cauchy-Riemann operator enable a geometric definition of (standard) Gromov-Witten invariants of semi-positive symplectic manifolds in arbitrary genera. We describe an analogue of these deformations compatible with our recent construction of real Gromov-Witten invariants in arbitrary genera. Our approach avoids the need for an embedding of the universal curve into a smooth manifold and systematizes the deformation-obstruction setup behind constructions of Gromov-Witten invariants.

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1 Introduction

The introduction of J-holomorphic curves techniques into symplectic topology in [14] led to definitions of (complex) Gromov-Witten (or GW-) invariants of semi-positive symplectic manifolds in genus 0 in [18] and in arbitrary genera in [23, 24] as actual counts of simple J-holomorphic maps. Local versions of the inhomogeneous deformations of the $\bar{\partial}_J$-equation pioneered in [23, 24] were later

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used to endow the moduli space of (complex) J-holomorphic maps with a so-called virtual fundamental class (or VFC) in [16, 6] and thus to define GW-invariants for arbitrary symplectic manifolds.

A real symplectic manifold (X,ω,ϕ) is a symplectic manifold (X,ω) with a smooth involution $\phi\colon X\longrightarrow X$ such that $\phi^*\omega=-\omega$. Invariant signed counts of genus 0 real curves, i.e. those preserved by ϕ, were defined for semi-positive real symplectic 4- and 6-manifolds in [30, 31] in the general spirit of [18]. The interpretation of these counts in [29] in the general spirit of [16] removed the need for the semi-positive restriction and made them amendable to the standard computational techniques of GW-theory; see [21], for example. Building on the perspectives in [17, 29], genus 0 real GW-invariants for many other real symplectic manifolds were later defined in [8, 5]. The recent work [10, 11, 12] sets up the theory of real GW-invariants in arbitrary genera with conjugate pairs of insertions and in genus 1 with arbitrary point insertions in the general spirit of [16]. Following a referee's suggestion, we now describe these invariants in the spirit of [23, 24]; this description is more geometric and should lead more readily to a tropical perspective on these invariants that has proved very powerful in studying the genus 0 real GW-invariants of [30, 31].

A conjugation on a complex vector bundle $V\longrightarrow X$ lifting an involution ϕ on X is a vector bundle involution $\varphi\colon V\longrightarrow V$ covering ϕ such that the restriction of φ to each fiber is anti-complex linear. A real bundle pair $(V,\varphi)\longrightarrow(X,\phi)$ consists of a complex vector bundle $V\longrightarrow X$ and a conjugation φ on V lifting ϕ. For example,

$$(TX,\mathrm{d}\phi)\longrightarrow(X,\phi) \qquad \text{and} \qquad (X\times\mathbb{C}^n,\phi\times\mathfrak{c})\longrightarrow(X,\phi),$$

where $\mathfrak{c}\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n$ is the standard conjugation on $\mathbb{C}^n$, are real bundle pairs. For any real bundle pair $(V,\varphi)\longrightarrow(X,\phi)$, we denote by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)=(\Lambda^{\mathrm{top}}_{\mathbb{C}}V,\Lambda^{\mathrm{top}}_{\mathbb{C}}\varphi)$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation. A real symplectic manifold (X,ω,ϕ) is real-orientable if there exists a rank 1 real bundle pair $(L,\widetilde{\phi})$ over (X,ϕ) such that

$$w_2(TX^\phi)=w_1(L^{\widetilde{\phi}})^2 \qquad \text{and} \qquad \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)\approx(L,\widetilde{\phi})^{\otimes 2}. \tag{1.1}$$

Definition 1.1. A real orientation on a real-orientable symplectic manifold (X,ω,ϕ) consists of

(RO1) a rank 1 real bundle pair $(L,\widetilde{\phi})$ over (X,ϕ) satisfying (1.1),

(RO2) a homotopy class of isomorphisms of real bundle pairs in (1.1), and

(RO3) a spin structure on the real vector bundle $TX^\phi\oplus2(L^*)^{\widetilde{\phi}^*}$ over X^ϕ compatible with the orientation induced by (RO2).

By [10, Theorem 1.3], a real orientation on (X,ω,ϕ) orients the moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ of genus g degree B real J-holomorphic maps with l conjugate pairs of marked points whenever the "complex" dimension of X is odd. By the proof of [10, Theorem 1.5], it also orients the moduli space $\overline{\mathfrak{M}}_{1,l;k}(X,B;J)^\phi$ of genus 1 maps with k real marked points outside of certain codimension 1 strata. In general, these moduli spaces are not smooth and the above orientability statements should be viewed in the usual moduli-theoretic (or virtual) sense.

The description in [17] of versal families of deformations of symmetric Riemann surfaces provides the necessary ingredient for adapting the interpretation of Gromov's topology in [16] from the complex to the real setting and eliminates the (virtual) boundary of $\overline{\mathfrak{M}}_{g,l;k}(X,B;J)^\phi$. A Kuranishi atlas for this moduli space is then obtained by carrying out the constructions of [16, 6] in a ϕ-invariant manner; see [29, Section 7] and [7, Appendix]. If oriented, this atlas determines a VFC for $\overline{\mathfrak{M}}_{g,l;k}(X,B;J)^\phi$ and thus gives rise to genus g real GW-invariants of (X,ω,ϕ); see [10, Theorem 1.4]. If this atlas is oriented only on the complement of some codimension 1 strata, real GW-invariants can still be obtained in some special cases by adapting the principle of [4, 29] to show that the problematic strata are avoided by a generic path; [10, Theorem 1.5]. In some important situations, the real genus g GW-invariants arising from [10, Theorem 1.3] can be described as actual counts of curves in the spirit of [23, 24].

For a manifold X, denote by

$$H_2^S(X;\mathbb{Z}) \equiv \big\{u_*[S^2]\colon u\in C(S^2;X)\big\} \subset H_2(X;\mathbb{Z})$$

the subset of spherical classes. There are two topological types of anti-holomorphic involutions on $\mathbb{P}^1$; they are represented by

$$\tau,\eta\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad z \longrightarrow \frac{1}{\bar z}, -\frac{1}{\bar z}\,.$$

For a manifold X with an involution ϕ, denote by

$$H_2^\sigma(X;\mathbb{Z})^\phi \equiv \big\{u_*[S^2]\colon u\in C(S^2;X),\ u\circ\sigma=\phi\circ u\big\} \qquad \text{for } \sigma=\tau,\eta,$$
$$H_2^{\mathbb{R}S}(X;\mathbb{Z})^\phi \equiv H_2^\tau(X;\mathbb{Z})^\phi \cup H_2^\eta(X;\mathbb{Z})^\phi \subset \big\{B\in H_2(X;\mathbb{Z})\colon \phi_*B=-B\big\}$$

the subsets of σ-spherical classes and real spherical classes.

Definition 1.2. A symplectic $2n$-manifold (X,ω) is semi-positive if

$$\langle c_1(X),B\rangle \geqslant 0 \qquad \forall\ B\in H_2^S(X;\mathbb{Z}) \ \text{ s.t. } \langle\omega,B\rangle>0,\ \langle c_1(X),B\rangle\geqslant 3-n.$$

A real symplectic $2n$-manifold (X,ω,ϕ) is semi-positive if (X,ω) is semi-positive and

$$\begin{aligned}\langle c_1(X),B\rangle &\geqslant \delta_{n2} &&\forall\ B\in H_2^{\mathbb{R}S}(X;\mathbb{Z})^\phi \ \text{ s.t. } \langle\omega,B\rangle>0,\ \langle c_1(X),B\rangle\geqslant 2-n,\\ \langle c_1(X),B\rangle &\geqslant 1 &&\forall\ B\in H_2^\tau(X;\mathbb{Z})^\phi \ \text{ s.t. } \langle\omega,B\rangle>0,\ \langle c_1(X),B\rangle\geqslant 2-n,\end{aligned}$$

where $\delta_{n2}=1$ if $n=2$ and 0 otherwise.

The stronger middle bound in the $n=2$ case above rules out the appearance of real degree B J-holomorphic spheres with $\langle c_1(X),B\rangle=0$ for a generic one-parameter family of real almost complex structures on a real symplectic manifold (X,ω,ϕ) and provides for the second bound in (3.63). The latter in turn ensures that the expected dimension of the moduli space of complex degree B J-holomorphic spheres in such a family of almost complex structures is not smaller than the expected dimension of the moduli space of real degree B J-holomorphic spheres.

Monotone symplectic manifolds, including all projective spaces and Fano hypersurfaces, are semi-positive. The maps

$$\begin{aligned} \tau_n\colon \mathbb{P}^{n-1} &\longrightarrow \mathbb{P}^{n-1}, & [Z_1,\ldots,Z_n] &\longrightarrow [\overline{Z}_1,\ldots,\overline{Z}_n],\\ \eta_{2m}\colon \mathbb{P}^{2m-1} &\longrightarrow \mathbb{P}^{2m-1}, & [Z_1,Z_2,\ldots,Z_{2m-1},Z_{2m}] &\longrightarrow \big[-\overline{Z}_2,\overline{Z}_1,\ldots,-\overline{Z}_{2m},\overline{Z}_{2m-1}\big], \end{aligned}$$

are anti-symplectic involutions with respect to the standard Fubini-Study symplectic forms ω_n on $\mathbb{P}^{n-1}$ and ω_{2m} on $\mathbb{P}^{2m-1}$, respectively. If

$$k\geqslant 0, \qquad \mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k\,,$$

and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n, then $\tau_{n;\mathbf{a}}\equiv\tau_n|_{X_{n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{n;\mathbf{a}}$ with respect to the symplectic form $\omega_{n;\mathbf{a}}=\omega_n|_{X_{n;\mathbf{a}}}$. Similarly, if $X_{2m;\mathbf{a}}\subset\mathbb{P}^{2m-1}$ is preserved by η_{2m}, then $\eta_{2m;\mathbf{a}}\equiv\eta_{2m}|_{X_{2m;\mathbf{a}}}$ is an anti-symplectic involution on $X_{2m;\mathbf{a}}$ with respect to the symplectic form $\omega_{2m;\mathbf{a}}=\omega_{2m}|_{X_{2m;\mathbf{a}}}$. The projective spaces $(\mathbb{P}^{2m-1},\tau_{2m-1})$ and $(\mathbb{P}^{4m-1},\eta_{4m-1})$, as well as many real complete intersections in these spaces, are real orientable; see [10, Proposition 2.1].

We show in this paper that the semi-positive property of Definition 1.2 for (X,ω,ϕ) plays the same role in the real GW-theory as the semi-positive property for (X,ω) plays in "classical" GW-theory. For each element (J,ν) of the space (3.3), the moduli space $\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^\phi$ of stable degree B genus g real (J,ν)-maps with l conjugate pairs of marked points and k real points is coarsely stratified by the subspaces $\mathfrak{M}_\gamma(J,\nu)^\phi$ of maps of the same combinatorial type; see (3.28). By Proposition 3.6, the open subspace

$$\mathfrak{M}^*_\gamma(J,\nu)^\phi\subset\mathfrak{M}_\gamma(J,\nu)^\phi$$

consisting of simple maps in the sense of Definition 3.2 is cut out transversely by the $\{\bar\partial_J-\nu\}$-operator for a generic pair (J,ν); thus, it is smooth and of the expected dimension. The image of

$$\mathfrak{M}^{\mathrm{mc}}_\gamma(J,\nu)^\phi\equiv\mathfrak{M}_\gamma(J,\nu)^\phi-\mathfrak{M}^*_\gamma(J,\nu)^\phi$$

under the product of the stabilization st and the evaluation map ev in (3.5) is covered by smooth maps from finitely many spaces $\mathfrak{M}^*_{\gamma'}(J,\nu')^\phi$ of simple degree B' genus g' real maps with $\omega(B')<\omega(B)$ and $g'\geqslant g$. By the proof of Proposition 3.10, the dimensions of the latter spaces are at least 2 less than the virtual dimension of $\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^\phi$ if (J,ν) is generic and (X,ω,ϕ) is semi-positive.

By Theorem 3.3(1), the restriction (3.7) of (3.5) is a pseudocycle for a generic pair (J,ν) in the space (3.3) whenever (X,ω,ϕ) is a semi-positive real symplectic manifold of odd "complex" dimension with a real orientation. Intersecting this pseudocycle with constraints in the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ of real curves and in X, we obtain an interpretation of the genus g real GW-invariants provided by [10, Theorem 1.4] as counts of real (J,ν)-curves in (X,ω,ϕ) which depend only on the homology classes of the constraints. A similar conclusion applies to the genus 1 real GW-invariants with real marked points provided by [10, Theorem 1.5]; see Remark 3.4.

For the purposes of Theorem 3.3(1), the $2-n$ inequalities in Definition 1.2 could be replaced by $3-n$ (which would weaken it). This would make its restrictions vacuous if $n=2$, i.e. $\dim_{\mathbb{R}}X=4$. The $2-n$ condition ensures that the conclusion of Proposition 3.10 remains valid for a generic

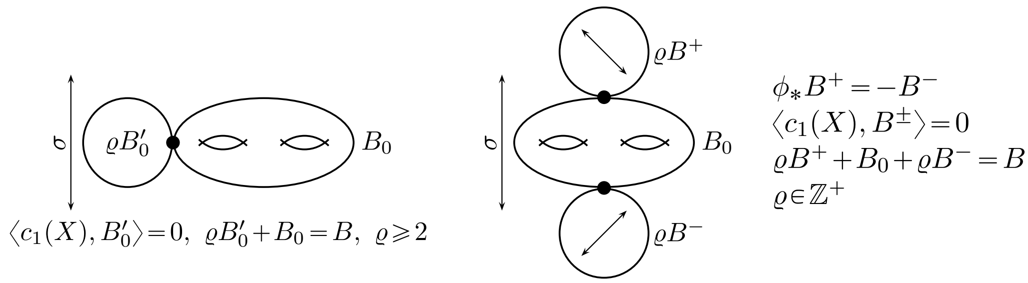

Figure 1: Typical elements of subspaces of $\mathfrak{M}^{\mathrm{mc}}_{\gamma}(\alpha)^{\phi}$ with codimension-one images under $\mathsf{st}\times\mathsf{ev}$ for a generic one-parameter family α of real Ruan-Tian deformations (J,ν) on a real symplectic 4-manifold (X,ω,ϕ). The degrees of the maps on the irreducible components of the domains are shown next to the corresponding components. The double-headed arrows labeled by σ indicate the involutions on the entire domains of the maps. The smaller double-headed arrows indicate the involutions on the real images of the corresponding irreducible components of the domain.

one-parameter family of elements (J,ν) in the space (3.3) and that the homology class determined by the pseudocycle (3.7) is independent of the choice of (J,ν); see Proposition 3.11 and the first statement of Theorem 3.3(2). For $n=2$, this condition excludes the appearance of stable maps represented by the two diagrams of Figure 1 for a generic one-parameter family of (J,ν). The maps of the first type are not regular solutions of the $(\bar\partial_J-\nu)$-equation if $\varrho\geqslant 2$. The maps of the second type are not regular solutions of the $(\bar\partial_J-\nu)$-equation if the images of the top and bottom irreducible components are the same (i.e. each of them is preserved by ϕ) and $\varrho\in\mathbb{Z}^+$. If maps of either type exist, their images under $\mathsf{st}\times\mathsf{ev}$ form a subspace of real codimension 1 in the image of (3.5).

Remark 1.3. Real symplectic 4-manifolds (X,ω,ϕ) with classes $B\in H_2^{\mathbb{R}S}(X;\mathbb{Z})^{\phi}$ such that $\langle\omega,B\rangle>0$ and $\langle c_1(X),B\rangle=0$ are not excluded from the constructions of genus 0 real GW-invariants in [30, 29]. However, the geometric proofs of the invariance of the curve counts defined in these papers neglect to consider stable maps as in Figure 1. The second Hirzebruch surface $\mathbb{F}_2\longrightarrow\mathbb{P}^1$ contains two natural section classes, C_2 and E, with normal bundles of degrees 2 and -2, respectively. Along with the fiber class F, either of them generates $H_2(\mathbb{F}_2;\mathbb{Z})$. There are algebraic families $p\colon\mathcal{C}\longrightarrow S$ and $\pi\colon\mathfrak{X}\longrightarrow S$, where S is a neighborhood of $0\in\mathbb{C}$, such that

$$p^{-1}(0)=\mathbb{P}^1\vee\mathbb{P}^1,\qquad \pi^{-1}(0)=\mathbb{F}_2,\qquad p^{-1}(z)=\mathbb{P}^1,\quad \pi^{-1}(z)=\mathbb{P}^1\times\mathbb{P}^1\quad \forall\ z\in S-\{0\}.$$

The projection π can be viewed as an algebraic family of algebraic structures on $\mathbb{F}_2$. By [1, Proposition 3.2.1], a morphism f from $p^{-1}(0)$ of degree $D=aC_2+bF$, with $a\in\mathbb{Z}^+$ and $b\in\mathbb{Z}^{\geqslant 0}$, to $\pi^{-1}(0)$ that passes through $4a+2b-1$ general points in $\mathbb{F}_2$ and extends to a morphism $\widehat{f}\colon\mathcal{C}\longrightarrow\mathfrak{X}$ restricts to an isomorphism from a component of $p^{-1}(0)$ to E. The end of the proof of [32, Proposition 2.9] cites [1] as establishing this conclusion for a generic one-parameter family of real almost complex structures J on blowups of $(\mathbb{F}_2,E)$ away from E. This is used to claim that multiply covered disk bubbles of Maslov index 0 do not appear in a one-parameter family of almost complex structures in the proof of [32, Theorem 0.1] and that maps as in the first diagram of Figure 1 do not appear in the proof of [30, Theorem 0.1]; see [32, Remark 2.12]. The potential appearance of maps as in the second diagram of Figure 1 is not even discussed in any geometric argument we are aware of. On the other hand, these maps create no difficulties in the virtual class approach of [29, Section 7].

Section 2 sets up the relevant notation for the moduli spaces of complex and real curves and for their covers. Section 3.1 introduces a real version of the perturbations of [24] and concludes with the main theorem. The strata $\mathfrak{M}_\gamma(J,\nu)^\phi$ splitting the moduli space $\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^\phi$ based on the combinatorial type of the map are described in Section 3.2. As summarized in Section 3.3, the subspaces $\mathfrak{M}^*_\gamma(J,\nu)^\phi$ of these strata consisting of simple maps are smooth manifolds. We use the regularity statements of this section, Propositions 3.6 and 3.7, to establish the main theorem in Section 3.5. The two propositions are proved in Sections 4.1 and 4.2. The first of these sections introduces suitable deformation-obstruction settings and then shows that the deformations of real Ruan-Tian pairs (J,ν) suffice to cover the obstruction space in all relevant cases; see Lemmas 4.1 and 4.2. By Section 4.2, Lemmas 4.1 and 4.2 ensure the smoothness of the universal moduli space of simple (J,ν)-maps from a domain of each topological type; see Theorem 4.3. As is well-known, the latter implies the smoothness of the corresponding stratum of the moduli space of simple (J,ν)-maps for a generic pair (J,ν) and thus concludes the proof of Proposition 3.6. In the process of establishing Theorem 3.3, we systematize and streamline the constructions of GW-pseudocycles in [19, 24].

The author would like to thank P. Georgieva and J. Starr for enlightening discussions on the Deligne-Mumford moduli of curves.

2 Terminology and notation

Ruan-Tian's deformations ν are obtained by passing to a regular cover of the Deligne-Mumford space $\overline{\mathcal{M}}_{g,l}$ of stable genus g complex curves with l marked points. After recalling such covers in Section 2.1, we describe their analogues suitable for real GW-theory.

2.1 Moduli spaces of complex curves

For $l\in\mathbb{Z}^{\geqslant 0}$, let

$$[l]\equiv\{i\in\mathbb{Z}^+ : i\leqslant l\}.$$

For $g\in\mathbb{Z}^{\geqslant 0}$, we denote by $\mathcal{D}_g$ the group of diffeomorphisms of a smooth compact connected orientable genus g surface Σ and by $\mathcal{J}_g$ the space of complex structures on Σ. If in addition $l\in\mathbb{Z}^{\geqslant 0}$ and $2g+l\geqslant 3$, let

$$\mathcal{M}_{g,l}\subset\overline{\mathcal{M}}_{g,l}$$

be the open subspace of smooth curves in the Deligne-Mumford moduli space of genus g complex curves with l marked points and define

$$\mathcal{J}_{g,l}=\{(\mathfrak{j},z_1,\ldots,z_l)\in\mathcal{J}_g\times\Sigma^l : z_i\neq z_j\ \forall\, i\neq j\}.$$

The group $\mathcal{D}_g$ acts on $\mathcal{J}_{g,l}$ by

$$h\cdot(\mathfrak{j},z_1,\ldots,z_l)=(h^*\mathfrak{j},h^{-1}(z_1),\ldots,h^{-1}(z_l)).$$

Denote by $\mathscr{T}_{g,l}$ the Teichmüller space of Σ with l punctures and by $\mathscr{G}_{g,l}$ the corresponding mapping class group. Thus,

$$\mathcal{M}_{g,l}=\mathcal{J}_{g,l}/\mathcal{D}_g=\mathscr{T}_{g,l}/\mathscr{G}_{g,l}\,. \tag{2.1}$$

Let

$$\mathfrak{f}_{g,l}:\overline{\mathcal{U}}_{g,l}=\overline{\mathcal{M}}_{g,l+1}\longrightarrow\overline{\mathcal{M}}_{g,l} \tag{2.2}$$

be the forgetful morphism dropping the last marked point; it determines the universal family over $\overline{\mathcal{M}}_{g,l}$.

For a tuple $\mathscr{D}\equiv(g_1,S_1;g_2,S_2)$ consisting of $g_1,g_2\in\mathbb{Z}^{\geqslant 0}$ with $g=g_1+g_2$ and $S_1,S_2\subset[l]$ with $[l]=S_1\sqcup S_2$, denote by

$$\overline{\mathcal{M}}_{\mathscr{D}}\subset\overline{\mathcal{M}}_{g,l}$$

the closure of the subspace of marked curves with two irreducible components Σ_1 and Σ_2 of genera g_1 and g_2, respectively, and carrying the marked points indexed by S_1 and S_2, respectively. Let

$$\iota_{\mathscr{D}}:\overline{\mathcal{M}}_{g_1,|S_1|+1}\times\overline{\mathcal{M}}_{g_2,|S_2|+1}\longrightarrow\overline{\mathcal{M}}_{g,l}$$

be the natural node-identifying immersion with image $\overline{\mathcal{M}}_{\mathscr{D}}$ (it sends the first $|S_i|$ marked points of the i-th factor to the marked points indexed by S_i in the order-preserving fashion). We denote by $\mathrm{Div}_{g,l}$ the set of tuples $\mathscr{D}$ above.

For each involution σ on the set $[l]$, define

$$\widetilde{\sigma}:\big[l+1\big]\longrightarrow\big[l+1\big],\qquad \widetilde{\sigma}(i)=\begin{cases}\sigma(i), & \text{if } i\in[l];\\ i, & \text{if } i=l+1;\end{cases}$$
$$\sigma_g:\mathcal{J}_{g,l}\longrightarrow\mathcal{J}_{g,l},\qquad \sigma_g\big(\mathfrak{j},z_1,\ldots,z_l\big)=\big(-\mathfrak{j},z_{\sigma(1)},\ldots,z_{\sigma(l)}\big).$$

Since the last involution commutes with the action of $\mathcal{D}_g$, it descends to an involution on the quotient (2.1). The latter extends to an involution

$$\sigma_g:\overline{\mathcal{M}}_{g,l}\longrightarrow\overline{\mathcal{M}}_{g,l}\qquad\text{s.t.}\quad \sigma_g\circ\mathfrak{f}_{g,l}=\mathfrak{f}_{g,l}\circ\widetilde{\sigma}_g\,. \tag{2.3}$$

A genus g complex curve $\mathcal{C}$ is cut out by polynomial equations in some $\mathbb{P}^{N-1}$ (N can be taken to be the same for all elements of $\overline{\mathcal{M}}_{g,l}$). The standard involution τ_N on $\mathbb{P}^{N-1}$ sends $\mathcal{C}$ to another genus g curve $\overline{\mathcal{C}}$. If $\mathcal{C}$ is smooth, τ_N identifies $\mathcal{C}$ and $\overline{\mathcal{C}}$ as smooth surfaces reversing the complex structure. The conjugation τ_N thus induces the involution (2.3).

Since $\mathscr{T}_{g,l}$ is simply connected, the involution σ_g on $\overline{\mathcal{M}}_{g,l}$ lifts to a $\mathscr{G}_{g,l}$-equivariant involution

$$\sigma_g:\mathscr{T}_{g,l}\longrightarrow\mathscr{T}_{g,l}\,. \tag{2.4}$$

Such a lift can be described as follows. Let $\Sigma_{g,l}$ be a smooth compact connected oriented genus g surface Σ with l distinct marked points $z_1,\ldots,z_l$ and $\mathcal{D}_{g,l}\subset\mathcal{D}_g$ be the subgroup of diffeomorphisms of $\Sigma_{g,l}$ isotopic to the identity (and preserving the marked points). Choose an orientation-reversing involution $\sigma_{g,l}$ on $\Sigma_{g,l}$ that restricts to σ on the marked points. An element of $\mathscr{T}_{g,l}$ is the $\mathcal{D}_{g,l}$-orbit $[\mathfrak{j}]$ of an element $\mathfrak{j}\in\mathcal{J}_g$ compatible with the orientation of $\Sigma_{g,l}$. A lift as in (2.4) can be obtained by defining

$$\sigma_g:\mathscr{T}_{g,l}\longrightarrow\mathscr{T}_{g,l},\qquad [\mathfrak{j}]\longrightarrow\big[-\sigma_{g,l}^*\mathfrak{j}\big]\,.$$

This description is standard in the analytic perspective on the moduli spaces of curves; see [26, Section 2], for example.

Definition 2.1. Let $g, l \in \mathbb{Z}^{\geqslant 0}$ with $2g+l \geqslant 3$ and

$$p\colon \widetilde{\mathcal{M}}_{g,l} \longrightarrow \overline{\mathcal{M}}_{g,l} \tag{2.5}$$

be a finite branched cover in the orbifold category. A universal curve over $\widetilde{\mathcal{M}}_{g,l}$ is a tuple

$$\big(\pi : \widetilde{\mathcal{U}}_{g,l} \longrightarrow \widetilde{\mathcal{M}}_{g,l}, s_1, \ldots, s_l\big),$$

where $\widetilde{\mathcal{U}}_{g,l}$ is a projective variety and π is a projective morphism with disjoint sections $s_1, \ldots, s_l$, such that for each $\widetilde{\mathcal{C}} \in \widetilde{\mathcal{M}}_{g,l}$ the tuple $(\pi^{-1}(\widetilde{\mathcal{C}}), s_1(\widetilde{\mathcal{C}}), \ldots, s_l(\widetilde{\mathcal{C}}))$ is a stable genus g curve with l marked points whose equivalence class is $p(\widetilde{\mathcal{C}})$.

Definition 2.2. Let $g, l \in \mathbb{Z}^{\geqslant 0}$ with $2g+l \geqslant 3$. A cover (2.5) is regular if

- it admits a universal curve,
- each topological component of $p^{-1}(\mathcal{M}_{g,l})$ is the quotient of $\mathscr{T}_{g,l}$ by a subgroup of $\mathscr{G}_{g,l}$, and
- for every element $\mathscr{D} \equiv (g_1, S_1; g_2, S_2)$ of $\mathrm{Div}_{g,l}$,

  $$\big(\overline{\mathcal{M}}_{g_1,|S_1|+1} \times \overline{\mathcal{M}}_{g_2,|S_2|+1}\big) \times_{(\iota_{\mathscr{D}},p)} \widetilde{\mathcal{M}}_{g,l} \approx \widetilde{\mathcal{M}}_{g_1,|S_1|+1} \times \widetilde{\mathcal{M}}_{g_2,|S_2|+1}$$

  for some covers $\widetilde{\mathcal{M}}_{g_i,|S_i|+1}$ of $\overline{\mathcal{M}}_{g_i,|S_i|+1}$.

The moduli space $\overline{\mathcal{M}}_{0,l}$ is smooth and the universal family over it satisfies the requirement of Definition 2.1. For $g \geqslant 2$, [2, Theorems 2.2,3.9] provide covers (2.5) satisfying the last two requirements of Definition 2.2 so that the orbifold fiber product

$$\pi : \widetilde{\mathcal{U}}_{g,l} \equiv \widetilde{\mathcal{M}}_{g,l} \otimes_{\overline{\mathcal{M}}_{g,l}} \overline{\mathcal{U}}_{g,l} \longrightarrow \widetilde{\mathcal{M}}_{g,l} \tag{2.6}$$

satisfies the requirement of Definition 2.1; see also [22, Section 2.2]. The same reasoning applies in the $g=1$ case if $l \geqslant 1$.

Lemma 2.3. *If (2.5) satisfies the second condition in Definition 2.2 and σ is an involution on $[l]$, then the involutions σ_g on $\overline{\mathcal{M}}_{g,l}$ and $\widetilde{\sigma}_g$ on $\overline{\mathcal{U}}_{g,l}$ lift to involutions*

$$\sigma_g : \widetilde{\mathcal{M}}_{g,l} \longrightarrow \widetilde{\mathcal{M}}_{g,l}, \quad \widetilde{\sigma}_g : \widetilde{\mathcal{U}}_{g,l} \longrightarrow \widetilde{\mathcal{U}}_{g,l} \qquad s.t. \;\; \sigma_g \circ \pi = \pi \circ \widetilde{\sigma}_g \,. \tag{2.7}$$

Proof. If (2.5) satisfies the second condition in Definition 2.2, then the involution (2.4) descends to an involution on $p^{-1}(\mathcal{M}_{g,l})$. Since every point $[\mathcal{C}] \in \overline{\mathcal{M}}_{g,l}$ has an arbitrary small neighborhood $U_{\mathcal{C}}$ such that $U_{\mathcal{C}} \cap \mathcal{M}_{g,l}$ is connected and dense in $U_{\mathcal{C}}$, the last involution extends to an involution σ_g as in (2.7). By the identity in (2.3), the involution $\widetilde{\sigma}_g$ on $\overline{\mathcal{U}}_{g,l}$ lifts to an involution $\widetilde{\sigma}_g$ as in (2.7) over the projection $\widetilde{\mathcal{U}}_{g,l} \longrightarrow \overline{\mathcal{U}}_{g,l}$ so that the identity in (2.7) holds. □

2.2 Moduli spaces of real curves

A symmetric surface (Σ,σ) is a nodal compact connected orientable surface Σ (manifold of real dimension 2 with distinct pairs of points identified) with an orientation-reversing involution σ. If Σ is smooth, then the fixed locus Σ^σ of σ is a disjoint union of circles. There are $\left\lfloor\frac{3g+4}{2}\right\rfloor$ different topological types of orientation-reversing involutions σ on a smooth surface Σ; see [20, Corollary 1.1]. We denote the set of these types by $\mathscr{I}_g^-$.

For an orientation-reversing involution σ on a smooth compact connected orientable genus g surface Σ, let

$$\mathcal{D}_g^\sigma=\big\{h\in\mathcal{D}_g\colon h\circ\sigma=\sigma\circ h\big\},\qquad \mathcal{J}_g^\sigma=\big\{\mathfrak{j}\in\mathcal{J}_g\colon \sigma^*\mathfrak{j}=-\mathfrak{j}\big\}.$$

If in addition $l,k\in\mathbb{Z}^{\geqslant0}$, define

$$\mathcal{J}_{g,l;k}^\sigma=\big\{\big(\mathfrak{j},(z_i^+,z_i^-)_{i\in[l]},(z_i)_{i\in[k]}\big)\in\mathcal{J}_{g,2l+k}\colon \mathfrak{j}\in\mathcal{J}_g^\sigma,\ \sigma(z_i^\pm)=z_i^\mp\ \forall i\in[l],\ \sigma(z_i)=z_i\ \forall i\in[k]\big\}.$$

An element of $\mathcal{J}_{g,l;k}^\sigma$ is a smooth real curve of genus g with l conjugate pairs of marked points and k real marked points.

The action of $\mathcal{D}_g$ on $\mathcal{J}_{g,2l+k}$ restricts to an action of $\mathcal{D}_g^\sigma$ on $\mathcal{J}_{g,l;k}^\sigma$. Let

$$\mathcal{M}_{g,l;k}^\sigma\equiv\mathcal{J}_{g,l;k}^\sigma/\mathcal{D}_g^\sigma\,.$$

If $2(g+l)+k\geqslant3$, the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l;k}$ of real genus g curves with l conjugate pairs of marked points and k real marked points is a compactification of

$$\mathbb{R}\mathcal{M}_{g,l;k}\equiv\bigsqcup_{\sigma\in\mathscr{I}_g^-}\mathcal{M}_{g,l;k}^\sigma$$

with strata of equivalence classes of stable nodal real curves of genus g with l conjugate pairs of marked points and k real marked points. This moduli space is topologized via versal deformations of real curves as described in [17, Section 3.2].

Fix $g,l\in\mathbb{Z}^{\geqslant0}$ and define

$$\sigma\colon\big[2l+k\big]\longrightarrow\big[2l+k\big],\qquad \sigma(i)=\begin{cases}i+1, & \text{if } i\leqslant2l,\ i\notin2\mathbb{Z};\\ i-1, & \text{if } i\leqslant2l,\ i\in2\mathbb{Z};\\ i, & \text{if } i>2l.\end{cases}$$

There is a natural morphism

$$\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\longrightarrow\overline{\mathcal{M}}_{g,2l+k}\,. \tag{2.8}$$

A genus g symmetric surface (Σ,σ_Σ) is cut out by real polynomial equations in some $\mathbb{P}^{N-1}$ so that $\sigma_\Sigma=\tau_N|_\Sigma$. The morphism (2.8) sends the equivalence class of (Σ,σ_Σ) to the equivalence class of Σ. The image of (2.8) is contained in the fixed locus $\overline{\mathcal{M}}_{g,2l+k}^{\sigma_g}$ of the involution σ_g. For $g=0$, (2.8) is an isomorphism onto $\overline{\mathcal{M}}_{g,2l+k}^{\sigma_g}$. In general, (2.8) is neither injective nor surjective onto $\overline{\mathcal{M}}_{g,2l+k}^{\sigma_g}$; see [25, Section 6.2].

Let p be as in (2.5) with l replaced by $2l+k$. Define

$$p_{\mathbb{R}}: \mathbb{R}\widetilde{\mathcal{M}}_{g,l;k} \equiv \mathbb{R}\overline{\mathcal{M}}_{g,l;k} \times_{\overline{\mathcal{M}}_{g,2l+k}} \widetilde{\mathcal{M}}_{g,2l+k} \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l;k}, \tag{2.9}$$

$$\pi_{\mathbb{R}}: \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k} \equiv \mathbb{R}\widetilde{\mathcal{M}}_{g,l;k} \times_{\widetilde{\mathcal{M}}_{g,2l+k}} \widetilde{\mathcal{U}}_{g,2l+k} \longrightarrow \mathbb{R}\widetilde{\mathcal{M}}_{g,l;k} \tag{2.10}$$

be the orbifold fiber products of the morphisms (2.8) and p and of the projection to the second component in (2.9) and (2.6), respectively. Suppose in addition that (2.5) satisfies the second condition in Definition 2.2. Since the image of (2.8) is contained in $\overline{\mathcal{M}}_{g,2l+k}^{\sigma_g}$, an involution $\widetilde{\sigma}_g$ on $\widetilde{\mathcal{U}}_{g,2l+k}$ provided by Lemma 2.3 then lifts to an involution

$$\widetilde{\sigma}_{\mathbb{R}}: \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k} \longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k} \tag{2.11}$$

which preserves the fibers of $\pi_{\mathbb{R}}$.

3 Real Ruan-Tian pseudocycles

Building on the approach in [8, Section 2.1] from the $g=0$ real setting case, we introduce a real analogue of the geometric perturbations of [24] in Section 3.1. Theorem 3.3 provides an interpretation of the arbitrary-genus real GW-invariants of [10, Theorem 1.4] for semi-positive targets in the style of [24]. A similar interpretation of the genus 1 real GW-invariants of [10, Theorem 1.5] is obtained by combining its proof with the portions of the proof of Theorem 3.3 not specific to the $k=0$ case; see Remark 3.4.

The covers (2.5) of the Deligne-Mumford moduli spaces of curves provided by [2] are branched over the boundaries of the moduli spaces. The total spaces of the universal curves (2.6) over these covers thus have singularities around the nodal points of the fibers of the from

$$\big\{(t,x,y)\in\mathbb{C}^3: xy=t^m\big\} \longrightarrow \mathbb{C}, \qquad (t,x,y)\longrightarrow t;$$

see the proof of [2, Proposition 1.4]. The approach of [24, Section 2] to deal with these singularities is to embed the universal curve (2.6) into some $\mathbb{P}^N$. Standard, though delicate, algebro-geometric arguments provide an embedding of the real universal curve (2.10) into $\mathbb{P}^N$ suitable for carrying out the approach of [24] in the relevant real settings. Following [16], we bypass such an embedding by using perturbations supported away from the nodes.

For a symplectic manifold (X,ω), we denote by $\mathcal{J}_\omega$ the space of ω-compatible almost complex structures on X. If (X,ω,ϕ) is a real symplectic manifold, let

$$\mathcal{J}_\omega^\phi = \big\{J\in\mathcal{J}_\omega:\ \phi^*J=-J\big\}. \tag{3.1}$$

For an almost complex structure J on a smooth manifold X, a complex structure $\mathfrak{j}$ on a nodal surface Σ, and a smooth map $u\colon\Sigma\longrightarrow X$, let

$$\bar\partial_{J,\mathfrak{j}}u = \frac{1}{2}\big(\mathrm{d}u + J\circ\mathrm{d}u\circ\mathfrak{j}\big): (T\Sigma,-\mathfrak{j}) \longrightarrow u^*(TX,J)\,.$$

Such a map is called J-holomorphic if $\bar\partial_{J,\mathfrak{j}}u=0$. If Σ is a smooth connected orientable surface, a C^1-map $u\colon\Sigma\longrightarrow X$ is

- somewhere injective if there exists $z\in\Sigma$ such that $u^{-1}(u(z))=\{z\}$ and $\mathrm{d}_z u\neq 0$,
- multiply covered if $u=u'\circ h$ for some smooth connected orientable surface Σ', branched cover $h\colon\Sigma\longrightarrow\Sigma'$ of degree different from ±1, and a smooth map $u'\colon\Sigma'\longrightarrow X$,
- simple if it is not multiply covered.

By [19, Proposition 2.5.1], a somewhere injective J-holomorphic map is simple. For an involution ϕ on X and an involution σ on Σ, a real map

$$u\colon(\Sigma,\sigma)\longrightarrow(X,\phi)$$

is a map $u\colon\Sigma\longrightarrow X$ such that $u\circ\sigma=\phi\circ u$.

The pseudocycle constructions in [19, 24] are based on showing that

(1) the open subspace $\mathfrak{M}^*_\gamma(J)$ of each stratum $\mathfrak{M}_\gamma(J)$ of the moduli space of J-holomorphic maps to X consisting of simple maps in the sense of Definition 3.2 is smooth for a generic choice of $J\in\mathcal{J}_\omega$ (of a compatible pair (J,ν) in [24]),

(2) the image of $\mathfrak{M}_\gamma(J)-\mathfrak{M}^*_\gamma(J)$ under the natural evaluation map ev (and the stabilization map st in [24]) is covered by smooth maps ev (and $\mathsf{st}_{\gamma',\varpi}$) from some other smooth spaces $\mathcal{Z}^*_{\gamma',\varpi}(J)$ of dimension at least 2 less than the dimension of the top stratum of the moduli space.

Our proof of Theorem 3.3 provides a systematic perspective on the reasoning in [19, 24]. We specify all spaces and maps relevant to (2) above. The regularity of these spaces, which include the spaces in (1) as special cases, is the subject of Propositions 3.6 and 3.7; they are proved in Section 4.

3.1 Main statement

Let $g,l,k\in\mathbb{Z}^{\geqslant0}$ with $2(g+l)+k\geqslant3$ and (2.5) be a regular cover. This implies that the fibers of (2.10) are stable genus g real curves with l conjugate pairs of marked points and k real marked points. We denote by

$$\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$$

the complement of the nodes of the fibers of $\pi_{\mathbb{R}}$ and by

$$\mathcal{T}_{g,l;k}=\ker\mathrm{d}\big(\pi_{\mathbb{R}}|_{\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}}\big)\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}$$

the vertical tangent bundle. The latter is a complex line bundle; let $\mathfrak{j}_{\mathcal{U}}$ denote its complex structure. The action of the differential of (2.11) reverses $\mathfrak{j}_{\mathcal{U}}$.

Let (X,J,ϕ) be an almost complex manifold with an involution ϕ reversing the almost complex structure J. Denote by

$$\pi_1,\pi_2\colon\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}\times X\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k},X$$

the projection maps. Let

$$\Gamma^{0,1}_{g,l;k}(X;J)=\big\{\nu\in\Gamma\big(\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}\times X;\pi_1^*(\mathcal{T}_{g,l;k},-\mathfrak{j}_{\mathcal{U}})^*\otimes_{\mathbb{C}}\pi_2^*(TX,J)\big)\colon\mathrm{supp}(\nu)\subset\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}\times X\big\},$$

where $\mathrm{supp}(\nu)$ is the closure of the set

$$\big\{(z,x)\in\mathbb{R}\widetilde{\mathcal{U}}^*_{g,l;k}\times X:\,\nu(z,x)\neq 0\big\}$$

in $\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}\times X$. Define

$$\Gamma^{0,1}_{g,l;k}(X;J)^{\phi}=\big\{\nu\in\Gamma^{0,1}_{g,l;k}(X;J)\colon \mathrm{d}\phi\circ\nu=\nu\circ\mathrm{d}\widetilde{\sigma}_{\mathbb{R}}\big\},\tag{3.2}$$

$$\mathcal{H}^{\omega,\phi}_{g,l;k}(X)=\big\{(J,\nu)\colon J\in\mathcal{J}^{\phi}_{\omega},\ \nu\in\Gamma^{0,1}_{g,l;k}(X;J)^{\phi}\big\}.\tag{3.3}$$

Definition 3.1. Suppose $g,l,k\in\mathbb{Z}^{\geqslant 0}$ with $2(g+l)+k\geqslant 3$, (X,J,ϕ) is an almost complex manifold with an involution ϕ reversing J, $\nu\in\Gamma^{0,1}_{g,l;k}(X;J)^{\phi}$, and $B\in H_2(X;\mathbb{Z})$. A degree B genus g real (l,k)-marked (J,ν)-map is a tuple

$$\mathbf{u}\equiv\big(u_{\mathcal{M}}\colon\Sigma\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k},u\colon\Sigma\longrightarrow X,(z_i^+,z_i^-)_{i\in[l]},(x_i)_{i\in[k]},\sigma,\mathfrak{j}\big),\tag{3.4}$$

where $\big(\Sigma,(z_i^+,z_i^-)_{i\in[l]},(x_i)_{i\in[k]},\sigma,\mathfrak{j}\big)$ is a nodal real genus g curve with l conjugate pairs of points and k real points, $u_{\mathcal{M}}$ is a $(\widetilde{\sigma}_{\mathbb{R}},\sigma)$-real $(\mathfrak{j}_{\mathcal{U}},\mathfrak{j})$-holomorphic map onto a fiber of $\pi_{\mathbb{R}}$ preserving the marked points, and u is a (ϕ,σ)-real map such that

$$\bar\partial_{J,\mathfrak{j}}u\big|_z=\nu\big(u_{\mathcal{M}}(z),u(z)\big)\circ\mathrm{d}_zu_{\mathcal{M}}\in\big(T_z\Sigma,-\mathfrak{j}\big)^*\otimes_{\mathbb{C}}(T_{u(z)}X,J)\quad\forall\,z\in\Sigma,\quad u_*[\Sigma]=B\in H_2(X;\mathbb{Z}).$$

Definition 3.2. Suppose g,l,k, (X,J,ϕ), ν, and B are as in Definition 3.1. A (J,ν)-map $\mathbf{u}$ as in (3.4) is simple if the restriction of u to each irreducible component Σ' of Σ contracted by $u_{\mathcal{M}}$ is simple whenever $u|_{\Sigma'}$ is not constant and the images of any two such components Σ' under u are distinct.

Following the standard terminology, we call the irreducible components contracted by $u_{\mathcal{M}}$ the contracted components of (3.4). Every such component Σ' is smooth and of genus 0; the restriction of u to Σ' is J-holomorphic. The last condition in Definition 3.2 implies that the image curve $u(\Sigma')\subset X$ is not real if $\mathbf{u}$ is a simple map, $u|_{\Sigma'}$ is not constant, and $\Sigma'\subset\Sigma$ is not a real component. In particular, the maps represented by the second diagram in Figure 1 are not simple.

A (J,ν)-map $\mathbf{u}$ as in (3.4) is equivalent to another (J,ν)-map

$$\mathbf{u}\equiv\big(u'_{\mathcal{M}}\colon\Sigma'\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k},u'\colon\Sigma'\longrightarrow X,(z_i'^+,z_i'^-)_{i\in[l]},(x'_i)_{i\in[k]},\sigma',\mathfrak{j}'\big)$$

if there exists a biholomorphic map $h\colon(\Sigma,\mathfrak{j})\longrightarrow(\Sigma',\mathfrak{j}')$ such that

$$h\circ\sigma=\sigma'\circ h,\quad h(z_i^+)=z_i'^+\quad\forall i\in[l],\quad h(x_i)=x'_i\quad\forall i\in[k],\quad(u_{\mathcal{M}},u)=\big(u'_{\mathcal{M}}\circ h,u'\circ h\big).$$

A (J,ν)-map $\mathbf{u}$ is stable if its group of automorphisms is finite. This is the case if and only if the degree of the restriction of u to every contracted component Σ' of Σ containing only one or two special (nodal or marked) points is not zero.

For $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$, we denote by $\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}$ the moduli space of equivalences classes of stable degree B genus g real (l,k)-marked (J,ν)-maps. It is topologized as in [16, Section 3] using maps from families of real curves described in [17, Section 3]. The map

$$\begin{aligned}\mathsf{st}\times\mathsf{ev}\colon\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}&\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\times\big(X^l\times(X^{\phi})^k\big),\\ \big[u_{\mathcal{M}},u,(z_i^+,z_i^-)_{i\in[l]},(x_i)_{i\in[k]},\sigma,\mathfrak{j}\big]&\longrightarrow\big(p_{\mathbb{R}}(\pi_{\mathbb{R}}(\mathrm{Im}\,u_{\mathcal{M}})),(u(z_i^+))_{i\in[l]},(u(x_i))_{i\in[k]}\big),\end{aligned}\tag{3.5}$$

is continuous with respect to this topology. Let

$$\mathfrak{M}^{\star}_{g,l;k}(X,B;J,\nu)^{\phi} \subset \overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi} \tag{3.6}$$

be the subspace of simple maps from domains with at most one node.

Theorem 3.3. *Suppose $n\notin 2\mathbb{Z}$, (X,ω,ϕ) is a compact semi-positive real symplectic $2n$-manifold endowed with a real orientation, $g,l\in\mathbb{Z}^{\geqslant 0}$ with $g+l\geqslant 2$, and $B\in H_2(X;\mathbb{Z})$.*

(1) There exists a Baire subset $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)\subset\mathcal{H}^{\omega,\phi}_{g,l;0}(X)$ of second category such that the restriction

$$\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi} \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l \tag{3.7}$$

is a pseudocycle of dimension

$$\dim_{\mathbb{R}} \mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi} = \big\langle c_1(TX),B\big\rangle+(n-3)(1-g)+2l \tag{3.8}$$

for every $(J,\nu)\in\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)$.

(2) The homology class on $\mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l$ determined by this pseudocycle is independent of the choice of (J,ν) in $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)$. The class

$$\frac{1}{\deg p}\Big[\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi} \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l\Big] \in H_*\Big(\mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l;\mathbb{Q}\Big) \tag{3.9}$$

is also independent of the choice of a regular cover (2.5).

The same conclusions apply with the $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ factor dropped everywhere. Identical notions of pseudocycle with target in a manifold M appear in [33, Section 1.1] and in [19, Definition 6.5.1]. They readily extend to orbifold targets M and include the more elaborate and less convenient notion of pseudo-manifold of [24, Definition 4.1]. By [33, Theorem 1.1], the group of pseudocycles into a manifold M modulo equivalence is naturally isomorphic to $H_*(M;\mathbb{Z})$. The same reasoning applies to orbifold targets.

The proof of Theorem 3.3 in the rest of the paper follows the same general principles as the proofs of [19, Theorem 6.6.1] and [24, Propositions 2.3,2.5]. However, their implementation in the real case requires more care. For example, the proof of the crucial transversality statements Propositions 3.6 and 3.7 requires classifying the irreducible components of the domains of the elements of each stratum of the moduli space into five types, instead of one in [19] and two in [24], based on whether they are contracted or not and whether they are real or not.

Remark 3.4. The only steps in the proof of Theorem 3.3 dependent on the $k=0$ assumption are Corollaries 3.8 and 3.9. A geometric interpretation of the genus 1 real GW-invariants of [10, Theorem 1.5] is obtained by combining its proof with the remaining steps in the proof of Theorem 3.3.

3.2 Strata of stable real maps

The moduli spaces of real curves and maps are stratified based on the topological type of the domain and the distribution of the map degree between its irreducible components in the latter case. These data correspond to certain decorated graphs. Because of the contraction operations on these graphs that are central to our proof of Theorem 3.3, we define such graphs based on the perspective in [3, Section 2.1.1].

For $l,k\in\mathbb{Z}^{\geqslant0}$, define

$$\sigma_{l;k}\colon S_{l;k}\equiv\{i^+\colon i\in[l]\}\sqcup\{i^-\colon i\in[l]\}\sqcup\{i\colon i\in[k]\}\longrightarrow S_{l;k},\quad \sigma_{l;k}(f)=\begin{cases}i^{\mp}, & \text{if } f=i^{\pm}, i\in[l];\\ f, & \text{if } f\in[k].\end{cases}$$

An (l,k)-marked graph is a tuple

$$\overline{\gamma}\equiv\big(\mathfrak{g}\colon\overline{\mathrm{Ver}}\longrightarrow\mathbb{Z}^{\geqslant0},\overline{\varepsilon}\colon S_{l;k}\sqcup\overline{\mathrm{Fl}}\longrightarrow\overline{\mathrm{Ver}},\overline{\vartheta}\colon\overline{\mathrm{Fl}}\longrightarrow\overline{\mathrm{Fl}},\overline{\sigma}\colon\overline{\mathrm{Ver}}\sqcup\overline{\mathrm{Fl}}\longrightarrow\overline{\mathrm{Ver}}\sqcup\overline{\mathrm{Fl}}\big),\tag{3.10}$$

where $\overline{\mathrm{Ver}}$ and $\overline{\mathrm{Fl}}$ are finite sets (of vertices and flags, respectively) and $\overline{\vartheta}$ and $\overline{\sigma}$ are involutions such that

$$\overline{\vartheta}(v)\neq v\quad\forall\,v\in\overline{\mathrm{Fl}},\quad \mathfrak{g}\circ\overline{\sigma}|_{\overline{\mathrm{Ver}}}=\mathfrak{g},\quad \overline{\varepsilon}\circ\big\{\sigma_{l;k}\sqcup\overline{\sigma}|_{\overline{\mathrm{Fl}}}\big\}=\overline{\sigma}|_{\overline{\mathrm{Ver}}}\circ\overline{\varepsilon},\quad \overline{\vartheta}\circ\overline{\sigma}|_{\overline{\mathrm{Fl}}}=\overline{\sigma}|_{\overline{\mathrm{Fl}}}\circ\overline{\vartheta}\,.$$

For $f\in S_{l;k}$, let $\overline{\sigma}(f)=\sigma_{l;k}(f)$. We denote by $\mathrm{Aut}(\overline{\gamma})$ the group of automorphisms of $\overline{\gamma}$, i.e. pairs of automorphisms of the sets $\overline{\mathrm{Ver}}$ and $\overline{\mathrm{Fl}}$ commuting with the maps $\mathfrak{g}$, $\overline{\varepsilon}$, and $\overline{\sigma}$. Define

$$\mathrm{V}_{\mathbb{R}}(\overline{\gamma})=\big\{v\in\overline{\mathrm{Ver}}\colon\overline{\sigma}(v)=v\big\},\qquad \mathrm{V}_{\mathbb{C}}(\overline{\gamma})=\big\{v\in\mathrm{Ver}\colon\overline{\sigma}(v)\neq v\big\}\,.\tag{3.11}$$

The set of edges of $\overline{\gamma}$ as in (3.10) is

$$\mathrm{E}(\overline{\gamma})\equiv\big\{e=\{f,\overline{\vartheta}(f)\}\colon f\in\overline{\mathrm{Fl}}\big\}.$$

The involution $\overline{\sigma}|_{\overline{\mathrm{Fl}}}$ induces an involution on $\mathrm{E}(\overline{\gamma})$, which we also denote by $\overline{\sigma}$. The graph $\overline{\gamma}$ is connected if for all $v,v'\in\overline{\mathrm{Ver}}$ distinct there exist

$$m\in\mathbb{Z}^+,\ f_1^-,f_1^+,\ldots,f_m^-,f_m^+\in\overline{\mathrm{Fl}}\qquad\text{s.t.}$$
$$\overline{\varepsilon}\big(f_1^-\big)=v,\ \overline{\varepsilon}\big(f_m^+\big)=v',\ \overline{\varepsilon}\big(f_i^+\big)=\overline{\varepsilon}\big(f_{i+1}^-\big)\ \forall\,i\in[m-1],\ \big\{f_i^-,f_i^+\big\}\in\mathrm{E}(\overline{\gamma})\ \forall\,i\in[m].$$

Define

$$\begin{gathered}\mathrm{E}_{\mathbb{C}}(\overline{\gamma})\equiv\big\{e\in\mathrm{E}(\overline{\gamma})\colon\overline{\sigma}(e)\neq e\big\},\quad \mathrm{E}_{\mathbb{RC}}(\overline{\gamma})\equiv\big\{e=\{f,\overline{\vartheta}(f)\}\colon f\in\overline{\mathrm{Fl}},\,\overline{\sigma}(f)=\overline{\vartheta}(f)\big\},\\ \text{and}\quad \mathrm{E}_{\mathbb{RR}}(\overline{\gamma})\equiv\big\{e=\{f,\overline{\vartheta}(f)\}\colon f\in\overline{\mathrm{Fl}},\,\overline{\sigma}(f)=f,\,\overline{\sigma}(\overline{\vartheta}(f))=\overline{\vartheta}(f)\big\}.\end{gathered}\tag{3.12}$$

For each $v\in\overline{\mathrm{Ver}}$, let

$$S_{v;\mathbb{R}}(\overline{\gamma})=\big\{f\in\overline{\varepsilon}^{-1}(v)\colon\overline{\sigma}(f)=f\big\},\qquad S_{v;\mathbb{C}}(\overline{\gamma})=\big\{f\in\overline{\varepsilon}^{-1}(v)\colon\overline{\sigma}(f)\neq f\big\}.\tag{3.13}$$

If $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$, the involution $\overline{\sigma}$ restricts to an involution $\overline{\sigma}_v$ on $S_{v;\mathbb{R}}(\overline{\gamma})$ and $S_{v;\mathbb{C}}(\overline{\gamma})$. If $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, $S_{v;\mathbb{R}}(\overline{\gamma})=\varnothing$ and the involution $\overline{\sigma}$ restricts to an involution $\overline{\sigma}_v=\overline{\sigma}_{\overline{\sigma}(v)}$ on $S_{v;\mathbb{C}}(\overline{\gamma})\cup S_{\overline{\sigma}(v);\mathbb{C}}(\overline{\gamma})$.

Let $\overline{\gamma}$ be as in (3.10). A vertex $v\in\overline{\mathrm{Ver}}$ of $\overline{\gamma}$ is trivalent if

$$2\overline{\mathfrak{g}}(v)+\big|\overline{\varepsilon}^{-1}(v)\big| \geqslant 3. \tag{3.14}$$

The graph $\overline{\gamma}$ is trivalent if all its vertices are trivalent. For $g\in\mathbb{Z}^{\geqslant0}$, we denote the (finite) set of (equivalence classes of) connected trivalent graphs $\overline{\gamma}$ as in (3.10) such that

$$1+|\mathrm{E}(\overline{\gamma})|-|\overline{\mathrm{Ver}}|+\sum_{v\in\overline{\mathrm{Ver}}}\overline{\mathfrak{g}}(v) = g \tag{3.15}$$

by $\mathcal{A}_{g,l;k}$.

Suppose $g,l,k\in\mathbb{Z}^{\geqslant0}$ with $2(g+l)+k\geqslant3$, $\overline{\gamma}$ is a connected graph as in (3.10) satisfying (3.15), and $v\in\overline{\mathrm{Ver}}$ is a vertex not satisfying (3.14). The vertices v and $\overline{\sigma}(v)$ can then be contracted to obtain another (l,k)-marked graph

$$\overline{\gamma}' \equiv \big(\overline{\mathfrak{g}}'\!:\overline{\mathrm{Ver}}'\longrightarrow\mathbb{Z}^{\geqslant0},\overline{\varepsilon}':S_{l;k}\sqcup\overline{\mathrm{Fl}}'\longrightarrow\overline{\mathrm{Ver}}',\overline{\vartheta}':\overline{\mathrm{Fl}}'\longrightarrow\overline{\mathrm{Fl}}',\overline{\sigma}':\overline{\mathrm{Ver}}'\sqcup\overline{\mathrm{Fl}}'\longrightarrow\overline{\mathrm{Ver}}'\sqcup\overline{\mathrm{Fl}}'\big)$$

satisfying (3.15) and

$$\begin{gathered}\overline{\mathrm{Ver}}' = \overline{\mathrm{Ver}}-\big\{v,\overline{\sigma}(v)\big\}, \quad \overline{\mathfrak{g}}' = \overline{\mathfrak{g}}|_{\overline{\mathrm{Ver}}'}, \quad \overline{\mathrm{Fl}}'\subset\overline{\mathrm{Fl}}\cap\varepsilon^{-1}\big(\overline{\mathrm{Ver}}'\big), \quad \sigma\big(\overline{\mathrm{Fl}}'\big)=\overline{\mathrm{Fl}}', \quad \overline{\sigma}' = \overline{\sigma}|_{\overline{\mathrm{Ver}}'\sqcup\overline{\mathrm{Fl}}'},\\ \overline{\varepsilon}' = \overline{\varepsilon} \;\text{ on }\; \big(S_{l;k}\cap\overline{\varepsilon}^{-1}(\overline{\mathrm{Ver}}')\big)\sqcup\overline{\mathrm{Fl}}', \quad \overline{\vartheta}' = \overline{\vartheta} \;\text{ on }\; \big\{f\in\overline{\mathrm{Fl}}':\overline{\varepsilon}(\overline{\vartheta}(f))\neq v,\overline{\sigma}(v)\big\}\end{gathered}$$

as follows. We take

$$\overline{\mathrm{Fl}}' = \begin{cases}\overline{\mathrm{Fl}}\cap\overline{\varepsilon}^{-1}\big(\overline{\mathrm{Ver}}'\big), & \text{if } |\overline{\mathrm{Fl}}\cap\overline{\varepsilon}^{-1}(v)|=2;\\ \big\{f\in\overline{\mathrm{Fl}}\cap\overline{\varepsilon}^{-1}\big(\overline{\mathrm{Ver}}'\big):\overline{\varepsilon}\big(\overline{\vartheta}(f)\big)\neq v,\overline{\sigma}(v)\big\}, & \text{if } |\overline{\mathrm{Fl}}\cap\overline{\varepsilon}^{-1}(v)|=1.\end{cases}$$

In the case $|\overline{\mathrm{Fl}}\cap\overline{\varepsilon}^{-1}(v)|=2$, we extend $\overline{\vartheta}'$ from its specification above by

$$\overline{\vartheta}'(f_1) = f_2 \quad \text{if } f_1,f_2\in\overline{\mathrm{Fl}}',\; f_1\neq f_2,\; \overline{\varepsilon}\big(\overline{\vartheta}(f_1)\big)=\overline{\varepsilon}\big(\overline{\vartheta}(f_2)\big)\in\big\{v,\overline{\sigma}(v)\big\}.$$

In the case $|S_{l;k}\cap\overline{\varepsilon}^{-1}(v)|=1$, we extend $\overline{\varepsilon}'$ from its specification above by

$$\overline{\varepsilon}'(f_1) = \overline{\varepsilon}\big(\overline{\vartheta}(f_2)\big) \quad \text{if } f_1\in S_{l;k},\; f_2\in\overline{\mathrm{Fl}}',\; \overline{\varepsilon}(f_1)=\overline{\varepsilon}(f_2)\in\big\{v,\overline{\sigma}(v)\big\}.$$

By the assumption that v does not satisfy (3.14), these extensions are well-defined.

Let $g,l,k\in\mathbb{Z}^{\geqslant0}$ with $2(g+l)+k\geqslant3$. For each $\overline{\gamma}\in\mathcal{A}_{g,l;k}$, denote by

$$\mathbb{R}\mathcal{M}_{\overline{\gamma}}\subset\mathbb{R}\overline{\mathcal{M}}_{g,l;k}$$

the subspace parametrizing marked real curves

$$\mathcal{C}\equiv\big(\Sigma,(z_i^+,z_i^-)_{i\in[l]},(z_i)_{i\in[k]},\sigma,\mathfrak{j}\big)$$

with dual graph $\overline{\gamma}$. Thus, the irreducible components Σ_v and the nodes z_e of Σ are indexed by the elements of $\overline{\mathrm{Ver}}$ and $\mathrm{E}(\overline{\gamma})$, respectively. The node z_e corresponding to $e=\{f,\overline{\vartheta}(f)\}$ is obtained by identifying a point $z_f\in\Sigma_{\overline{\varepsilon}(f)}$ with a point $z_{\overline{\vartheta}(f)}\in\Sigma_{\overline{\varepsilon}(\overline{\vartheta}(f))}$. The marked point z_f corresponding to $f\in S_{l;k}$ is carried by the irreducible component $\Sigma_{\overline{\varepsilon}(f)}$. The involution σ sends Σ_v to $\Sigma_{\overline{\sigma}(v)}$ and z_e

to $z_{\overline{\sigma}(e)}$. The two sets in (3.11) correspond to the real components of Σ, i.e. those fixed by the involution, and the conjugate components of Σ, i.e. those interchanged by the involution. The three sets in (3.12) correspond to the C, E, and H-nodes of Σ; see [13, Section 3] for the terminology. Let

$$\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}=p_{\mathbb{R}}^{-1}\big(\mathbb{R}\mathcal{M}_{\overline{\gamma}}\big)\subset\mathbb{R}\widetilde{\mathcal{M}}_{g,l;k}. \tag{3.16}$$

The stratification of $\mathbb{R}\widetilde{\mathcal{M}}_{g,l;k}$ by the subspaces in (3.16) is analogous to the stratification [24, (3.2)] in the complex case.

For $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$, let $\mathbb{R}\mathcal{M}_{\overline{\gamma};v}$ denote the moduli space of smooth real genus $\overline{\mathfrak{g}}(v)$ connected curves with the real and conjugate marked points indexed by $S_{v;\mathbb{R}}(\overline{\gamma})$ and $S_{v;\mathbb{C}}(\overline{\gamma})$, respectively. For $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, let $\mathcal{M}^{\bullet}_{\overline{\gamma};v}$ denote the moduli space of smooth real curves with two genus $\overline{\mathfrak{g}}(v)$ topological components, Σ_v and $\Sigma_{\overline{\sigma}(v))}$, interchanged by the involution and carrying the marked points indexed by $S_{v;\mathbb{C}}(\overline{\gamma})$, and $S_{\overline{\sigma}(v);\mathbb{C}}(\overline{\gamma})$, respectively; we call such curves real doublets. The image of the immersion

$$\prod_{v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})}\mathbb{R}\mathcal{M}_{\overline{\gamma};v}\times\prod_{\{v,\overline{\sigma}(v)\}\subset\mathrm{V}_{\mathbb{C}}(\overline{\gamma})}\mathcal{M}^{\bullet}_{\overline{\gamma};v}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l;k} \tag{3.17}$$

identifying the marked point z_f with $z_{\overline{\vartheta}(f)}$ for each $f\in\overline{\mathrm{Fl}}$ is $\mathbb{R}\mathcal{M}_{\overline{\gamma}}$. This immersion descends to an isomorphism from the quotient of its domain by the natural $\mathrm{Aut}(\overline{\gamma})$ action to $\mathbb{R}\mathcal{M}_{\overline{\gamma}}$.

By the last requirement in Definition 2.2, there exist covers

$$\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma};v}\longrightarrow\mathbb{R}\mathcal{M}_{\overline{\gamma};v},\quad v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma}),\quad\text{and}\quad\widetilde{\mathcal{M}}^{\bullet}_{\overline{\gamma};v}\longrightarrow\mathcal{M}^{\bullet}_{\overline{\gamma};v},\quad\big\{v,\overline{\sigma}(v)\big\}\subset\mathrm{V}_{\mathbb{C}}(\overline{\gamma}),$$

with universal curves

$$\mathbb{R}\widetilde{\mathcal{U}}_{\overline{\gamma};v}\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma};v},\quad v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma}),\quad\text{and}\quad\widetilde{\mathcal{U}}^{\bullet}_{\overline{\gamma};v}\longrightarrow\widetilde{\mathcal{M}}^{\bullet}_{\overline{\gamma};v},\quad\big\{v,\overline{\sigma}(v)\big\}\subset\mathrm{V}_{\mathbb{C}}(\overline{\gamma}),$$

and an immersion

$$\widetilde{\iota}_{\overline{\gamma}}\colon\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}\equiv\prod_{v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})}\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma};v}\times\prod_{\{v,\overline{\sigma}(v)\}\subset\mathrm{V}_{\mathbb{C}}(\overline{\gamma})}\widetilde{\mathcal{M}}^{\bullet}_{\overline{\gamma};v}\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{g,l;k} \tag{3.18}$$

lifting (3.17).

Let $B\in H_2(X;\mathbb{Z})$. An (l,k)-marked degree B graph is a tuple

$$\begin{split}\gamma\equiv\Big((\mathfrak{g},\mathfrak{d})\colon\mathrm{Ver}\longrightarrow\mathbb{Z}^{\geqslant0}\oplus H_2(X;\mathbb{Z}),\varepsilon\colon S_{l;k}\sqcup\mathrm{Fl}\longrightarrow\mathrm{Ver},\vartheta\colon\mathrm{Fl}\longrightarrow\mathrm{Fl},\\ \sigma\colon\mathrm{Ver}\sqcup\mathrm{Fl}\longrightarrow\mathrm{Ver}\sqcup\mathrm{Fl}\Big)\end{split} \tag{3.19}$$

such that the tuple

$$\gamma_{\mathcal{M}}\equiv\big(\mathfrak{g}\colon\mathrm{Ver}\longrightarrow\mathbb{Z}^{\geqslant0},\varepsilon\colon S_{l;k}\sqcup\mathrm{Fl}\longrightarrow\mathrm{Ver},\vartheta\colon\mathrm{Fl}\longrightarrow\mathrm{Fl},\sigma\colon\mathrm{Ver}\sqcup\mathrm{Fl}\longrightarrow\mathrm{Ver}\sqcup\mathrm{Fl}\big)$$

is an (l,k)-marked graph and

$$\mathfrak{d}\circ\sigma=-\phi_*\circ\mathfrak{d},\qquad\sum_{v\in\mathrm{Ver}}\mathfrak{d}(v)=B,\qquad\big\langle\omega,\mathfrak{d}(v)\big\rangle\geqslant0\ \ \forall\ v\in\mathrm{Ver}\,. \tag{3.20}$$

Denote by $\mathrm{Aut}(\gamma)$ the group of automorphisms of γ, i.e. the subgroup of automorphisms of $\gamma_{\mathcal{M}}$ preserving $\mathfrak{d}$. Let

$$\begin{gathered}\mathrm{V}_{\mathbb{R}}(\gamma)=\mathrm{V}_{\mathbb{R}}(\gamma_{\mathcal{M}}),\quad \mathrm{V}_{\mathbb{C}}(\gamma)=\mathrm{V}_{\mathbb{C}}(\gamma_{\mathcal{M}}),\quad S_{v;\mathbb{R}}(\gamma)=S_{v;\mathbb{R}}(\gamma_{\mathcal{M}}),\ S_{v;\mathbb{C}}(\gamma)=S_{v;\mathbb{C}}(\gamma_{\mathcal{M}})\ \ \forall\, v\in\mathrm{Ver},\\ \mathrm{E}(\gamma)=\mathrm{E}(\gamma_{\mathcal{M}}),\ \ \mathrm{E}_{\mathbb{C}}(\gamma)=\mathrm{E}_{\mathbb{C}}(\gamma_{\mathcal{M}}),\ \ \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma)=\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma_{\mathcal{M}}),\ \ \mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma)=\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma_{\mathcal{M}}),\ \ |\gamma|=\big|\mathrm{E}(\gamma)\big|.\end{gathered}$$

We call γ **connected** if $\gamma_{\mathcal{M}}$ is connected and a vertex $v\in\mathrm{Ver}$ **trivalent** if it satisfies (3.14) with the overlines dropped.

Let $g,l,k\in\mathbb{Z}^{\geqslant 0}$ with $2(g+l)+k\geqslant 3$ and $B\in H_2(X;\mathbb{Z})$. We denote the set of (equivalence classes of) connected graphs γ as in (3.19) such that (3.15) with the overlines dropped holds and

$$\mathfrak{d}(v)=0,\ \ 2\mathfrak{g}(v)+\big|\varepsilon^{-1}(v)\big|\geqslant 3\qquad \forall\ v\in\mathrm{Ver}\ \text{s.t.}\ \big\langle\omega,\mathfrak{d}(v)\big\rangle=0$$

by $\mathcal{A}^{\phi}_{g,l;k}(B)$. As described below, the moduli space on the left-hand side of (3.5) is stratified by the subspaces $\mathfrak{M}_\gamma(J,\nu)$ that are indexed by $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$ and consist of maps from domains of the same topological type $\gamma_{\mathcal{M}}$.

Let $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$ be as in (3.19). The **stabilization** $\overline{\gamma}\in\mathcal{A}_{g,l;k}$ of γ is the trivalent graph as in (3.10) obtained by contracting the non-trivalent vertices of γ until all vertices become trivalent. The set $\overline{\mathrm{Ver}}$ thus consists of trivalent vertices of γ. It contains all vertices $v\in\mathrm{Ver}$ with $\mathfrak{g}(v)>0$, but may be missing some vertices v with $\mathfrak{g}(v)=0$ and $|\varepsilon^{-1}(v)|\geqslant 3$. Let

$$\aleph(\gamma)\equiv\mathrm{Ver}-\overline{\mathrm{Ver}},\quad \aleph_{\mathbb{R}}(\gamma)=\big\{v\in\aleph(\gamma)\colon\sigma(v)=v\big\},\quad\text{and}\quad \aleph_{\mathbb{C}}(\gamma)=\big\{v\in\aleph(\gamma)\colon\sigma(v)=v\big\}$$

denote the set of vertices contracted by the stabilization, its subset fixed by the involution on the graph, and the complement of this subset.

Define

$$\begin{aligned}\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;v}\equiv\mathbb{R}\widetilde{\mathcal{U}}_{\overline{\gamma};v}&\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}\equiv\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma};v}&&\text{if } v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma}),\\ \widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\equiv\widetilde{\mathcal{U}}^{\bullet}_{\overline{\gamma};v}&\longrightarrow\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}\equiv\widetilde{\mathcal{M}}^{\bullet}_{\overline{\gamma};v}&&\text{if } v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma}).\end{aligned}$$

If $v\in\aleph_{\mathbb{R}}(\gamma)$ with $|\varepsilon^{-1}(v)|\geqslant 3$, we take $\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}=\mathbb{R}\mathcal{M}_{\gamma;v}$ with $\mathbb{R}\mathcal{M}_{\gamma;v}$ defined as before and

$$\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;v}\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}$$

to be the universal curve. If $v\in\aleph_{\mathbb{C}}(\gamma)$ with $|\varepsilon^{-1}(v)|\geqslant 3$, we take $\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}=\mathcal{M}^{\bullet}_{\gamma;v}$ with $\mathcal{M}^{\bullet}_{\gamma;v}$ defined as before and

$$\widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\longrightarrow\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}$$

to be the universal curve. For $v\in\aleph_{\mathbb{R}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$ and $v\in\aleph_{\mathbb{C}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$, denote by $\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}$ and $\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}$ the one-point spaces. Let

$$\mathbb{R}\widetilde{\mathcal{M}}_{\gamma}=\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}\,.$$

Denote by

$$p_{\gamma;v}\colon\mathbb{R}\widetilde{\mathcal{M}}_{\gamma}\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v},\quad v\in\mathrm{V}_{\mathbb{R}}(\gamma),\qquad p^{\bullet}_{\gamma;v}\colon\mathbb{R}\widetilde{\mathcal{M}}_{\gamma}\longrightarrow\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}\ \ v\in\mathrm{V}_{\mathbb{C}}(\gamma),$$

the projection maps.

Let $\mathrm{Aut}(\mathbb{P}^1)$ be the group of holomorphic automorphisms of $\mathbb{P}^1$. For $v\in\aleph_{\mathbb{C}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$, let

$$\widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\equiv\Sigma_v\sqcup\Sigma_{\sigma(v)}\equiv\mathbb{P}^1\sqcup\mathbb{P}^1$$

be the genus 0 real doublet with the involution $\tau_1\colon\Sigma_v\longrightarrow\Sigma_{\sigma(v)}$ and the marked points indexed by $\varepsilon^{-1}(v)\cup\varepsilon^{-1}(\sigma(v))$ so that the marked points on Σ_v are given by

$$\big\{z_f\colon f\in\varepsilon^{-1}(v)\big\}=\begin{cases}\{\infty\}, & \text{if } |\varepsilon^{-1}(v)|=1;\\ \{\infty,0\}, & \text{if } |\varepsilon^{-1}(v)|=2.\end{cases}\tag{3.21}$$

Define

$$G_v=\begin{cases}\{h\in\mathrm{Aut}(\mathbb{P}^1)\colon h(\infty)=\infty\}, & \text{if } |\varepsilon^{-1}(v)|=1;\\ \{h\in\mathrm{Aut}(\mathbb{P}^1)\colon h(\infty)=\infty, h(0)=0\}, & \text{if } |\varepsilon^{-1}(v)|=2.\end{cases}$$

For $\rho=\tau,\eta$, let $\mathrm{Aut}_\rho(\mathbb{P}^1)$ be the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ consisting of the automorphisms that commute with ρ. Denote by $\mathrm{Inv}(\gamma)$ the set of maps

$$\rho\colon\big\{v\in\aleph_{\mathbb{R}}(\gamma)\colon|\varepsilon^{-1}(v)|\leqslant 2\big\}\longrightarrow\big\{\tau,\eta\big\}\qquad\text{s.t.}\quad\rho(v)=\tau\ \text{ if }\ S_{v;\mathbb{R}}(\gamma)\neq\varnothing.$$

For $\rho\in\mathrm{Inv}(\gamma)$ and $v\in\aleph_{\mathbb{R}}(\gamma)$ such that $|\varepsilon^{-1}(v)|\leqslant 2$, let

$$\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}\equiv\Sigma_v\equiv\mathbb{P}^1$$

be the genus 0 real curve with the involution $\rho(v)$ and the marked points

$$\big\{z_f\colon f\in\varepsilon^{-1}(v)\big\}=\begin{cases}\{1\}, & \text{if } |\varepsilon^{-1}(v)|=1;\\ \{1,-1\}, & \text{if } |S_{v;\mathbb{R}}(\gamma)|=2;\\ \{\infty,0\}, & \text{if } |S_{v;\mathbb{C}}(\gamma)|=2.\end{cases}\tag{3.22}$$

Define

$$G_{\rho;v}=\begin{cases}\{h\in\mathrm{Aut}_\tau(\mathbb{P}^1)\colon h(1)=1\}, & \text{if } |\varepsilon^{-1}(v)|=1;\\ \{h\in\mathrm{Aut}_\tau(\mathbb{P}^1)\colon h(1)=1, h(-1)=-1\}, & \text{if } |S_{v;\mathbb{R}}(\gamma)|=2;\\ \{h\in\mathrm{Aut}_{\rho(v)}(\mathbb{P}^1)\colon h(\infty)=\infty, h(0)=0\}, & \text{if } |S_{v;\mathbb{C}}(\gamma)|=2.\end{cases}$$

For each $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$, we have thus constructed a fibration

$$\pi^{\bullet}_{\gamma;v}\equiv\pi_{\gamma;v}\sqcup\pi_{\gamma;\sigma(v)}\colon\widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\equiv\widetilde{\mathcal{U}}_{\gamma;v}\sqcup\widetilde{\mathcal{U}}_{\gamma;\sigma(v)}\longrightarrow\widetilde{\mathcal{M}}^{\bullet}_{\gamma;v}\tag{3.23}$$

by smooth genus $\mathfrak{g}(v)$ real marked doublets. For $\rho\in\mathrm{Inv}(\gamma)$ and $v\in\mathrm{V}_{\mathbb{R}}(\gamma)$, we have constructed a fibration

$$\pi_{\gamma;\rho;v}\colon\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}\tag{3.24}$$

by smooth genus $\mathfrak{g}(v)$ real marked curves (described above without the ρ subscript except for $v\in\aleph_{\mathbb{R}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$). Define

$$\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}=\Bigg(\bigsqcup_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}p^*_{\gamma;v}\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}\times\bigsqcup_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}p^{\bullet\,*}_{\gamma;v}\widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\Bigg)\Big/\!\sim\qquad\text{with}$$

$$\big((\mathcal{C}_v)_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)},(\mathcal{C}^{\bullet}_v)_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)};z_f\big)\sim\big((\mathcal{C}_v)_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)},(\mathcal{C}^{\bullet}_v)_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)};z_{\vartheta(f)}\big)\quad\forall\,f\in\mathrm{Fl},$$

i.e. we identify marked points in a fiber of

$$\bigsqcup_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)} p_{\gamma;v}^*\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v} \times \bigsqcup_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)} p_{\gamma;v}^{\bullet\,*}\widetilde{\mathcal{U}}_{\gamma;v}^{\bullet} \longrightarrow \mathbb{R}\widetilde{\mathcal{M}}_\gamma$$

if they correspond to flags interchanged by the involution ϑ on Fl. The natural projection

$$\pi_{\gamma;\rho}\colon \mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho} \longrightarrow \mathbb{R}\widetilde{\mathcal{M}}_\gamma \tag{3.25}$$

is a fibration whose fibers are nodal marked real curves with dual graph $\gamma_{\mathcal{M}}$; the irreducible components Σ_v of these fibers are indexed by the set Ver. Let $\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}^*\subset\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}$ be the complement of the nodes of the fibers of $\pi_{\gamma;\rho}$.

The involutions σ and ρ induce an involution $\widetilde{\sigma}_{\gamma;\rho}$ on $\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}$. Let

$$\mathcal{T}_{\gamma;\rho} \equiv \ker \mathrm{d}\big(\pi_{\gamma;\rho}|_{\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}^*}\big) \longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}^*$$

be the vertical tangent bundle. Denote by $\mathfrak{j}_{\gamma;\rho}$ the complex structure on this complex line bundle. The group

$$G_{\gamma;\rho}^\circ \equiv \prod_{\substack{v\in\aleph_{\mathbb{R}}(\gamma)\\ |\varepsilon^{-1}(v)|\leqslant 2}} G_{\rho;v} \times \prod_{\substack{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma)\\ |\varepsilon^{-1}(v)|\leqslant 2}} G_v$$

acts on $\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}$ by reparametrizing the irreducible components Σ_v of the fibers with $|\varepsilon^{-1}(v)|\leqslant 2$. Let

$$q_{\gamma;\rho}\colon \mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho} \longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}\big|_{\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}} \tag{3.26}$$

be the surjection covering the composition of the projection $\mathbb{R}\widetilde{\mathcal{M}}_\gamma \longrightarrow \mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}$ with (3.18) and contracting the irreducible components of the fibers of $\pi_{\gamma;\rho}$ indexed by $\aleph(\gamma)$. Denote by $G_{\gamma;\rho}$ the group of holomorphic automorphisms of $q_{\gamma;\rho}$ that commute with $\widetilde{\sigma}_{\gamma;\rho}$ and preserve the marked points. The identity component of $G_{\gamma;\rho}$ is $G_{\gamma;\rho}^\circ$; the group $G_{\gamma;\rho}/G_{\gamma;\rho}^\circ$ is naturally isomorphic to $\mathrm{Aut}(\gamma)$.

For $J\in\mathcal{J}_\omega^\phi$, define

$$\Gamma_{\gamma;\rho}^{0,1}(X;J)^\phi = \Big\{\nu\in\Gamma\big(\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}^*\times X; \pi_1^*(\mathcal{T}_{\gamma;\rho},-\mathfrak{j}_{\gamma;\rho})^*\otimes_{\mathbb{C}}\pi_2^*(TX,J)\big)\colon \mathrm{supp}(\nu)\subset\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho}^*\times X,\\ \mathrm{d}\phi\circ\nu=\nu\circ\mathrm{d}\widetilde{\sigma}_{\gamma;\rho}\Big\}.$$

For $\nu\in\Gamma_{\gamma;\rho}^{0,1}(X;J)^\phi$, let $\widetilde{\mathfrak{M}}_{\gamma;\rho}(J,\nu)$ be the space of tuples

$$\mathbf{u} \equiv \big(u\colon\Sigma\longrightarrow X, (z_f)_{f\in S_{l;k}}, \sigma, \mathfrak{j}\big), \tag{3.27}$$

where $(\Sigma,(z_f)_{f\in S_{l;k}},\sigma,\mathfrak{j})$ is a fiber of $\pi_{\gamma;\rho}$ and u is a (ϕ,σ)-real map such that

$$\begin{aligned} \bar\partial_{J,\mathfrak{j}}u\big|_z &= \nu\big(z,u(z)\big) \in \big(T_z\Sigma,-\mathfrak{j}\big)^*\otimes_{\mathbb{C}}(T_{u(z)}X,J) && \forall\, z\in\Sigma,\\ u_*[\Sigma_v] &= \mathfrak{d}(v)\in H_2(X;\mathbb{Z}) && \forall\, v\in\mathrm{Ver}\,. \end{aligned}$$

For $J\in\mathcal{J}_\omega^\phi$ and $\nu\in\Gamma_{g,l;k}^{0,1}(X;J)^\phi$, let

$$\nu_{\gamma;\rho} = \big\{q_{\gamma;\rho}\times\mathrm{id}_X\big\}^*\nu \in \Gamma_{\gamma;\rho}^{0,1}(X;J)^\phi\,, \qquad \widetilde{\mathfrak{M}}_{\gamma;\rho}(J,\nu) = \widetilde{\mathfrak{M}}_{\gamma;\rho}(J,\nu_{\gamma;\rho})\,.$$

The group $G_{\gamma;\rho}$ acts on $\widetilde{\mathfrak{M}}_{\gamma;\rho}(J,\nu)$ by reparametrizing the domains of maps as usual. Define

$$\mathfrak{M}_{\gamma}(J,\nu)=\bigsqcup_{\rho\in\mathrm{Inv}(\gamma)}\widetilde{\mathfrak{M}}_{\gamma;\rho}(J,\nu)/G_{\gamma;\rho}\subset\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}\,. \tag{3.28}$$

The stratification by the subspaces (3.28) is analogous to the stratification by the subspaces in [19, (5.1.5)] and on the right-hand side of [24, (3.25)]. The number of nodes of the domains in the stratum (3.28) is $|\gamma|$. The dual graph of the element $\mathsf{st}(\mathbf{u})$ of $\mathbb{R}\overline{\mathcal{M}}_{g,l;k}$ for any $\mathbf{u}\in\mathfrak{M}_{\gamma}(J,\nu)$ is $\overline{\gamma}$. By [19, Proposition 4.1.5], the set

$$\mathcal{A}^{\phi}_{g,l;k}(B;J,\nu)\equiv\big\{\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)\colon\mathfrak{M}_{\gamma}(J,\nu)\neq\varnothing\big\}$$

is finite for each pair (J,ν).

3.3 Strata of simple real maps: definitions

Let γ be as in (3.19). For each subset $\mathscr{V}\subset\mathrm{Ver}$, let

$$\begin{aligned}\mathscr{V}_0(\gamma)&=\big\{v\in\mathscr{V}\colon\mathfrak{d}(v)=0\big\}, & \mathscr{E}_{\gamma}(\mathscr{V})&=\big\{\{f_1,f_2\}\in\mathrm{E}(\gamma)\colon\varepsilon(f_1),\varepsilon(f_2)\in\mathscr{V}\big\},\\ \mathscr{F}_{\gamma}(\mathscr{V})&=\big\{f\in\mathrm{Fl}\colon\varepsilon(f),\varepsilon(\vartheta(f))\in\mathscr{V}\big\}, & \mathscr{F}^*_{\gamma}(\mathscr{V})&=\big\{f\in\mathrm{Fl}\colon\varepsilon(f)\in\mathscr{V},\,\varepsilon(\vartheta(f))\notin\mathscr{V}\big\}.\end{aligned} \tag{3.29}$$

The tuple

$$\gamma_{\mathscr{V}}\equiv\big(\mathfrak{g}\colon\mathscr{V}\longrightarrow\mathbb{Z}^{\geqslant0},\varepsilon\colon(S_{l;k}\cap\varepsilon^{-1}(\mathscr{V}))\sqcup\mathscr{F}_{\gamma}(\mathscr{V})\longrightarrow\mathscr{V},\vartheta\colon\mathscr{F}_{\gamma}(\mathscr{V})\longrightarrow\mathscr{F}_{\gamma}(\mathscr{V})\big)$$

is then an $S_{l;k}\cap\varepsilon^{-1}(\mathscr{V})$-marked graph (without an involution $\overline{\sigma}$ as in (3.10)). Let $\pi_0(\gamma,\mathscr{V})$ be the set of connected components of $\gamma_{\mathscr{V}}$ and

$$\ell(\gamma,\mathscr{V})\equiv\big|\mathscr{E}_{\gamma}(\mathscr{V})\big|-|\mathscr{V}|+\big|\pi_0(\gamma,\mathscr{V})\big|$$

be the number of loops in $\gamma_{\mathscr{V}}$, i.e. the sum of the genera of its connected components.

For $\mathscr{V}\subset\mathrm{Ver}$ as above, let

$$\mathscr{F}^{\circ}_{\gamma}(\mathscr{V})\subset\mathrm{Fl}\cap\varepsilon^{-1}\big(\mathrm{Ver}-\mathscr{V}\big)$$

be the collection of the flags f such that the edge $e\equiv\{f,\vartheta(f)\}$ disconnects a connected component $\gamma_{\mathscr{V}'}$ of $\gamma_{\mathscr{V}}$ from the rest of γ. Denote by

$$\mathscr{F}^{\dagger}_{\gamma}(\mathscr{V})\subset\mathscr{F}^{\circ}_{\gamma}(\mathscr{V})$$

the subcollection of the flags f such that $S_{l;k}\cap\varepsilon^{-1}(\mathscr{V}')=\varnothing$ for the subset $\mathscr{V}'\subset\mathscr{V}$ of the vertices separated from the remainder of γ by the edge $\{f,\vartheta(f)\}$. In particular,

$$\begin{split}\big|\mathrm{E}(\gamma)-\mathscr{E}_{\gamma}(\mathscr{V})-\mathscr{E}_{\gamma}(\mathrm{Ver}-\mathscr{V})\big|+\big|\mathscr{F}^{\dagger}_{\gamma}(\mathscr{V})\big|+\big|S_{l;k}\cap\varepsilon^{-1}(\mathscr{V})\big|&\\ \geqslant\big|\mathrm{E}(\gamma)-\mathscr{E}_{\gamma}(\mathscr{V})-\mathscr{E}_{\gamma}(\mathrm{Ver}-\mathscr{V})\big|+\big|\mathscr{F}^{\circ}_{\gamma}(\mathscr{V})\big|\geqslant2\big|\pi_0(\gamma,\mathscr{V})\big|\,.&\end{split} \tag{3.30}$$

For $\rho\in\mathrm{Inv}(\gamma)$, let

$$\begin{split}\Gamma^{0,1}_{\gamma;\rho;\mathscr{V}}(X;J)=\Big\{\nu\in\Gamma^{0,1}_{\gamma;\rho}(X;J)^{\phi}\colon\nu\big|_{(p^*_{\gamma;v}\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}\cap\mathbb{R}\widetilde{\mathcal{U}}^*_{\gamma;\rho})\times X}=0\;\forall\,v\in\mathrm{V}_{\mathbb{R}}(\gamma)\cap\mathscr{V},\\ \nu\big|_{(p^{\bullet*}_{\gamma;v}\mathbb{R}\widetilde{\mathcal{U}}^{\bullet}_{\gamma;v}\cap\mathbb{R}\widetilde{\mathcal{U}}^*_{\gamma;\rho})\times X}=0\;\forall\,v\in\mathrm{V}_{\mathbb{C}}(\gamma)\cap\mathscr{V}\Big\}.\end{split}$$

Let $S(\gamma)$ denote the collections of subsets $\mathscr{V}\subset\mathrm{Ver}$ such that

$$\aleph(\gamma)\subset\mathscr{V}, \qquad \sigma(\mathscr{V})=\mathscr{V}, \qquad \mathfrak{g}(v)\!=\!0 \ \ \forall\, v\!\in\!\mathscr{V}\,.$$

For each $\mathscr{V}\in S(\gamma)$, define

$$\mathbb{R}\mathcal{M}_{\gamma;\mathscr{V}} = \prod_{v\in\mathscr{V}_0\cap\mathrm{V}_{\mathbb{R}}(\gamma)}\!\!\mathbb{R}\mathcal{M}_{\gamma;\rho;v} \ \times \prod_{\{v,\sigma(v)\}\subset\mathscr{V}_0\cap\mathrm{V}_{\mathbb{C}}(\gamma)}\!\!\mathcal{M}^{\bullet}_{\gamma;v}\,.$$

By (3.30) with $\mathscr{V}$ replaced by $\mathscr{V}_0$,

$$\big(\dim_{\mathbb{R}}\mathbb{R}\mathcal{M}_{\gamma;\mathscr{V}}+\big|\mathscr{F}_\gamma^\dagger(\mathscr{V}_0)\big|\big)+\big|\mathscr{E}_\gamma(\mathscr{V}_0)\big|+\big|\pi_0(\gamma,\mathscr{V}_0)\big|\geqslant 3\ell(\gamma,\mathscr{V}_0). \tag{3.31}$$

Let $\overline{\gamma}\in\mathcal{A}_{g,l;k}$ be as in (3.10). Denote by $\mathcal{A}(\overline{\gamma})$ the collection of pairs (γ,ϖ), where $\gamma\in\mathcal{A}^{\phi}_{g',l;k}(B)$ is as in (3.19) such that

$$\begin{gathered}\overline{\mathrm{Ver}}\subset\mathrm{Ver},\quad \aleph(\gamma,\overline{\gamma})\equiv\mathrm{Ver}-\overline{\mathrm{Ver}}\in S(\gamma),\quad \mathscr{F}^\circ_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big)\subset\mathscr{F}_\gamma\big(\aleph(\gamma,\overline{\gamma})\big), \\ \mathscr{F}_\gamma\big(\overline{\mathrm{Ver}}\big)\subset\overline{\mathrm{Fl}}\subset\mathscr{F}_\gamma\big(\overline{\mathrm{Ver}}\big)\cup\mathscr{F}^*_\gamma\big(\overline{\mathrm{Ver}}\big),\quad \sigma(\overline{\mathrm{Fl}})=\overline{\mathrm{Fl}}, \\ \overline{\mathfrak{g}}=\mathfrak{g}|_{\overline{\mathrm{Ver}}},\quad \overline{\varepsilon}\big|_{(S_{l;k}\cap\varepsilon^{-1}(\overline{\mathrm{Ver}}))\sqcup\overline{\mathrm{Fl}}}=\varepsilon\big|_{(S_{l;k}\cap\varepsilon^{-1}(\overline{\mathrm{Ver}}))\sqcup\overline{\mathrm{Fl}}},\quad \overline{\vartheta}\big|_{\mathscr{F}_\gamma(\overline{\mathrm{Ver}})}=\vartheta\big|_{\mathscr{F}_\gamma(\overline{\mathrm{Ver}})},\quad \overline{\sigma}=\sigma|_{\overline{\mathrm{Ver}}\sqcup\overline{\mathrm{Fl}}},\end{gathered} \tag{3.32}$$

and $\varpi\colon S_{l;k}\cap\varepsilon^{-1}(\aleph(\gamma,\overline{\gamma}))\longrightarrow\mathscr{F}^*_\gamma(\overline{\mathrm{Ver}})-\overline{\mathrm{Fl}}$ is a $(\sigma,\sigma_{l;k})$-equivariant injective map such that

$$\overline{\varepsilon}|_{S_{l;k}\cap\varepsilon^{-1}(\aleph(\gamma,\overline{\gamma}))}=\varepsilon\circ\varpi\colon S_{l;k}\cap\varepsilon^{-1}\big(\aleph(\gamma,\overline{\gamma})\big)\longrightarrow\overline{\mathrm{Ver}}.$$

Thus, $\overline{\gamma}$ is obtained from γ by

- dropping every vertex $v\in\aleph(\gamma,\overline{\gamma})$,
- attaching each marked point $f\in S_{l;k}$ carried by a vertex in $\aleph(\gamma,\overline{\gamma})$ to the vertex $\varepsilon(\varpi(f))\in\overline{\mathrm{Ver}}$,
- identifying some pairs of the flags in $\mathscr{F}^*_\gamma(\overline{\mathrm{Ver}})$ into the edges of $\overline{\gamma}$ that are not contained in $\mathscr{E}_\gamma(\overline{\mathrm{Ver}})$.

Define

$$\begin{gathered}\aleph(\gamma)_0=\aleph(\gamma)_0(\gamma),\qquad \aleph(\gamma)_\bullet=\aleph(\gamma)-\aleph(\gamma)_0,\qquad \aleph(\gamma)_0^c=\mathrm{Ver}-\aleph(\gamma)_0,\\ \aleph(\gamma,\overline{\gamma})_0=\aleph(\gamma,\overline{\gamma})_0(\gamma),\quad \aleph(\gamma,\overline{\gamma})_\bullet=\aleph(\gamma,\overline{\gamma})-\aleph(\gamma,\overline{\gamma})_0,\quad \aleph(\gamma,\overline{\gamma})_0^c=\mathrm{Ver}-\aleph(\gamma,\overline{\gamma})_0\,.\end{gathered}$$

If $\overline{\gamma}$ is the stabilization of γ, then $g'=g$, $\aleph(\gamma,\overline{\gamma})=\aleph(\gamma)$, and $(\gamma,\varpi_\gamma)\in\mathcal{A}(\overline{\gamma})$ for a unique injective map

$$\varpi_\gamma\colon S_{l;k}\cap\varepsilon^{-1}\big(\aleph(\gamma)\big)\longrightarrow\mathscr{F}^*_\gamma(\overline{\mathrm{Ver}})-\overline{\mathrm{Fl}}\,. \tag{3.33}$$

A more elaborate example is depicted in Figure 2.

Let $(\gamma,\varpi)\in\mathcal{A}(\overline{\gamma})$ and $\rho\in\mathrm{Inv}(\gamma)$. The reduction of γ to $\overline{\gamma}$ described above determines a smooth map

$$\mathsf{st}_{\gamma,\varpi}\colon\mathbb{R}\widetilde{\mathcal{M}}_\gamma\longrightarrow\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}\subset\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\,. \tag{3.34}$$

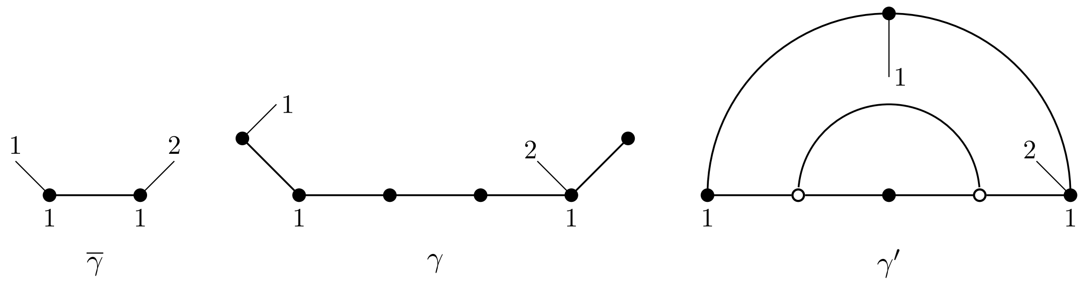

Figure 2: A graph $\overline{\gamma}$ as in (3.10) and graphs γ,γ' as in (3.19) such that (γ,ϖ_γ) and (γ',ϖ') are elements of $\mathcal{A}(\overline{\gamma})$ for some ϖ'. All vertices, edges, and marked points are taken to be real (i.e. σ acts trivially). The value of $\mathfrak{g}$ on the vertices with the number 1 next to them is 1; its value on the remaining vertices is 0. The value of $\mathfrak{d}'$ on each of the unshaded vertices is 0; the values of $\mathfrak{d}$ on all vertices in the middle diagram and of $\mathfrak{d}'$ on the shaded vertices in the last diagram are not 0. The graph $\overline{\gamma}$ is the stabilization of γ, but not of γ'.

This map lifts to smooth maps

$$\begin{aligned} q_{\gamma,\varpi;v}\colon p_{\gamma;v}^*\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v} &\longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}, \qquad v\in \mathrm{V}_{\mathbb{R}}(\overline{\gamma}),\\ q_{\gamma,\varpi;v}^\bullet\colon p_{\gamma;v}^{\bullet *}\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;v}^\bullet &\longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}, \qquad \{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\overline{\gamma}). \end{aligned}$$

Each map $q_{\gamma,\varpi;v}$ restricts to a degree one map from a fiber of

$$p_{\gamma;v}^*\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}\big|_{\mathsf{st}_{\gamma,\varpi}^{-1}(\mathcal{C})} \longrightarrow \mathsf{st}_{\gamma,\varpi}^{-1}(\mathcal{C})\subset \mathbb{R}\widetilde{\mathcal{M}}_\gamma$$

onto a real irreducible component of a fiber of (2.10) over $\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}$. Each map $q_{\gamma,\varpi;v}^\bullet$ restricts to a degree one map from a connected component of a fiber

$$p_{\gamma;v}^{\bullet *}\mathbb{R}\widetilde{\mathcal{U}}_{\gamma;v}^\bullet\big|_{\mathsf{st}_{\gamma,\varpi}^{-1}(\mathcal{C})} \longrightarrow \mathsf{st}_{\gamma,\varpi}^{-1}(\mathcal{C})\subset \mathbb{R}\widetilde{\mathcal{M}}_\gamma$$

onto a conjugate irreducible component of a fiber of (2.10) over $\mathbb{R}\widetilde{\mathcal{M}}_{\overline{\gamma}}$. In both cases, the marked and nodal points of the domain are preserved. However, in general these maps do not induce a continuous map even over a fiber of (3.25).

For $J\in\mathcal{J}_\omega^\phi$ and $\nu\in\Gamma_{g,l;k}^{0,1}(X;J)^\phi$, define $\nu_{\gamma,\varpi;\rho}\in\Gamma^{0,1}_{\gamma;\rho;\aleph(\gamma,\overline{\gamma})}(X;J)^\phi$ by

$$\begin{aligned} \nu_{\gamma,\varpi;\rho}\big|_{(p_{\gamma;v}^*\mathbb{R}\tilde{\mathcal{U}}_{\gamma;\rho;v}\cap\mathbb{R}\tilde{\mathcal{U}}_{\gamma;\rho}^*)\times X} &= \begin{cases} \{q_{\gamma,\varpi;v}\times\mathrm{id}_X\}^*\nu, & \text{if } v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma});\\ 0, & \text{if } v\in\mathrm{V}_{\mathbb{R}}(\gamma)\cap\aleph(\gamma,\overline{\gamma}); \end{cases}\\ \nu_{\gamma,\varpi;\rho}\big|_{p_{\gamma;v}^{\bullet *}\mathbb{R}\tilde{\mathcal{U}}_{\gamma;v}^\bullet\cap\mathbb{R}\tilde{\mathcal{U}}_{\gamma;\rho}^*)\times X} &= \begin{cases} \{q_{\gamma,\varpi;v}^\bullet\times\mathrm{id}_X\}^*\nu, & \text{if } v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma});\\ 0, & \text{if } v\in\mathrm{V}_{\mathbb{C}}(\gamma)\cap\aleph(\gamma,\overline{\gamma}). \end{cases} \end{aligned}$$

Let

$$\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}(J,\nu)=\widetilde{\mathfrak{M}}_{\gamma;\rho}\big(J,\nu_{\gamma,\varpi;\rho}\big), \qquad \mathfrak{M}_{\gamma,\varpi}(J,\nu)=\bigsqcup_{\rho\in\mathrm{Inv}(\gamma)}\widetilde{\mathfrak{M}}_{\gamma;\varpi;\rho}(J,\nu)\big/G_{\gamma;\rho}^\circ\,.$$

The maps (3.34) with $\rho\in\mathrm{Inv}(\gamma)$ and the evaluation morphism ev determine a continuous map

$$\mathsf{st}_{\gamma,\varpi}\times\mathsf{ev}\colon\mathfrak{M}_{\gamma,\varpi}(J,\nu)\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\times\big(X^l\times(X^\phi)^k\big)\,. \tag{3.35}$$

For $\mathbf{u}\in\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}(J,\nu)$ as in (3.27) and $v\in\aleph(\gamma,\overline{\gamma})_0$, the restriction u_v of u to the irreducible component $\Sigma_v\subset\Sigma$ corresponding to v is constant. Thus,

$$\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}(J,\nu)\approx\mathbb{R}\mathcal{M}_{\gamma;\aleph(\gamma,\overline{\gamma})}\times\widetilde{\mathcal{Z}}'_{\gamma,\varpi;\rho}(J,\nu) \tag{3.36}$$

for some space $\widetilde{\mathcal{Z}}'_{\gamma,\varpi;\rho}(J,\nu)$ of tuples $(u_v)_{v\in\aleph(\gamma,\overline{\gamma})_0^c}$ of $\varepsilon^{-1}(v)$-marked maps with matching conditions at the points indexed by $\mathrm{Fl}-\mathscr{F}^{\circ}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)$. By the last assumption in (3.32),

$$\varepsilon(f)\notin\overline{\mathrm{Ver}},\quad \mathfrak{d}\big(\varepsilon(v)\big)\neq 0\qquad\forall\ f\in\mathscr{F}^{\circ}_{\gamma}\big(\aleph(\gamma,\overline{\gamma})_0\big).$$

Dropping the marked points z_f with $f\in\mathscr{F}^{\dagger}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)$, we thus obtain a fiber bundle

$$\widetilde{\mathcal{Z}}'_{\gamma,\varpi;\rho}(J,\nu)\longrightarrow\widetilde{\mathcal{Z}}_{\gamma,\varpi;\rho}(J,\nu) \tag{3.37}$$

with $|\mathscr{F}^{\dagger}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)|$-dimensional fibers for some space $\widetilde{\mathcal{Z}}_{\gamma,\varpi;\rho}(J,\nu)$ of tuples $(u_v)_{v\in\aleph(\gamma,\overline{\gamma})_0^c}$ of maps with marked points indexed by $\varepsilon^{-1}(v)-\mathscr{F}^{\dagger}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)$ and with the same matching conditions as before.

The $G^{\circ}_{\gamma;\rho}$-action on the left-hand side of (3.36) corresponds to an action on the last factor on the right-hand side. The latter in turn descends to an action on the right-hand side of (3.37). Let

$$\mathcal{Z}'_{\gamma,\varpi}(J,\nu)=\bigsqcup_{\rho\in\mathrm{Inv}(\gamma)}\widetilde{\mathcal{Z}}'_{\gamma,\varpi;\rho}(J,\nu)/G^{\circ}_{\gamma;\rho}\qquad\text{and}\qquad\mathcal{Z}_{\gamma,\varpi}(J,\nu)=\bigsqcup_{\rho\in\mathrm{Inv}(\gamma)}\widetilde{\mathcal{Z}}_{\gamma,\varpi;\rho}(J,\nu)/G^{\circ}_{\gamma;\rho}\,.$$

Thus,

$$\mathfrak{M}_{\gamma,\varpi}(J,\nu)\approx\mathbb{R}\mathcal{M}_{\gamma;\aleph(\gamma,\overline{\gamma})}\times\mathcal{Z}'_{\gamma,\varpi}(J,\nu) \tag{3.38}$$

and the fibers of the projection

$$\mathcal{Z}'_{\gamma,\varpi}(J,\nu)\longrightarrow\mathcal{Z}_{\gamma,\varpi}(J,\nu) \tag{3.39}$$

are $|\mathscr{F}^{\dagger}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)|$-dimensional. The map (3.35) factors through the projection from the left-hand side of (3.38) to the right-hand side in (3.39) and a continuous map

$$\mathsf{st}_{\gamma,\varpi}\times\mathsf{ev}\colon\mathcal{Z}_{\gamma,\varpi}(J,\nu)\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\times\big(X^l\times(X^{\phi})^k\big)\,. \tag{3.40}$$

Denote by

$$\mathfrak{M}^*_{\gamma,\varpi}(J,\nu)\subset\mathfrak{M}_{\gamma,\varpi}(J,\nu)\qquad\text{and}\qquad\widetilde{\mathfrak{M}}^*_{\gamma,\varpi;\rho}(J,\nu)\subset\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}(J,\nu) \tag{3.41}$$

the subspaces consisting of maps as in (3.27) such that u_v is simple for every $v\in\aleph(\gamma,\overline{\gamma})_{\bullet}$ and

$$u(\Sigma_{v_1})\neq u(\Sigma_{v_2})\qquad\forall\ v_1,v_2\in\aleph(\gamma,\overline{\gamma})_{\bullet},\ v_1\neq v_2.$$

Denote by

$$\mathcal{Z}'^*_{\gamma,\varpi}(J,\nu)\subset\mathcal{Z}'_{\gamma,\varpi}(J,\nu)\qquad\text{and}\qquad\mathcal{Z}^*_{\gamma,\varpi}(J,\nu)\subset\mathcal{Z}_{\gamma,\varpi}(J,\nu)$$

the image of $\mathfrak{M}^*_{\gamma,\varpi}(J,\nu)$ under the projection to the last component in (3.38) and the image of $\mathcal{Z}'^*_{\gamma,\varpi}(J,\nu)$ under (3.39). The splitting (3.38) and the fibration (3.40) restrict to a splitting

$$\mathfrak{M}^*_{\gamma,\varpi}(J,\nu)\approx\mathbb{R}\mathcal{M}_{\gamma;\aleph(\gamma,\overline{\gamma})}\times\mathcal{Z}'^*_{\gamma,\varpi}(J,\nu)$$

and fibration

$$\mathcal{Z}'^*_{\gamma,\varpi}(J,\nu)=\mathcal{Z}'_{\gamma,\varpi}(J,\nu)\big|_{\mathcal{Z}^*_{\gamma,\varpi}(J,\nu)}\longrightarrow\mathcal{Z}^*_{\gamma,\varpi}(J,\nu)$$

with $|\mathscr{F}^{\dagger}_{\gamma}(\aleph(\gamma,\overline{\gamma})_0)|$-dimensional fibers. The elements of the base of this fibration are real analogues of reduced GU-maps of [24, Definition 3.10].

3.4 Strata of simple real maps: properties

Let $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$ be as in (3.19) and $\overline{\gamma}$ be the stabilization of γ. Denote by $\mathcal{A}(\gamma)\subset\mathcal{A}(\overline{\gamma})$ the subset of pairs (γ',ϖ) with

$$\begin{aligned}\gamma' \equiv \Big((\mathfrak{g}',\mathfrak{d}')\colon \mathrm{Ver}'\longrightarrow\mathbb{Z}^{\geqslant0}\oplus H_2(X;\mathbb{Z}),\varepsilon'\colon S_{l;k}\sqcup\mathrm{Fl}'\longrightarrow\mathrm{Ver}',\vartheta'\colon\mathrm{Fl}'\longrightarrow\mathrm{Fl}',\\ \sigma'\colon\mathrm{Ver}'\sqcup\mathrm{Fl}'\longrightarrow\mathrm{Ver}'\sqcup\mathrm{Fl}'\Big)\in\mathcal{A}^{\phi}_{g',l;k}(B')\end{aligned} \tag{3.42}$$

such that

$$\mathfrak{d}'\big|_{\overline{\mathrm{Ver}}}=\mathfrak{d}\big|_{\overline{\mathrm{Ver}}}, \qquad g'+\big|\aleph(\gamma',\overline{\gamma})_{\bullet}\big|=g+\big|\aleph(\gamma)_{\bullet}\big|, \tag{3.43}$$

and there are maps

$$\kappa\colon\aleph(\gamma)_{\bullet}\longrightarrow\aleph(\gamma',\overline{\gamma})_{\bullet} \qquad\text{and}\qquad \varrho\colon\aleph(\gamma)_{\bullet}\longrightarrow\mathbb{Z}^{+} \tag{3.44}$$

so that κ is surjective and (σ',σ)-equivariant, ϱ is σ-invariant, and

$$\varrho(v)\mathfrak{d}'\big(\kappa(v)\big)=\mathfrak{d}(v) \qquad \forall\,v\in\aleph(\gamma)_{\bullet}\,. \tag{3.45}$$

In particular,

$$\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)\leqslant g'-g, \quad B'=B-\sum_{v'\in\aleph(\gamma',\overline{\gamma})_{\bullet}}\bigg(\sum_{v\in\kappa^{-1}(v')}\varrho(v)\;-1\bigg)\mathfrak{d}'(v')\,. \tag{3.46}$$

For example, $(\gamma,\varpi_{\gamma})\in\mathcal{A}(\gamma)$. The map κ is the identity in this case, while ϱ is the constant function with value 1. For suitable choices of the values of $\mathfrak{d}$ and $\mathfrak{d}'$ on the shaded vertices in the middle and last diagrams in Figure 2, $(\gamma',\varpi')\in\mathcal{A}(\gamma)$.

Lemma 3.5. *Suppose (X,ω,ϕ) is a real symplectic manifold, $g,l,k\in\mathbb{Z}^{\geqslant0}$ with $2(g+l)+k\geqslant3$, $B\in H_2(X;\mathbb{Z})$, and $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$. Let $\overline{\gamma}\in\mathcal{A}_{g,l;k}$ be the stabilization of γ and $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$. For every element $\mathbf{u}$ of $\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}$, there exist*

$$(\gamma',\varpi)\in\mathcal{A}(\gamma),\ \mathbf{u}'\in\mathcal{Z}^{*}_{\gamma',\varpi}(J,\nu) \qquad s.t. \quad \big\{\mathsf{ev}\times\mathsf{st}\big\}(\mathbf{u})=\big\{\mathsf{ev}\times\mathsf{st}_{\gamma',\varpi}\big\}(\mathbf{u}'). \tag{3.47}$$

Let γ, $\overline{\gamma}$, and $\mathbf{u}$ be as in (3.19), (3.10), and (3.27), respectively. We construct (γ',ϖ), with γ' as in (3.42),

$$\mathbf{u}'\equiv\big(u'\colon\Sigma'\longrightarrow X,(z'_f)_{f\in S_{l;k}},\sigma',\mathfrak{j}'\big)\equiv\big(u'_v\colon\Sigma'_v\longrightarrow X,(z'_f)_{f\in\varepsilon'^{-1}(v)},\sigma'_v,\mathfrak{j}'\big)_{v\in\mathrm{Ver}'}\in\mathfrak{M}^{*}_{\gamma',\varpi}(J,\nu),$$

and associated maps (3.44) explicitly below. The image of this element $\mathbf{u}'$ of $\mathfrak{M}^{*}_{\gamma',\varpi}(J,\nu)$ under the projection to the first component in (3.38) and (3.39) is a desired element of $\mathcal{Z}^{*}_{\gamma',\varpi}(J,\nu)$. The map $\mathbf{u}'$ keeps the irreducible components u_v of $\mathbf{u}$ with their marked and nodal points that either correspond to the domains Σ_v preserved by the stabilization (3.26) or are constant maps to X. These components are indexed by the vertices v in $\overline{\mathrm{Ver}}\sqcup\aleph(\gamma)_0$; the remaining irreducible components of $\mathbf{u}$ are indexed by the vertices v in $\aleph(\gamma)_{\bullet}$.

We replace each u_v with $v\in\aleph(\gamma)_\bullet$ by simple map u'_v with the same image and every set of such maps that have the same image in X by a single simple map $u'_{v'}$. The collection of maps $u'_{v'}$ obtained in this way is indexed by the vertices v' in the set $\aleph(\gamma',\overline{\gamma})_\bullet$ below, which is thus a quotient of $\aleph(\gamma)_\bullet$. This two-step replacement may send distinct marked and nodal points z_f of possibly different components u_v of $\mathbf{u}$ into the same point of the domain $\Sigma_{v'}\approx\mathbb{P}^1$ of $u'_{v'}$. We resolve such an accumulation by adding an extra contracted bubble $\Sigma_{v''}\approx\mathbb{P}^1$. The collection of the extra bubbles is indexed by the set $\mathscr{V}_0$ below.

The description above identifies the flags Fl of γ with the subset of the flags Fl' not forming the edges $e=\{v',v''\}$ as in the previous paragraph. We identify these flags into nodes of $\mathbf{u}'$ in the same way as for $\mathbf{u}$. This procedure preserves the evaluation maps at the marked points. Since the irreducible components u_v and u'_v with their marked points are the same whenever $v\in\overline{\mathrm{Ver}}$, $\mathbf{u}'$ remembers $\mathsf{st}(\mathbf{u})$.

An analogue of this procedure in the complex case without the resolution step is sketched after [24, Definition 3.10]. A similar procedure with $g=0$ and $\nu=0$ is described in the proof of [19, Proposition 6.1.2]. The tuple of maps produced by our two-step replacement procedure is in fact a real analogue of reduced GU-map of [24, Definition 3.10] and is a desired element of $\mathcal{Z}^*_{\gamma',\varpi}(J,\nu)$. The resolution step demonstrates that this tuple is indeed an element of $\mathcal{Z}^*_{\gamma',\varpi}(J,\nu)$ by producing an associated stable map in $\mathfrak{M}^*_{\gamma',\varpi}(J,\nu)$.

Proof of Lemma 3.5. By definition,

$$\begin{gathered}\mathrm{Ver}=\overline{\mathrm{Ver}}\sqcup\aleph(\gamma)_\bullet\sqcup\aleph(\gamma)_0,\qquad \mathrm{Ver}'=\overline{\mathrm{Ver}}\sqcup\aleph(\gamma',\overline{\gamma})_\bullet\sqcup\aleph(\gamma',\overline{\gamma})_0,\\ (\mathfrak{g}',\mathfrak{d}')|_{\overline{\mathrm{Ver}}}=(\mathfrak{g},\mathfrak{d})|_{\overline{\mathrm{Ver}}},\quad \mathfrak{g}'|_{\aleph(\gamma',\overline{\gamma})_\bullet}=0,\quad (\mathfrak{g}',\mathfrak{d}')\big|_{\aleph(\gamma',\overline{\gamma})_0}=(0,0),\quad \sigma'|_{\overline{\mathrm{Ver}}}=\sigma|_{\overline{\mathrm{Ver}}}.\end{gathered}$$

We produce γ' so that

$$\begin{gathered}\aleph(\gamma',\overline{\gamma})_0=\aleph(\gamma)_0\sqcup\mathscr{V}_0,\quad \mathrm{Fl}'=\mathrm{Fl}\sqcup\mathscr{F}_0,\quad |\mathscr{F}_0|=2|\mathscr{V}_0|,\quad \vartheta'|_{\mathrm{Fl}}=\vartheta,\quad \sigma'|_{\aleph(\gamma)_0\sqcup\mathrm{Fl}}=\sigma|_{\aleph(\gamma)_0\sqcup\mathrm{Fl}},\\ \varepsilon'\big|_{(S_{l;k}\sqcup\mathrm{Fl})-\varepsilon^{-1}(\aleph(\gamma)_\bullet)}=\varepsilon\big|_{(S_{l;k}\sqcup\mathrm{Fl})-\varepsilon^{-1}(\aleph(\gamma)_\bullet)},\quad \varepsilon'\big(\varepsilon^{-1}(\aleph(\gamma)_\bullet)\sqcup\mathscr{F}_0\big)\subset\aleph(\gamma',\overline{\gamma})_\bullet\sqcup\mathscr{V}_0,\end{gathered}$$

for some finite (possibly empty) sets $\mathscr{V}_0$ of additional degree 0 vertices and $\mathscr{F}_0$ of additional flags. Along with the surjectivity of κ, the condition $|\mathscr{F}_0|=2|\mathscr{V}_0|$ implies the second property in (3.43).

For each $v\in\overline{\mathrm{Ver}}\sqcup\aleph(\gamma)_0$, define

$$u'_v=u_v:\Sigma_v\longrightarrow X,\qquad z'_f=z_f\;\;\forall\,f\in\varepsilon'^{-1}(v)=\varepsilon^{-1}(v). \tag{3.48}$$

Thus, the components u_v and u'_v of $\mathbf{u}$ and $\mathbf{u}'$, respectively, corresponding to each element v of $\overline{\mathrm{Ver}}\sqcup\aleph(\gamma)_0$ are the same and carry the same marked and nodal points.

For each $v\in\aleph(\gamma)_\bullet$, there exist a branched cover $h_v:\mathbb{P}^1\longrightarrow\mathbb{P}^1$ and a simple J-holomorphic map $u'_v:\mathbb{P}^1\longrightarrow X$ such that $u_v=u'_v\circ h_v$; see [19, Proposition 2.5.1]. If u''_v is another simple J-holomorphic map such that $u''_v(\mathbb{P}^1)=u'_v(\mathbb{P}^1)$, then $u''_v=u'_v\circ h$ for some $h\in\mathrm{Aut}(\mathbb{P}^1)$. Let

$$\kappa:\aleph(\gamma)_\bullet\longrightarrow\aleph(\gamma',\overline{\gamma})_\bullet\equiv\aleph(\gamma)_\bullet/\!\sim,\quad v_1\sim v_2\;\text{ if }\;u_{v_1}(\mathbb{P}^1)=u_{v_2}(\mathbb{P}^1),$$

be the quotient map. Define

$$\begin{aligned}\varrho:\aleph(\gamma)_\bullet&\longrightarrow\mathbb{Z}^+, & \varrho(v)&=\deg h_v,\\ \mathfrak{d}':\aleph(\gamma',\overline{\gamma})_\bullet&\longrightarrow H_2(X;\mathbb{Z}), & \mathfrak{d}'(v')&=\big\{u'_{v*}\big\}_*[\mathbb{P}^1]\;\text{ if }\kappa(v)=v'.\end{aligned}$$

These two maps are well-defined, i.e. independent of the choices of h_v, u'_v, and $v\in\kappa^{-1}(v')$, and satisfy (3.45). The condition

$$\phi\big(u_{v_1}(\mathbb{P}^1)\big)=u_{v_2}\big(\mathbb{P}^1\big)\qquad\forall\ v_1\in\kappa^{-1}(v'),\ v_2\in\kappa^{-1}\big(\sigma'(v')\big),\ v'\in\aleph(\gamma',\overline{\gamma})_\bullet$$

determines an involution σ' on $\aleph(\gamma',\overline{\gamma})_\bullet$.

For each $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$, we pick $v\in\kappa^{-1}(v')$ and first set

$$u'_{v'}=u'_v\colon\mathbb{P}^1\longrightarrow X\,.$$

For each $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$, the map $\phi\circ u'_{v'}\circ\tau_1$ is simple and J-holomorphic and has the same image as $u'_{\sigma'(v')}$. If $\sigma'(v')\neq v'$, we can thus assume that

$$u'_{\sigma'(v')}=\phi\circ u'_{v'}\circ\tau_1\colon\mathbb{P}^1\longrightarrow X\,. \tag{3.49}$$

If $\sigma'(v')=v'$, the involution ϕ on $u'_{v'}(\mathbb{P}^1)$ determines a anti-holomorphic involution $\sigma'_{v'}$ on $\mathbb{P}^1$ such that

$$\phi\circ u'_{v'}=u'_{v'}\circ\sigma'_{v'}\,. \tag{3.50}$$

By replacing $u'_{v'}$ with $u'_{v'}\circ h$ for some $h\in\mathrm{Aut}(\mathbb{P}^1)$, we can assume that $\sigma'_{v'}\in\{\tau,\eta\}$. We set $\sigma'_{v'}=\tau_1$ if $\sigma'(v')\neq v'$.

For all $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$ and $v\in\kappa^{-1}(v')$, $u'_{v'}(\mathbb{P}^1)=u_v(\mathbb{P}^1)$. For every $f\in\varepsilon^{-1}(v)$, there thus exists $\check{z}'_f\in\mathbb{P}^1$ such that $u'_{v'}(\check{z}'_f)=u_v(z_f)$. Since

$$u'_{\sigma'(v')}\big(\check{z}'_{\sigma(f)}\big)=u_{\sigma(v)}(z_{\sigma(f)})=\phi\big(u_v(z_f)\big)=\phi\big(u'_{v'}(\check{z}'_f)\big)=u'_{\sigma'(v')}\big(\sigma'_{v'}(\check{z}'_f)\big),$$

these points $\check{z}'_f$ can be chosen so that

$$\check{z}'_{\sigma(f)}=\sigma'_{v'}\big(\check{z}'_f\big)\qquad\forall\ f\in\varepsilon^{-1}(v),\ v\in\kappa^{-1}(v'),\ v'\in\aleph(\gamma',\overline{\gamma})_\bullet. \tag{3.51}$$

For each $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$, let

$$\varepsilon^{-1}\big(\kappa^{-1}(v')\big)=\bigsqcup_{w\in\mathscr{V}_{0;v'}}\mathscr{F}_w$$

be the decomposition so that

$$\begin{aligned}\check{z}'_{f_1}&=\check{z}'_{f_2}\quad\text{if}\ \ f_1,f_2\in\mathscr{F}_w,\ w\in\mathscr{V}_{0;v'},\\ \check{z}'_{f_1}&\neq\check{z}'_{f_2}\quad\text{if}\ \ f_1\in\mathscr{F}_{w_1},\ f_2\in\mathscr{F}_{w_2},\ w_1,w_2\in\mathscr{V}_{0;v'},\ w_1\neq w_2.\end{aligned}$$

By (3.51), there is an involution σ'

$$\sigma'\colon\mathscr{V}_0\equiv\bigsqcup_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}\big\{w\in\mathscr{V}_{0;v'}\colon|\mathscr{F}_w|\geqslant2\big\}\longrightarrow\mathscr{V}_0$$

such that $\sigma(f)\in\mathscr{F}_{\sigma'(w)}$ for all $f\in\mathscr{F}_w$ and $w\in\mathscr{V}_0$.

Define

$$\sigma'\colon \mathscr{F}_0 \equiv \bigsqcup_{w\in\mathscr{V}_0}\{f_w^-,f_w^+\} \longrightarrow \mathscr{F}_0, \qquad \sigma'\big(f_w^\pm\big)=f_{\sigma'(w)}^\pm,$$

$$\varepsilon'\colon \mathscr{F}_0\sqcup\varepsilon^{-1}(\aleph(\gamma)_\bullet)\longrightarrow \mathrm{Ver}', \quad \varepsilon'(f)=\begin{cases} v', & \text{if } w\in\mathscr{V}_{0;v'},\, f=f_w^- \text{ or } f\in\mathscr{F}_w,\ |\mathscr{F}_w|=1;\\ w, & \text{if } w\in\mathscr{V}_{0;v'},\, f=f_w^+ \text{ or } f\in\mathscr{F}_w,\ |\mathscr{F}_w|\geqslant 2.\end{cases}$$

This completes the specification of γ'. We take

$$\varpi=\varpi_\gamma\colon S_{l;k}\cap\varepsilon'^{-1}\big(\aleph(\gamma',\overline{\gamma})\big)=S_{l;k}\cap\varepsilon^{-1}\big(\aleph(\gamma)\big)\longrightarrow \mathscr{F}_\gamma^*(\overline{\mathrm{Ver}})-\overline{\mathrm{Fl}}=\mathscr{F}_{\gamma'}^*(\overline{\mathrm{Ver}})-\overline{\mathrm{Fl}}$$

to be the injective map corresponding to the contraction of γ to $\overline{\gamma}$. This defines $(\gamma',\varpi)\in\mathcal{A}(\gamma)$.

Let $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$ and $w\in\mathscr{V}_{0;v'}$. If $\mathscr{F}_w=\{f\}$ consists of a single element, we take

$$z'_f=\check{z}'_f\in\Sigma_{\varepsilon'(f)}\equiv\Sigma_{v'}=\mathbb{P}^1.$$

If $|\mathscr{F}_w|\geqslant 2$ and $f\in\mathscr{F}_w$, we take

$$z'_{f_w^-}=\check{z}'_f\in\Sigma_{\varepsilon'(f_w^-)}\equiv\Sigma_{v'}=\mathbb{P}^1, \qquad z'_{f_w^+}=1\in\Sigma_{\varepsilon'(f_w^+)}\equiv\Sigma_w=\mathbb{P}^1,$$
$$u'_w\colon\Sigma_w\longrightarrow X,\ u'_w(z)=u'_{v'}(\check{z}'_f)\quad\forall\, z\in\Sigma_w,$$
$$\sigma'_w=\tau\colon\Sigma_w\longrightarrow\Sigma_w \ \text{ if } \sigma'(w)=w, \quad \sigma'_w=\tau_1\colon\Sigma_w\longrightarrow\Sigma_{\sigma'(w)} \ \text{ if } \sigma'(w)\neq w.$$

These definitions are independent of the choice of $f\in\mathscr{F}_w$. In this case, we also choose distinct points

$$z'_f\in\Sigma_w-\{1\}\subset\Sigma_{\varepsilon'(f)}, \qquad f\in\mathscr{F}_w.$$

They can be chosen so that

$$\sigma'_w\big(z'_f\big)=z'_{\sigma(f)}\qquad\forall\ f\in\mathscr{F}_w,\, w\in\mathscr{V}_0.$$

The maps $u_{v'}\colon\mathbb{P}^1\longrightarrow X$ with $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$ such that $|\varepsilon'^{-1}(v')|\leqslant 2$ can be reparametrized to achieve (3.21) and (3.22) with z_f,γ,ε replaced by $z'_f,\gamma',\varepsilon'$, while preserving the conditions (3.49) and (3.50). This completes the specification of a map $\mathbf{u}'$ with dual graph γ' that satisfies (3.21) and (3.22). Since

$$u'_{\varepsilon'(f)}(z'_f)=u_{\varepsilon(f)}(z_f)=u_{\varepsilon(\vartheta(f))}\big(z_{\vartheta(f)}\big)=u'_{\varepsilon'(\vartheta'(f))}\big(z'_{\vartheta'(f)}\big)\quad\forall\, f\in\mathrm{Fl},$$
$$u'_{v'}\big(z'_{f_w^-}\big)=u'_w\big(z'_{f_w^+}\big)\quad\forall\ w\in\mathscr{V}_0,$$

this map is continuous at the nodes. Furthermore,

$$u'_{\varepsilon'(f)}(z'_f)=u_{\varepsilon(f)}(z_f)\qquad\forall\ f\in S_{l;k}. \tag{3.52}$$

Let $\rho'\in\mathcal{A}(\gamma')$ be given by $\rho'(v')=\sigma'_{v'}$ whenever $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$, $\sigma'(v')=v'$, and $|\varepsilon'^{-1}(v')|\leqslant 2$. By (3.48) and the choice of ϖ,

$$\begin{aligned} q_{\gamma',\varpi;v}&=q_{\gamma,\varpi_\gamma;v}\colon\Sigma_v\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}\quad\forall\ v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma}),\\ q^\bullet_{\gamma',\varpi;v}&=q^\bullet_{\gamma,\varpi_\gamma;v}\colon\Sigma_v\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}\quad\forall\ v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma}).\end{aligned} \tag{3.53}$$

Thus,

$$\nu_{\gamma,\varpi';\rho'}\big|_{\Sigma_v}=\nu_{\gamma,\varpi;\rho}|_{\Sigma_v},\quad \bar\partial_{J,\mathfrak j}u'_v\big|_z=\bar\partial_{J,\mathfrak j}u_v\big|_z=\nu_{\gamma,\varpi;\rho}\big(z,u_v(z)\big)=\nu_{\gamma',\varpi;\rho'}\big(z,u'_v(z)\big)\quad\forall\,z\in\Sigma_v$$

for all $v\in\mathrm{Ver}'$. By the construction, $u'_{v'}$ is a simple map for every $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$ and

$$u'_{v'_1}(\mathbb P^1)\neq u'_{v'_2}(\mathbb P^1)\qquad\forall\ v'_1,v'_2\in\aleph(\gamma',\overline{\gamma})_\bullet,\ v'_1\neq v'_2.$$

Thus, $\mathbf u'\in\widetilde{\mathfrak M}^*_{\gamma',\varpi;\rho'}(J,\nu)$. By (3.52) and (3.53), $[\mathbf u']$ satisfies the condition in (3.47). □

Let $\mathbb I=[0,1]$. For (J,ν) and (J',ν') in $\mathcal H^{\omega,\phi}_{g,l;k}(X)$, define

$$\mathscr P(J,\nu;J',\nu')=\big\{\alpha\colon[0,1]\longrightarrow\mathcal H^{\omega,\phi}_{g,l;k}(X)\colon\alpha(0)=(J,\nu),\ \alpha(1)=(J',\nu')\big\}$$

to be the space of paths from (J,ν) and (J',ν'). For any such path α, let

$$\begin{aligned}\overline{\mathfrak M}_{g,l;k}(B;\alpha)^\phi&=\big\{\big(t,[\mathbf u]\big)\colon t\in\mathbb I,\ [\mathbf u]\in\overline{\mathfrak M}_{g,l;k}\big(X,B;\alpha(t)\big)^\phi\big\},\\ \mathfrak M^\star_{g,l;k}(B;\alpha)^\phi&=\big\{\big(t,[\mathbf u]\big)\colon t\in\mathbb I,\ [\mathbf u]\in\mathfrak M^\star_{g,l;k}\big(X,B;\alpha(t)\big)^\phi\big\}.\end{aligned}$$

If in addition $\overline{\gamma}\in\mathcal A_{g,l;k}$ and $(\gamma,\varpi)\in\mathcal A(\overline{\gamma})$, let

$$\mathcal Z^*_{\gamma,\varpi}(\alpha)=\big\{\big(t,[\mathbf u]\big)\colon t\in\mathbb I,\ [\mathbf u]\in\mathcal Z^*_{\gamma,\varpi}\big(\alpha(t)\big)\big\}\,.$$

These spaces are again topologized as in [16, Section 3] so that the maps

$$\begin{aligned}\mathsf{st}\times\mathsf{ev}\colon\mathfrak M^\star_{g,l;k}(B;\alpha)^\phi&\longrightarrow\mathbb R\overline{\mathcal M}_{g,l;k}\times\big(X^l\times(X^\phi)^k\big),&(3.54)\\ \mathsf{st}_{\gamma,\varpi}\times\mathsf{ev}\colon\mathcal Z^*_{\gamma,\varpi}(\alpha)&\longrightarrow\mathbb R\overline{\mathcal M}_{g,l;k}\times\big(X^l\times(X^\phi)^k\big),&(3.55)\end{aligned}$$

induced by (3.5) and (3.40) are continuous.

For $g,l,k\in\mathbb Z^{\geqslant0}$ and $B\in H_2(X;\mathbb Z)$, let

$$\dim_{g,l;k}(B)=\big\langle c_1(TX),B\big\rangle+(n-3)(1-g)+2l+k.\tag{3.56}$$

For $\overline{\gamma}\in\mathcal A_{g,l;k}$ and $(\gamma,\varpi)\in\mathcal A(\overline{\gamma})$ with $\gamma\in\mathcal A^\phi_{g',l;k}(B')$, let

$$\dim_{\gamma,\varpi}=\dim_{g',l;k}(B')-|\gamma|+n\,\ell\big(\gamma,\aleph(\gamma,\overline{\gamma})_0\big)-\big(\dim_{\mathbb R}\mathbb R\mathcal M_{\gamma;\aleph(\gamma,\overline{\gamma})}+\big|\mathscr F^\dagger_\gamma(\aleph(\gamma,\overline{\gamma})_0)\big|\big).\tag{3.57}$$

Propositions 3.6 and 3.7 below are analogous to [19, Theorem 6.2.6] and [24, Theorem 3.16]; they are established in Section 4.

Proposition 3.6. *Let (X,ω,ϕ) be a real symplectic $2n$-manifold. For all $g,l,k\in\mathbb Z^{\geqslant0}$ with $2(g+l)+k\geqslant3$ and $B\in H_2(X;\mathbb Z)$, there exists a Baire subset*

$$\widehat{\mathcal H}^{\omega,\phi}_{g,l;k}(X)\subset\mathcal H^{\omega,\phi}_{g,l;k}(X)\tag{3.58}$$

of second category such that for every $(J,\nu)\in\widehat{\mathcal H}^{\omega,\phi}_{g,l;k}(X)$

(1) $\mathfrak M^\star_{g,l;k}(X,B;J,\nu)^\phi$ is a smooth manifold of dimension (3.56) and the restriction of (3.5) to $\mathfrak M^\star_{g,l;k}(X,B;J,\nu)^\phi$ is a smooth map,

(2) $\mathcal{Z}^*_{\gamma,\varpi}(J,\nu)$ *is a smooth manifold of dimension (3.57) and the restriction of (3.40) to* $\mathcal{Z}^*_{\gamma,\varpi}(J,\nu)$ *is a smooth map for all* $(\gamma,\varpi)\in\mathcal{A}(\overline{\gamma})$ *satisfying* $\overline{\gamma}\in\mathcal{A}_{g,l;k}$, $\gamma\in\mathcal{A}^{\phi}_{g',l;k}(B')$, $B'\in H_2(X;\mathbb{Z})$, *and* $\omega(B')\leqslant\omega(B)$.

Proposition 3.7. *Let* (X,ω,ϕ), g,l,k, B, *and* $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l;k}(X)$ *be as in Proposition 3.6. For all elements* (J,ν) *and* (J',ν') *of* $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l;k}(X)$, *there exists a Baire subset*

$$\widehat{\mathscr{P}}(J,\nu;J',\nu')\subset\mathscr{P}(J,\nu;J',\nu') \tag{3.59}$$

of second category such that for every $\alpha\in\widehat{\mathscr{P}}(J,\nu;J',\nu')$

(1) $\mathfrak{M}^{\star}_{g,l;k}(\alpha)^{\phi}$ *is a smooth manifold with boundary of dimension* $\dim_{g,l;k}(B)+1$ *and (3.54) is a smooth map,*

(2) $\mathcal{Z}^*_{\gamma,\varpi}(\alpha)$ *is a smooth manifold with boundary of dimension* $\dim_{\gamma,\varpi}+1$ *and (3.55) is a smooth map for all* $(\gamma,\varpi)\in\mathcal{A}(\overline{\gamma})$ *satisfying* $\overline{\gamma}\in\mathcal{A}_{g,l;k}$, $\gamma\in\mathcal{A}^{\phi}_{g',l;k}(B')$, $B'\in H_2(X;\mathbb{Z})$, *and* $\omega(B')\leqslant\omega(B)$.

Corollary 3.8. *Let* $n\notin 2\mathbb{Z}$ *and* (X,ω,ϕ) *be a compact real symplectic* $2n$*-manifold endowed with a real orientation. For all* $g,l\in\mathbb{Z}^{\geqslant 0}$ *with* $g+l\geqslant 2$ *and* $B\in H_2(X;\mathbb{Z})$, *there exists a Baire subset* $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)\subset\mathcal{H}^{\omega,\phi}_{g,l;0}(X)$ *of second category such that* $\mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi}$ *is an oriented manifold of dimension (3.8).*

Proof. By Proposition 3.6(1), $\mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi}$ is a smooth manifold of the expected dimension. By [10, Theorem 1.3], a real orientation (X,ω,ϕ) determines an orientation on this space. □

Corollary 3.9. *Let* n, (X,ω,ϕ), g,l, B, *and* $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)$ *be in Corollary 3.8. For all elements* (J,ν) *and* (J',ν') *of* $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)$, *there exists a Baire subset of second category as in (3.59) such that* $\mathfrak{M}^{\star}_{g,l;0}(B;\alpha)^{\phi}$ *is an oriented manifold with boundary*

$$\partial\,\mathfrak{M}^{\star}_{g,l;0}(B;\alpha)^{\phi}=\mathfrak{M}^{\star}_{g,l;0}(X,B;J',\nu')^{\phi}-\mathfrak{M}^{\star}_{g,l;0}(X,B;J,\nu)^{\phi} \tag{3.60}$$

for every $\alpha\in\widehat{\mathscr{P}}(J,\nu;J',\nu')$.

Proof. By Proposition 3.7(1), $\mathfrak{M}^{\star}_{g,l;0}(B;\alpha)^{\phi}$ is a smooth manifold with boundary; its boundary is as specified by (3.60). By the proof of [10, Theorem 1.3], a real orientation (X,ω,ϕ) determines an orientation on $\mathfrak{M}^{\star}_{g,l;0}(B;\alpha)^{\phi}$. □

3.5 Proof of Theorem 3.3

We deduce Proposition 3.10 below from Proposition 3.6 primarily through dimension counting. Proposition 3.11 is obtained similarly from Proposition 3.7. We then combine these propositions with Corollaries 3.8 and 3.9 to establish Theorem 3.3.

Proposition 3.10. *Let* (X,ω,ϕ) *be a compact semi-positive real symplectic manifold. For all* $g,l,k\in\mathbb{Z}^{\geqslant 0}$ *with* $2(g+l)+k\geqslant 3$ *and* $B\in H_2(X;\mathbb{Z})$, *there exists a Baire subset of second category as in (3.58) such that*

$$\dim\big\{\mathsf{st}\times\mathsf{ev}\big\}\big(\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}-\mathfrak{M}^{\star}_{g,l;k}(X,B;J,\nu)^{\phi}\big)\leqslant\dim_{g,l;k}(B)-2 \tag{3.61}$$

for every $(J,\nu)\in\widehat{\mathcal{H}}^{\omega,\phi}_{g,l;k}(X)$.

Proposition 3.11. *Let (X,ω,ϕ), g,l,k, B, and $\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$ be as in Proposition 3.10. For all elements (J,ν) and (J',ν') of $\widehat{\mathcal{H}}^{\omega,\phi}_{g,l;k}(X)$, there exists a Baire subset of second category as in (3.59) such that*

$$\dim\big\{\mathsf{st}\times\mathsf{ev}\big\}\big(\overline{\mathfrak{M}}_{g,l;k}(B;\alpha)^{\phi}-\mathfrak{M}^{\star}_{g,l;k}(B;\alpha)^{\phi}\big)\leqslant \dim_{g,l;k}(B)-1$$

for every $\alpha\in\widehat{\mathscr{P}}(J,\nu;J',\nu')$.

As with the definitions of pseudocycle in [33, 19], (3.61) means that

$$\big\{\mathsf{st}\times\mathsf{ev}\big\}\big(\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^{\phi}-\mathfrak{M}^{\star}_{g,l;k}(X,B;J,\nu)\big)^{\phi}\subset\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\times\big(X^l\times(X^{\phi})^k\big)$$

is contained in the image of a smooth map from a manifold of dimension equal to the right-hand side of this inequality. Thus, the restriction

$$\mathsf{st}\times\mathsf{ev}\colon\mathfrak{M}^{\star}_{g,l;k}(X,B;J,\nu)^{\phi}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l;k}\times\big(X^l\times(X^{\phi})^k\big)$$

is a pseudocycle whenever $(J,\nu)\in\widehat{\mathcal{H}}^{\omega,\phi}_{g,l;k}(X)$ *and* the domain of this map is an oriented manifold.

Let $B\in H_2(X;\mathbb{Z})-\{0\}$. For $J\in\mathcal{J}^{\phi}_{\omega}$ and $\sigma=\tau,\eta$, we denote by

$$\mathfrak{M}^*_0(X,B;J)\subset\overline{\mathfrak{M}}_0(X,B;J)\qquad\text{and}\qquad\mathfrak{M}^*_0(X,B;J)^{\sigma,\phi}\subset\overline{\mathfrak{M}}_0(X,B;J)^{\sigma,\phi}$$

the moduli spaces of equivalences classes of simple degree B J-holomorphic maps from $\mathbb{P}^1$ to X and of simple real degree B J-holomorphic maps from $(\mathbb{P}^1,\sigma)$ to (X,ϕ), respectively. Let

$$\mathfrak{M}^*_0(X,B;J)^{\phi}\equiv\mathfrak{M}^*_0(X,B;J)^{\phi,\tau}\sqcup\mathfrak{M}^*_0(X,B;J)^{\phi,\eta}\subset\overline{\mathfrak{M}}_0(X,B;J)^{\phi}$$

be the space of all simple real degree B J-holomorphic maps from $\mathbb{P}^1$ to (X,ϕ). We note that

$$\mathfrak{M}^*_0(X,B;J)=\varnothing\quad\text{if}\ \ \omega(B)\leqslant 0,$$
$$\mathfrak{M}^*_0(X,B;J)^{\phi}=\varnothing\ \ \text{if}\ B\notin H_2^{\mathbb{R}S}(X;\mathbb{Z})^{\phi},\qquad \mathfrak{M}^*_0(X,B;J)^{\phi,\tau}=\varnothing\ \ \text{if}\ B\notin H_2^{\tau}(X;\mathbb{Z})^{\phi}.$$

The natural morphism

$$\overline{\mathfrak{M}}_0(X,B;J)^{\phi}\longrightarrow\overline{\mathfrak{M}}_0(X,B;J)$$

restricts to an embedding

$$\mathfrak{M}^*_0(X,B;J)^{\phi}\longrightarrow\mathfrak{M}^*_0(X,B;J);$$

we view $\mathfrak{M}^*_0(X,B;J)^{\phi}$ as a subspace of $\mathfrak{M}^*_0(X,B;J)$.

For $J,J'\in\mathcal{J}^{\phi}_{\omega}$, define

$$\mathscr{P}(J;J')=\big\{\alpha\colon[0,1]\longrightarrow\mathcal{J}^{\phi}_{\omega}\colon\alpha(0)=J,\ \alpha(1)=J'\big\}$$

to be the space of paths from J and J'. For any such path α and each $B\in H_2(X;\mathbb{Z})-\{0\}$, let

$$\mathfrak{M}^*_0(B;\alpha)=\big\{\big(t,[\mathbf{u}]\big)\colon t\in\mathbb{I},\ [\mathbf{u}]\in\mathfrak{M}^*_0\big(X,B;\alpha(t)\big)\big\},$$
$$\mathfrak{M}^*_0(B;\alpha)^{\phi}=\big\{\big(t,[\mathbf{u}]\big)\colon t\in\mathbb{I},\ [\mathbf{u}]\in\mathfrak{M}^*_0\big(X,B;\alpha(t)\big)^{\phi}\big\},$$
$$\mathfrak{M}^*_0(B;\alpha)^{\phi,\tau}=\big\{\big(t,[\mathbf{u}]\big)\colon t\in\mathbb{I},\ [\mathbf{u}]\in\mathfrak{M}^*_0\big(X,B;\alpha(t)\big)^{\phi,\tau}\big\}.$$

Lemma 3.12. *Let (X,ω,ϕ) be a compact real symplectic $2n$-manifold. For every $B\in H_2(X;\mathbb{Z})$, there exists a Baire subset $\widehat{\mathcal{J}}_\omega^\phi\subset\mathcal{J}_\omega^\phi$ of second category with the following property. If $J,J'\in\widehat{\mathcal{J}}_\omega^\phi$, there exists a Baire subset*

$$\widehat{\mathscr{P}}(J;J')\subset\mathscr{P}(J;J') \tag{3.62}$$

of second category such that for every $\alpha\in\widehat{\mathscr{P}}(J;J')$

$$\begin{aligned}
\mathfrak{M}_0^*(B';\alpha)&=\mathfrak{M}_0^*(B';\alpha)^\phi && \text{if } B'\in H_2(X;\mathbb{Z}),\ 0<\omega(B')\leqslant\omega(B),\ \langle c_1(TX),B'\rangle<3-n,\\
\mathfrak{M}_0^*(B';\alpha)^\phi&=\varnothing && \text{if } B'\in H_2(X;\mathbb{Z}),\ 0<\omega(B')\leqslant\omega(B),\ \langle c_1(TX),B'\rangle<2-n.
\end{aligned}$$

Proof. By Section 4, there exists a Baire subset $\widehat{\mathcal{J}}_\omega^\phi\subset\mathcal{J}_\omega^\phi$ of second category with the following property. If $J,J'\in\widehat{\mathcal{J}}_\omega^\phi$, there exists a Baire subset of second category as in (3.62) such that for all

$$\alpha\in\widehat{\mathscr{P}}(J;J') \qquad\text{and}\qquad B'\in H_2(X;\mathbb{Z}) \ \text{ with } \ 0<\omega(B')\leqslant\omega(B)$$

the moduli spaces $\mathfrak{M}_0^*(B';\alpha)-\mathfrak{M}_0^*(B';\alpha)^\phi$ and $\mathfrak{M}_0^*(B';\alpha)^\phi$ are smooth manifolds of dimensions

$$\begin{aligned}
\dim_{\mathbb{R}}\big(\mathfrak{M}_0^*(B';\alpha)-\mathfrak{M}_0^*(B';\alpha)^\phi\big)&=2\big(\langle c_1(TX),B'\rangle+n-3\big)+1,\\
\dim_{\mathbb{R}}\mathfrak{M}_0^*(B';\alpha)^\phi&=\langle c_1(TX),B'\rangle+n-3+1.
\end{aligned}$$

This implies the claim. □

Corollary 3.13. *Let (X,ω,ϕ) be a compact semi-positive real symplectic $2n$-manifold. For every $B\in H_2(X;\mathbb{Z})$, there exists a Baire subset $\widehat{\mathcal{J}}_\omega^\phi\subset\mathcal{J}_\omega^\phi$ of second category with the following property. If $J,J'\in\widehat{\mathcal{J}}_\omega^\phi$, there exists a Baire subset of second category as in (3.62) such that for every $\alpha\in\widehat{\mathscr{P}}(J;J')$*

$$\begin{aligned}
\langle c_1(TX),B'\rangle&\geqslant 0,3-n && \text{if } B'\in H_2(X;\mathbb{Z}),\ 0<\omega(B')\leqslant\omega(B),\ \mathfrak{M}_0^*(B';\alpha)\neq\varnothing, && (3.63)\\
\langle c_1(TX),B'\rangle&\geqslant\delta_{n2} && \text{if } B'\in H_2(X;\mathbb{Z}),\ 0<\omega(B')\leqslant\omega(B),\ \mathfrak{M}_0^*(B';\alpha)^\phi\neq\varnothing, && (3.64)\\
\langle c_1(TX),B'\rangle&\geqslant 1 && \text{if } B'\in H_2(X;\mathbb{Z}),\ 0<\omega(B')\leqslant\omega(B),\ \mathfrak{M}_0^*(B';\alpha)^{\phi,\tau}\neq\varnothing. && (3.65)
\end{aligned}$$

Proof. The first inequality in (3.63), (3.64), and (3.65) follow immediately from Lemma 3.12 and Definition 1.2. If

$$\mathfrak{M}_0^*(B';\alpha)\neq\varnothing \qquad\text{and}\qquad \langle c_1(TX),B'\rangle<3-n,$$

then the first statement of Lemma 3.12 gives

$$\mathfrak{M}_0^*(B';\alpha)^\phi=\mathfrak{M}_0^*(B';\alpha)\neq\varnothing\,.$$

By the second statements of Lemma 3.12 and of the present corollary, this implies that

$$2-n=\langle c_1(TX),B'\rangle\geqslant\delta_{n2}.$$

Thus, $n=1$ and $X=\mathbb{P}^1$ (X can be assumed to be connected). However, $\langle c_1(TX),B'\rangle$ is even for every homology class B' on $X=\mathbb{P}^1$. □

Suppose $g,l,k\in\mathbb{Z}^{\geqslant0}$ with $2(g+l)+k\geqslant3$, $B\in H_2(X;\mathbb{Z})$, and $\gamma\in\mathcal{A}_{g,l;k}^\phi(B)$. For $(\gamma',\varpi)\in\mathcal{A}'(\gamma)$, let

$$[\![\gamma',\varpi]\!]=|\gamma'|-\big|\mathscr{E}_{\gamma'}\big(\aleph(\gamma',\overline{\gamma})_0\big)\big|-\big|\pi_0\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)\big|\,.$$

Since γ' is connected,

$$[\![\gamma',\varpi]\!] \geqslant \big|\aleph(\gamma',\overline{\gamma})_\bullet\big|\,. \tag{3.66}$$

By the second condition in (3.43),

$$[\![\gamma',\varpi]\!] + \ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big) = g' + \big|\aleph(\gamma',\overline{\gamma})_\bullet\big| + \big|\overline{\mathrm{Ver}}\big| - 1 \geqslant \big|\aleph(\gamma)_\bullet\big|. \tag{3.67}$$

We denote $\mathcal{A}'(\gamma)\subset\mathcal{A}(\gamma)$ the subset of pairs (γ',ϖ) such that

$$|\varrho|_{v'} \equiv \sum_{v\in\kappa^{-1}(v')}\varrho(v) - 1 \;\geqslant 1 \tag{3.68}$$

for some $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$, where κ and ϱ are the maps (3.44) corresponding to (γ',ϖ).

Corollary 3.14. *Let (X,ω,ϕ) be a compact semi-positive real symplectic $2n$-manifold. For all $g,l,k\in\mathbb{Z}^{\geqslant 0}$ with $2(g+l)+k\geqslant 3$ and $B\in H_2(X;\mathbb{Z})$, there exists a Baire subset of second category as in (3.58) with the following property. For all elements (J,ν) and (J',ν') of this subset, there exists a Baire subset of second category as in (3.59) such that*

$$\big\langle c_1(TX), B-B'\big\rangle + (n-3)\big(g'-g-\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)\big) + [\![\gamma',\varpi]\!] \geqslant 2$$

for every $\alpha\in\widehat{\mathscr{P}}(J,\nu;J',\nu')$, $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$, and $(\gamma',\varpi)\in\mathcal{A}'(\gamma)$ with $\gamma'\in\mathcal{A}^\phi_{g',l;k}(B')$ and $\mathcal{Z}^*_{\gamma';\varpi}(\alpha)\neq\varnothing$.

Proof. Take the Baire subsets to be the preimages of the subsets

$$\widehat{\mathcal{J}}^\phi_\omega \subset \mathcal{J}^\phi_\omega \qquad \text{and} \qquad \widehat{\mathscr{P}}(J;J')\subset\mathscr{P}(J;J')$$

of Corollary 3.13 under the natural projections

$$\mathcal{H}^{\omega,\phi}_{g,l;k}(X) \longrightarrow \mathcal{J}^\phi_\omega \qquad \text{and} \qquad \mathscr{P}(J,\nu;J',\nu') \longrightarrow \mathscr{P}(J;J')\,. \tag{3.69}$$

Suppose $\alpha\in\widehat{\mathscr{P}}(J,\nu;J',\nu')$, $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$, and $(\gamma',\varpi)\in\mathcal{A}'(\gamma)$ with γ' as in (3.42) and $\mathfrak{M}^*_{\gamma';\varpi}(\alpha)\neq\varnothing$. Let $\alpha_{\mathcal{J}}$ be the image of α under the second projection in (3.69) and κ and ϱ be the maps (3.44) corresponding to (γ',ϖ).

We first note that

$$\sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet} |\varrho|_{v'}\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle + [\![\gamma',\varpi]\!] \geqslant 2, \tag{3.70}$$

$$\sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}\sum_{v\in\kappa^{-1}(v')}\big(\varrho(v)-1\big)\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle + [\![\gamma',\varpi]\!] + \ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big) \geqslant 2. \tag{3.71}$$

By the first inequality in (3.63),

$$\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle \geqslant 0 \qquad \forall\; v'\in\aleph(\gamma',\overline{\gamma})_\bullet\,. \tag{3.72}$$

If the inequality in (3.66) is an equality and $|\aleph(\gamma',\overline{\gamma})_\bullet|=1$, then there is a unique flag $f\in\mathrm{Fl}'$ such that

- $v'=\varepsilon'(f)$ is the unique element of $\aleph(\gamma',\overline{\gamma})_\bullet$ and

- the removal of the edge $e=\{f,\vartheta'(f)\}$ separates v' from all vertices $\overline{\mathrm{Ver}}\subset\mathrm{Ver}'$.

The unique flag f is then preserved by the involution σ' on γ'. It thus corresponds to a real point of the irreducible component $\Sigma_{v'}$ of the domain of any element of $\mathfrak{M}^*_{\gamma';\varpi}(\alpha)$. It follows that the moduli space $\mathfrak{M}^*_0(\mathfrak{d}'(v');\alpha_{\mathcal{J}})^{\phi,\tau}$ is not empty. The inequality (3.70) thus follows from (3.65), (3.66), and (3.72) in all cases. If $\varrho(v)\geqslant 2$ for some $v\in\aleph(\gamma)_\bullet$, (3.71) also follows from (3.65), (3.66), and (3.72). Otherwise, $|\aleph(\gamma)_\bullet|\geqslant 2$ and (3.71) follows from (3.67) and (3.72).

By the last statement in (3.46),

$$\begin{aligned}\big\langle c_1(TX),B-B'\big\rangle &= \sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}|\varrho|_{v'}\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle\\ &=\sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}\sum_{v\in\kappa^{-1}(v')}\big(\varrho(v)-1\big)\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle-(n-3)\big(|\aleph(\gamma)_\bullet|-|\aleph(\gamma',\overline{\gamma})_\bullet|\big)\\ &\quad+\sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}\big(|\kappa^{-1}(v')|-1\big)\big(\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle+n-3\big).\end{aligned}\tag{3.73}$$

For $n\geqslant 3$, the claim follows from the first equality above, the first statement in (3.46), and (3.70). For $n<3$, the second equality in (3.73), the second statement in (3.43), and the second inequality in (3.63) give

$$\begin{aligned}&\big\langle c_1(TX),B-B'\big\rangle+(n-3)\big(g'-g-\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)\big)\\ &\qquad\geqslant\sum_{v'\in\aleph(\gamma',\overline{\gamma})_\bullet}\sum_{v\in\kappa^{-1}(v')}\big(\varrho(v)-1\big)\big\langle c_1(TX),\mathfrak{d}'(v')\big\rangle+\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big).\end{aligned}$$

The claim now follows from (3.71). □

Proof of Proposition 3.10. For each $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$, let

$$\mathfrak{M}^*_\gamma(J,\nu)\subset\mathfrak{M}_\gamma(J,\nu)\quad\text{and}\quad\mathfrak{M}^{mc}_\gamma(J,\nu)\equiv\mathfrak{M}_\gamma(J,\nu)-\mathfrak{M}^*_\gamma(J,\nu)\subset\mathfrak{M}_\gamma(J,\nu)$$

denote the subspace of simple maps in the sense of Definition 3.2 and the subspace of multiply covered maps, respectively. In particular,

$$\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^\phi-\mathfrak{M}^\star_{g,l;k}(X,B;J,\nu)^\phi=\bigsqcup_{\substack{\gamma\in\mathcal{A}^\phi_{g,l;k}(B)\\|\gamma|\geqslant 2}}\mathfrak{M}^*_\gamma(J,\nu)\quad\sqcup\quad\bigsqcup_{\substack{\gamma\in\mathcal{A}^\phi_{g,l;k}(B)\\\aleph(\gamma)\neq\varnothing}}\mathfrak{M}^{mc}_\gamma(J,\nu)\,.$$

Since the map (3.5) is continuous and

$$\big|\big\{\gamma\in\mathcal{A}^\phi_{g,l;k}(B):\mathfrak{M}_\gamma(J,\nu)\neq\varnothing\big\}\big|<\infty\,,$$

it is thus sufficient to show that

$$\dim_{\mathbb{R}}\mathfrak{M}^*_\gamma(J,\nu)\leqslant\dim_{g,l;k}(B)-2\qquad\forall\ \gamma\in\mathcal{A}^\phi_{g,l;k}(B),\ |\gamma|\geqslant 2,\tag{3.74}$$

$$\dim\big\{\mathsf{st}\times\mathsf{ev}\big\}\big(\mathfrak{M}^{mc}_\gamma(J,\nu)\big)\leqslant\dim_{g,l;k}(B)-2\qquad\forall\ \gamma\in\mathcal{A}^\phi_{g,l;k}(B),\ \aleph(\gamma)\neq\varnothing.\tag{3.75}$$

Let $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$, $\overline{\gamma}\in\mathcal{A}_{g,l;k}$ be the contraction of γ, and $\varpi_\gamma\equiv\varpi$ be as in (3.33). Then,

$$\mathfrak{M}^*_\gamma(J,\nu)=\mathfrak{M}^*_{\gamma,\varpi_\gamma}(J,\nu)/\mathrm{Aut}(\gamma)=\big(\mathbb{R}\mathcal{M}_{\gamma;\aleph(\gamma,\overline{\gamma})}\times\mathcal{Z}'^*_{\gamma,\varpi_\gamma}(J,\nu)\big)/\mathrm{Aut}(\gamma)\,.$$

For a generic choice of (J,ν), the images of the irreducible components Σ_v of the domain Σ of any element $\mathbf{u}$ of $\mathcal{Z}'^*_{\gamma,\varpi_\gamma}(J,\nu)$ are distinct. Since $\aleph(\gamma,\overline{\gamma})=\aleph(\gamma)$ contains no loops, the $\mathrm{Aut}(\gamma)$-action on $\mathfrak{M}^*_\gamma(J,\nu)$ is thus free and $\ell(\gamma,\aleph(\gamma,\overline{\gamma})_0)=0$. Thus, (3.74) follows from Proposition 3.6(2).

Suppose $\gamma\in\mathcal{A}^{\phi}_{g,l;k}(B)$ and $\aleph(\gamma)\neq\varnothing$. By Lemma 3.5,

$$\{\mathsf{st}\times\mathsf{ev}\}\big(\mathfrak{M}^{mc}_\gamma(J,\nu)\big)\subset\bigsqcup_{(\gamma',\varpi)\in\mathcal{A}'(\gamma)}\{\mathsf{st}_{\gamma';\varpi}\times\mathsf{ev}\}\big(\mathcal{Z}^*_{\gamma';\varpi}(J,\nu)\big)\,.$$

For the purposes of establishing (3.75), it thus suffices to show that

$$\dim_{\mathbb{R}}\mathcal{Z}^*_{\gamma';\varpi}(J,\nu)\leqslant\dim_{g,l;k}(B)-2\qquad\forall\,(\gamma',\varpi)\in\mathcal{A}'(\gamma)\,.\tag{3.76}$$

Let $(\gamma',\varpi)\in\mathcal{A}'(\gamma)$ with $\gamma'\in\mathcal{A}^{\phi}_{g',l;k}(B')$ and $|\varrho|_{v'}$ for each $v'\in\aleph(\gamma',\overline{\gamma})_\bullet$ be as in (3.68). By Proposition 3.6(2) and (3.57),

$$\begin{split}\dim_{\mathbb{R}}\mathcal{Z}^*_{\gamma';\varpi}(J,\nu)=\dim_{g,l;k}(B)-\big(\langle c_1(TX),B-B'\rangle+(n-3)(g'-g)\big)\\ -|\gamma'|+n\,\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)-\big(\dim\mathbb{R}\mathcal{M}_{\gamma';\aleph(\gamma',\overline{\gamma})}+\big|\mathscr{F}^{\dagger}_{\gamma'}(\aleph(\gamma',\overline{\gamma})_0)\big|\big)\,.\end{split}$$

Along with (3.31) with $\mathscr{V}=\aleph(\gamma',\overline{\gamma})$, this implies that

$$\begin{split}\dim_{\mathbb{R}}\mathcal{Z}^*_{\gamma';\varpi}(J,\nu)\leqslant\dim_{g,l;k}(B)-\big(\langle c_1(TX),B-B'\rangle+(n-3)(g'-g)\big)\\ +(n-3)\ell\big(\gamma',\aleph(\gamma',\overline{\gamma})_0\big)-[\![\gamma',\varpi]\!]\,.\end{split}$$

The inequality (3.76) now follows from Corollary 3.14. □

Corollary 3.8 and Proposition 3.10 establish Theorem 3.3(1). Corollary 3.9 and Proposition 3.11 establish the first claim of Theorem 3.3(2).

It remains to establish the second claim of Theorem 3.3(2). Let

$$p\colon\widetilde{\mathcal{M}}_{g,l}\longrightarrow\overline{\mathcal{M}}_{g,l}\qquad\text{and}\qquad p'\colon\widetilde{\mathcal{M}}'_{g,l}\longrightarrow\overline{\mathcal{M}}_{g,l}$$

be regular covers. Then the cover

$$\widehat{p}\colon\widehat{\mathcal{M}}_{g,l}\equiv\widetilde{\mathcal{M}}_{g,l}\times_{\overline{\mathcal{M}}_{g,l}}\widetilde{\mathcal{M}}'_{g,l}\longrightarrow\overline{\mathcal{M}}_{g,l}$$

is also regular and is the composition of p and p' with covers

$$q\colon\widehat{\mathcal{M}}_{g,l}\longrightarrow\widetilde{\mathcal{M}}_{g,l}\qquad\text{and}\qquad q'\colon\widehat{\mathcal{M}}_{g,l}\longrightarrow\widetilde{\mathcal{M}}'_{g,l}\,,$$

respectively. It is thus sufficient to compare the class (3.9) with its $\widehat{p}$-analogue.

We denote by $\widehat{\Gamma}^{0,1}_{g,l;k}(X;J)^\phi$ the analogue of the space (3.2) for $\widehat{p}$. The projection q lifts to a cover

$$\widetilde{q}\colon \mathbb{R}\widehat{\mathcal{U}}_{g,l;k} \longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$$

between the real universal curves (2.10) determined by $\widehat{p}$ and p. This lift commutes with the involutions and is biholomorphic on each fiber of (2.10). Thus,

$$\widehat{\nu} \equiv \big\{\widetilde{q}\times\mathrm{id}_X\big\}^*\nu \in \widehat{\Gamma}^{0,1}_{g,l;k}(X;J)^\phi \qquad \forall\,\nu\in\Gamma^{0,1}_{g,l;k}(X;J)^\phi\,.$$

The composition of $u_{\mathcal{M}}$ in (3.4) with $\widetilde{q}$ determines a projection

$$\widetilde{q}\colon \mathfrak{M}^\star_{g,l;0}(X,B;J,\widehat{\nu})^\phi \longrightarrow \mathfrak{M}^\star_{g,l;0}(X,B;J,\nu)^\phi \tag{3.77}$$

of degree equal to the degree of q.

If $(J,\nu)\in\widehat{\mathcal{H}}^{\omega,\phi}_{g,l}(X)$, then

$$\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^\star_{g,l;0}(X,B;J,\widehat{\nu})^\phi \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l$$

is a pseudocycle representing the class (3.9) determined by $\widehat{p}$. It equals to the composition of the pseudocycle

$$\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^\star_{g,l;0}(X,B;J,\nu)^\phi \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l$$

with (3.77). Thus,

$$\begin{split}\Big[\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^\star_{g,l;0}(X,B;J,\widehat{\nu})^\phi \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l\Big]&\\ =\big(\deg q\big)\Big[\mathsf{st}\times\mathsf{ev}\colon \mathfrak{M}^\star_{g,l;0}(X,B;J,\nu)^\phi \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,l}\times X^l\Big].&\end{split}$$

Since the degree of $\widehat{p}$ is the product of the degrees of p and q, this establishes the second claim of Theorem 3.3(2).

4 Transversality

Proposition 3.6(2) comes down to the smoothness of the second subspace in (3.41) for a generic pair (J,ν). It holds for fundamentally the same reasons as [19, Theorem 6.2.6] and [24, Theorem 3.16]. However, we present these reasons more systematically. In Section 4.1, we describe a deformation-obstruction setup for each of the four types of irreducible components Σ_v of the domain of maps in this subspace for a typical element $(\gamma,\varpi)\in\mathcal{A}(\overline{\gamma})$. By Lemmas 4.1 and 4.2, the deformations of (J,ν) in (3.3) supported in an open set W intersecting the image of Σ_v cover the obstruction spaces in all four cases. In Section 4.2, we show that this lemma implies the smoothness of the universal moduli space (4.25). As usual, the latter in turn implies that the second subspace in (3.41) is smooth for a generic element (J,ν) of $\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$. As explained in the next paragraph, Proposition 3.6(2) implies Proposition 3.6(1). The proof of Proposition 3.7 is similar.

Let $\gamma_0\in\mathcal{A}^\phi_{g,l;k}(B)$ denote the unique element with $|\gamma_0|=0$. For each $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$, let

$$\mathfrak{M}^*_\gamma(J,\nu)\subset\mathfrak{M}_\gamma(J,\nu)$$

denote the subspace of simple maps in the sense of Definition 3.2. The subspace in (3.6) is stratified by the subspaces $\mathfrak{M}^*_\gamma(J,\nu)$ with $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$ such that $|\gamma|\leqslant 1$. For a generic pair (J,ν), each of the latter subspaces is cut transversely by the $(\bar\partial_J-\nu)$-operator. Any standard gluing construction, such as in [16, Section 3] or [19, Chapter 10], restricts to the real setting and provides a continuous map

$$\Phi_\gamma\colon\mathcal{N}'_\gamma\longrightarrow\mathfrak{M}^*_\gamma(J,\nu)\cup\mathfrak{M}_{\gamma_0}(J,\nu)\subset\mathfrak{M}^\star_{g,l;k}(X,B;J,\nu)^\phi\subset\overline{\mathfrak{M}}_{g,l;k}(X,B;J,\nu)^\phi$$

from a neighborhood $\mathcal{N}'_\gamma$ of the zero section $\mathfrak{M}^*_\gamma(J,\nu)$ in the (real) rank $|\gamma|$ bundle of smoothing parameters. It restricts to the identity on $\mathfrak{M}^*_\gamma(J,\nu)$ and to a diffeomorphism from its complement to an open subspace of $\mathfrak{M}_{\gamma_0}(J,\nu)$. Thus, the maps Φ_γ with $\gamma\in\mathcal{A}^\phi_{g,l;k}(B)$ such that $|\gamma|=1$ extend the canonical smooth structure on $\mathfrak{M}_{\gamma_0}(J,\nu)$ to a smooth structure on the entire subspace in (3.6).

For the remainder of this paper, fix (X,ω,ϕ), B, and g,l,k as in Proposition 3.6, $\overline{\gamma}\in\mathcal{A}_{g,l;k}$ as in (3.10), $(\gamma,\varpi)\in\mathcal{A}(\overline{\gamma})$ with $\gamma\in\mathcal{A}^\phi_{g',l;k}(B')$ as in (3.19), and $\rho\in\mathcal{A}(\gamma)$. With $\mathrm{V}_\mathbb{R}(\gamma)$ and $\mathrm{V}_\mathbb{C}(\gamma)$ as defined in (3.11), let

$$\begin{aligned}&\aleph_\mathbb{R}(\gamma,\overline{\gamma})_\bullet=\aleph(\gamma,\overline{\gamma})_\bullet\cap\mathrm{V}_\mathbb{R}(\gamma),\quad \aleph_\mathbb{R}(\gamma,\overline{\gamma})_0=\aleph(\gamma,\overline{\gamma})_0\cap\mathrm{V}_\mathbb{R}(\gamma),\quad \aleph_\mathbb{R}(\gamma,\overline{\gamma})_0^c=\aleph(\gamma,\overline{\gamma})_0^c\cap\mathrm{V}_\mathbb{R}(\gamma),\\ &\aleph_\mathbb{C}(\gamma,\overline{\gamma})_\bullet=\aleph(\gamma,\overline{\gamma})_\bullet\cap\mathrm{V}_\mathbb{C}(\gamma),\quad \aleph_\mathbb{C}(\gamma,\overline{\gamma})_0=\aleph(\gamma,\overline{\gamma})_0\cap\mathrm{V}_\mathbb{C}(\gamma),\quad \aleph_\mathbb{C}(\gamma,\overline{\gamma})_0^c=\aleph(\gamma,\overline{\gamma})_0^c\cap\mathrm{V}_\mathbb{C}(\gamma).\end{aligned}$$

For each $v\in\mathrm{Ver}$, let

$$S_{v;\mathbb{C}}(\gamma),S_{v;\mathbb{R}}(\gamma)\subset\varepsilon^{-1}(v)\subset S_{l;k}\sqcup\mathrm{Fl}$$

be as in (3.13) with $\overline{\gamma}$ replaced by γ.

4.1 Spaces of deformations and obstructions

Let Σ be a nodal surface. Its irreducible components Σ_v, the nodes z_e, and the preimages z_f of the nodes in the normalization $\widetilde{\Sigma}$ of Σ are indexed by the sets Ver of vertices, $\mathrm{E}(\gamma)$ of edges, and Fl of flags, respectively, of the dual graph γ. We call a continuous map $u\colon\Sigma\longrightarrow X$ smooth if the restriction u_v of u to each irreducible component Σ_v is smooth.

If $u_v\colon\Sigma_v\longrightarrow X$ is a smooth map, let

$$\Gamma(u_v)=\Gamma(\Sigma_v;u_v^*TX).$$

For a complex structure $\mathfrak{j}$ on Σ_v and an almost complex structure J on X, define

$$\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)=\Gamma\big(\Sigma_v;(T\Sigma_v,-\mathfrak{j})^*\otimes_\mathbb{C}u_v^*(TX,J)\big).$$

For a real map u_v from (Σ_v,σ) to (X,ϕ), define

$$\Gamma(u_v)^{\phi,\sigma}=\big\{\xi\in\Gamma(u_v)\colon\,\mathrm{d}\phi\circ\xi=\xi\circ\sigma\big\}.$$

If in addition $\mathfrak{j}$ is a complex structure on Σ_v reversed by σ and J is an almost complex structure on X reversed by ϕ, let

$$\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)^{\phi,\sigma}=\big\{\eta\in\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)\colon\,\mathrm{d}\phi\circ\eta=\eta\circ\mathrm{d}\sigma\big\}\,.$$

Denote by σ_v for $v\in V_{\mathbb{R}}(\gamma)$ and $v\in V_{\mathbb{C}}(\gamma)$ the involutions on (3.24) and (3.23), respectively. Let

$$\mathcal{T}_v \equiv \ker \mathrm{d}\pi_{\gamma;\rho;v} \longrightarrow \mathbb{R}\widetilde{\mathcal{U}}_{\gamma;\rho;v}, \quad v\in V_{\mathbb{R}}(\gamma), \qquad \mathcal{T}_v^\bullet \equiv \ker \mathrm{d}\pi^\bullet_{\gamma;v} \longrightarrow \widetilde{\mathcal{U}}^\bullet_{\gamma;v}, \quad v\in V_{\mathbb{C}}(\gamma),$$

be the vertical tangent line bundles with complex structures $\mathfrak{j}_v$ and $\mathfrak{j}_v^\bullet$, respectively. For $J\in\mathcal{J}_\omega^\phi$, define

$$\begin{aligned}\Gamma^{0,1}_v(X;J)^\phi &= \big\{\nu\in\Gamma\big(\mathbb{R}\widetilde{\mathcal{U}}_v\times X;\pi_1^*(\mathcal{T}_v,-\mathfrak{j}_v)^*\otimes_{\mathbb{C}}\pi_2^*(TX,J)\big)\colon \mathrm{d}\phi\circ\nu=\nu\circ\mathrm{d}\sigma_v\big\}, \quad v\in V_{\mathbb{R}}(\gamma),\\ \Gamma^{0,1}_{v;\bullet}(X;J)^\phi &= \big\{\nu\in\Gamma\big(\widetilde{\mathcal{U}}_v^\bullet\times X;\pi_1^*(\mathcal{T}_v^\bullet,-\mathfrak{j}_v^\bullet)^*\otimes_{\mathbb{C}}\pi_2^*(TX,J)\big)\colon \mathrm{d}\phi\circ\nu=\nu\circ\mathrm{d}\sigma_v\big\}, \quad v\in V_{\mathbb{C}}(\gamma).\end{aligned}$$

Let $v\in V_{\mathbb{R}}(\gamma)$. Denote by $\mathfrak{B}_v$ the space of tuples

$$\mathbf{u}_v \equiv \big(u_v\colon \Sigma_v\longrightarrow X, (z_f)_{f\in\varepsilon^{-1}(v)},\sigma,\mathfrak{j}\big) \tag{4.1}$$

so that $(\Sigma,(z_f)_{f\in\varepsilon^{-1}(v)},\sigma,\mathfrak{j})$ is a fiber of (3.24) and u_v is

- a smooth (ϕ,σ)-real degree $\mathfrak{d}(v)$ map if $v\in V_{\mathbb{R}}(\overline{\gamma})\cup\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$,
- is a constant (ϕ,σ)-real map if $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0$.

In the first case, let

$$\begin{aligned}\Gamma(\mathbf{u}_v) &= \Gamma(u_v)^{\phi,\sigma}, \qquad \Gamma^{0,1}_J(\mathbf{u}_v) = \Gamma^{0,1}_{J,\mathfrak{j}}(u_v)^{\phi,\sigma},\\ \Gamma_0(\mathbf{u}_v) &= \big\{\xi\in\Gamma(\mathbf{u}_v)\colon \xi(z_f)=0\ \forall\, f\in\varepsilon^{-1}(v)\big\}.\end{aligned} \tag{4.2}$$

In the second case, we take $\Gamma(\mathbf{u}_v)$ to be the space of constant real sections of u_v^*TX and $\Gamma^{0,1}_J(\mathbf{u}_v)$ and $\Gamma_0(\mathbf{u}_v)$ to be the zero vector spaces. For $f\in\varepsilon^{-1}(v)$, let

$$\mathsf{ev}_f\colon\mathfrak{B}_v\longrightarrow\begin{cases}X, & \text{if } f\in S_{v;\mathbb{C}}(\gamma);\\ X^\phi, & \text{if } f\in S_{v;\mathbb{R}}(\gamma);\end{cases} \qquad L_f\colon\Gamma(\mathbf{u}_v)\longrightarrow\begin{cases}T_{\mathsf{ev}_f(\mathbf{u}_v)}X, & \text{if } f\in S_{v;\mathbb{C}}(\gamma);\\ T_{\mathsf{ev}_f(\mathbf{u}_v)}X^\phi, & \text{if } f\in S_{v;\mathbb{R}}(\gamma);\end{cases}$$

be the evaluation maps at the marked point z_f corresponding to f.

For $J\in\mathcal{J}_\omega^\phi$ and $\nu\in\Gamma^{0,1}_v(X;J)^\phi$, let $\widetilde{\mathfrak{M}}_v(J,\nu)\subset\mathfrak{B}_v$ be the subspace of tuples (4.1) such that

$$\bar\partial_{J,\mathfrak{j}}u_v\big|_z = \nu\big(z,u_v(z)\big) \qquad \forall\, z\in\Sigma_v\,. \tag{4.3}$$

Denote by

$$\mathfrak{B}_v^*\subset\mathfrak{B}_v \qquad \text{and} \qquad \widetilde{\mathfrak{M}}_v^*(J,\nu)\subset\widetilde{\mathfrak{M}}_v(J,\nu) \tag{4.4}$$

- the subspaces of simple maps if $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$,
- the entire spaces if $v\in V_{\mathbb{R}}(\overline{\gamma})\cup\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0$.

For $\mathbf{u}_v\in\widetilde{\mathfrak{M}}_v(J,\nu)$, let

$$D_{J,\nu;u_v}\colon\Gamma(u_v)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u_v) \qquad \text{and} \qquad D^0_{J,\nu;\mathbf{u}_v}\colon\Gamma_0(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}_v)$$

be the linearization of $\bar\partial_J-\nu$ at $\mathbf{u}_v$ and its restriction. If $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})$ and $W\subset X$, define

$$\widehat\Gamma^{0,1}_{J,\nu;W}(\mathbf{u}_v) = \mathrm{Im}\, D^0_{J,\nu;\mathbf{u}_v} + \big\{A\circ\mathrm{d}u_v\circ\mathfrak{j}\colon A\in T_J\mathcal{J}_\omega^\phi,\ \mathrm{supp}(A)\subset W\big\}.$$

If $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$, $\iota_v\colon\Sigma_v\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ is a normalization of a real irreducible component of a fiber of (2.10), and $W\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$, define

$$\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)=\mathrm{Im}\,D^0_{J,\nu;\mathbf{u}_v}+\Big\{\{\iota_v\times u_v\}^*\nu'\colon\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^{\phi},\,\mathrm{supp}(\nu')\subset W\times X\Big\}.$$

Let $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$. Denote by $\mathfrak{B}^{\bullet}_v$ the space of tuples

$$\mathbf{u}^{\bullet}_v\equiv\big(u_v\sqcup u_{\sigma(v)}:\Sigma_v\sqcup\Sigma_{\sigma(v)}\longrightarrow X,(z_f)_{f\in\varepsilon^{-1}(v)\sqcup\varepsilon^{-1}(\sigma(v))},\sigma,\mathfrak{j}\big)\tag{4.5}$$

so that $(\Sigma_v\sqcup\Sigma_{\sigma(v)},(z_f)_{f\in\varepsilon^{-1}(v)\sqcup\varepsilon^{-1}(\sigma(v))},\sigma,\mathfrak{j})$ is a fiber of (3.23), $u_{\sigma(v)}=\phi\circ u_v\circ\sigma$, and u_v is

- a smooth degree $\mathfrak{d}(v)$ map if $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})\cup\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_{\bullet}$,
- a constant map if $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0$.

In the first case, let

$$\begin{aligned}\Gamma(\mathbf{u}^{\bullet}_v)&=\big\{(\xi,\xi')\in\Gamma(u_v)\oplus\Gamma(u_{\sigma(v)})\colon\xi'=\mathrm{d}\phi\circ\xi\circ\sigma\big\},\\ \Gamma^{0,1}_J(\mathbf{u}^{\bullet}_v)&=\big\{(\eta,\eta')\in\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)\oplus\Gamma^{0,1}_{J,\mathfrak{j}}(u_{\sigma(v)})\colon\eta'=\mathrm{d}\phi\circ\eta\circ\mathrm{d}\sigma\big\},\\ \Gamma_0(\mathbf{u}^{\bullet}_v)&=\big\{(\xi,\xi')\in\Gamma(\mathbf{u}^{\bullet}_v)\colon\,\xi(z_f)=0\ \forall\,f\in\varepsilon^{-1}(v)\big\}.\end{aligned}\tag{4.6}$$

In the second case, we take $\Gamma(\mathbf{u}^{\bullet}_v)$ to be the space of pairs (ξ,ξ') as above so that $\xi|_{\Sigma_v}$ is constant and $\Gamma^{0,1}_J(\mathbf{u}^{\bullet}_v)$ and $\Gamma_0(\mathbf{u}^{\bullet}_v)$ to be the zero vector spaces. For $f\in\varepsilon^{-1}(v)\cup\varepsilon^{-1}(\sigma(v))$, define

$$\mathsf{ev}_f\colon\mathfrak{B}^{\bullet}_v\longrightarrow X,\qquad \mathsf{ev}_f(\mathbf{u}^{\bullet}_v)=\begin{cases}u_v(z_f)&\text{if }f\in\varepsilon^{-1}(v);\\ u_{\sigma(v)}(z_f),&\text{if }f\in\varepsilon^{-1}(\sigma(v));\end{cases}$$

$$L_f\colon\Gamma(\mathbf{u}^{\bullet}_v)\longrightarrow T_{\mathsf{ev}_f(\mathbf{u}^{\bullet}_v)}X,\qquad L_f(\xi,\xi')=\begin{cases}\xi(z_f),&\text{if }f\in\varepsilon^{-1}(v);\\ \xi'(z_f),&\text{if }f\in\varepsilon^{-1}(\sigma(v)).\end{cases}$$

For $J\in\mathcal{J}^{\phi}_{\omega}$ and $\nu\in\Gamma^{0,1}_{v;\bullet}(X;J)^{\phi}$, let $\widetilde{\mathfrak{M}}^{\bullet}_v(J,\nu)\subset\mathfrak{B}^{\bullet}_v$ be the subspace of tuples (4.5) satisfying (4.3). Denote by

$$\mathfrak{B}^{\bullet;*}_v\subset\mathfrak{B}^{\bullet}_v\quad\text{and}\quad\widetilde{\mathfrak{M}}^{\bullet;*}_v(J,\nu)\subset\widetilde{\mathfrak{M}}^{\bullet}_v(J,\nu)\tag{4.7}$$

- the subspaces of simple maps such that $u_v(\Sigma_v)\neq u_{\sigma(v)}(\Sigma_{\sigma(v)})$ if $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_{\bullet}$,
- the entire spaces if $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})\cup\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0$.

For $\mathbf{u}^{\bullet}_v\in\widetilde{\mathfrak{M}}_v(J,\nu)$, let

$$D^{\bullet}_{J,\nu;u_v}\colon\Gamma(u_v)\oplus\Gamma\big(u_{\sigma(v)}\big)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)\oplus\Gamma^{0,1}_{J,\mathfrak{j}}\big(u_{\sigma(v)}\big)\quad\text{and}\quad D^0_{J,\nu;\mathbf{u}^{\bullet}_v}\colon\Gamma_0(\mathbf{u}^{\bullet}_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}^{\bullet}_v)$$

be the linearization of $\bar\partial_J-\nu$ at $\mathbf{u}^{\bullet}_v$ and its restriction. If $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})$ and $W\subset X$, define

$$\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}^{\bullet}_v)=\mathrm{Im}\,D^0_{J,\nu;\mathbf{u}^{\bullet}_v}+\big\{\big(A\circ\mathrm{d}u_v\circ\mathfrak{j},A\circ\mathrm{d}u_{\sigma(v)}\circ\mathfrak{j}\big)\colon A\in T_J\mathcal{J}^{\phi}_{\omega},\,\mathrm{supp}(A)\subset W\big\}.$$

If $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, $\iota^{\bullet}_v:\Sigma_v\cup\Sigma_{\sigma(v)}\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ is a normalization of a conjugate pair of irreducible components of a fiber of (2.10), and $W\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$, let

$$\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}^{\bullet}_v)=\mathrm{Im}\,D^0_{J,\nu;\mathbf{u}^{\bullet}_v}+\Big\{\{\iota^{\bullet}_v\times(u_v\sqcup u_{\sigma(v)})\}^*\nu'\colon\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^{\phi},\,\mathrm{supp}(\nu')\subset W\times X\Big\}.$$

Lemma 4.1. *Suppose* $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_\bullet\cup\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, $J\in\mathcal{J}_\omega^\phi$, $\nu\in\Gamma^{0,1}_{v;\bullet}(X;J)^\phi$, *and* $\mathbf{u}_v^\bullet\in\widetilde{\mathfrak{M}}_v^{\bullet,*}(J,\nu)$ *is as in (4.5). If* $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_\bullet$, *let* $W\subset X$ *be a* ϕ*-invariant open subset intersecting* $u_v(\Sigma_v)$. *If* $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, *let*

$$\iota_v^\bullet\colon\Sigma_v\cup\Sigma_{\sigma(v)}\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}\qquad\textit{and}\qquad W\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$$

be a normalization of a conjugate pair of irreducible components of a fiber of (2.10) and a $\widetilde{\sigma}_{\mathbb{R}}$*-invariant open subset intersecting* $\iota_v^\bullet(\Sigma_v)$, *respectively. Then* $\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v^\bullet)=\Gamma^{0,1}(\mathbf{u}_v^\bullet)$.

Proof. Denote by

$$D_{J,\nu;u_v}\colon\Gamma(u_v)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u_v),\quad D^0_{J,\nu;u_v}\colon\Gamma_0(\mathbf{u}_v)\equiv\big\{\xi\in\Gamma(u_v)\colon\,\xi(z_f)=0\ \forall\,f\in\varepsilon^{-1}(v)\big\}\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)$$

the restrictions of $D^\bullet_{J,\nu;\mathbf{u}_v}$ and $D^0_{J,\nu;\mathbf{u}_v^\bullet}$, respectively. If $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})$, define

$$\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)=\mathrm{Im}\,D^0_{J,\nu;\mathbf{u}_v}+\big\{A\circ\mathrm{d}u_v\circ\mathfrak{j}\colon A\in T_J\mathcal{J}_\omega^\phi,\ \mathrm{supp}(A)\subset W\big\}.$$

If $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, define

$$\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)=\mathrm{Im}\,D^0_{J,\nu;\mathbf{u}_v}+\Big\{\big\{\iota_v^\bullet|_{\Sigma_v}\times u_v\big\}^*\nu'\colon\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^\phi,\ \mathrm{supp}(\nu')\subset W\Big\}.$$

The projections

$$\Gamma_0(\mathbf{u}_v^\bullet)\longrightarrow\Gamma_0(\mathbf{u}_v)\qquad\text{and}\qquad\Gamma^{0,1}_{J,\mathfrak{j}}(\mathbf{u}_v^\bullet)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(u_v)$$

are isomorphisms intertwining $D^0_{J,\nu;\mathbf{u}_v^\bullet}$ and $D^0_{J,\nu;\mathbf{u}_v}$. The second projection maps $\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v^\bullet)$ onto $\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)$. Thus, the claim of the lemma is equivalent to $\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)=\Gamma^{0,1}(\mathbf{u}_v)$.

Denote by

$$\Gamma^{1,0}(\mathbf{u}_v)\equiv\Gamma^{1,0}_J(\mathbf{u}_v)\subset\Gamma\big(\Sigma_v-\varepsilon^{-1}(v);(T\Sigma_v,\mathfrak{j})^*\otimes_{\mathbb{C}}u_v^*(TX,J)^*\big)$$

the subspace of smooth sections η with at most simple poles at the points of $\varepsilon^{-1}(v)$. In other words, for every $z_0\in\varepsilon^{-1}(v)$ there exists a holomorphic coordinate w on a neighborhood U of z_0 in Σ_v such that

$$w(z_0)=0\qquad\text{and}\qquad w\cdot\eta|_U\in\Gamma\big(U;(T\Sigma_v,\mathfrak{j})^*\otimes_{\mathbb{C}}u^*(TX,J)^*\big).$$

By [27, Lemma 2.4.1] and [15, Lemma 2.3.2], the cokernel of $D^0_{J,\nu;\mathbf{u}_v}$ is isomorphic to the kernel of the formal adjoint $D^*_{\mathbf{u}_v}$ of $D_{J,\nu;u_v}$ on $\Gamma^{1,0}(\mathbf{u}_v)$ via the standard pairing of $(0,1)$- and $(1,0)$-forms on Σ_v; see also [34, Sections 2.1,2.2].

Let $\eta\in\ker D^*_{\mathbf{u}_v}-\{0\}$. The only property of $D^*_{\mathbf{u}_v}$ relevant for our purposes is

(P1) η does not vanish on any non-empty open subset of Σ_v.

As with [19, Proposition 3.2.1] and [24, Proposition 3.2], the proof comes down to showing that η pairs non-trivially with some element of $T_J\mathcal{J}_\omega^\phi$ if $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_\bullet$ and of $\Gamma^{0,1}_{g,l;k}(X;J)^\phi$ if $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$.

Suppose $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_\bullet$. Since u_v is simple and $\phi(u_v(\Sigma_v))\neq u_v(\Sigma_v)$, we can assume that there exist non-empty open subsets $U\subset\Sigma_v$ and $W'\subset X$ such that $u_v|_U$ is an embedding,

$$W=W'\sqcup\phi(W'),\qquad u_v(\Sigma_v)\cap W'=u_v(U),\qquad u_v(\Sigma_v)\cap\phi(W')=\varnothing\,.\tag{4.8}$$

The proof of [19, Proposition 3.2.1] provides $A'\in T_J\mathcal{J}_\omega$ such that

$$\mathrm{supp}(A')\subset W' \qquad\text{and}\qquad \int_{\Sigma_v}\big(A'\circ\mathrm{d}u_v\circ\mathfrak{j}\big)\wedge\eta\neq 0\,. \tag{4.9}$$

We define $A\in T_J\mathcal{J}_\omega^\phi$ by

$$A|_{W'}=A', \qquad A|_{\phi(W')}=-\phi^*A', \qquad A|_{X-W}=0\,. \tag{4.10}$$

By the last assumption in (4.8), A pairs non-trivially with η.

Let $v\in\overline{\mathrm{V}}_{\mathbb{C}}(\overline{\gamma})$. Since $\iota_v^\bullet$ is injective outside of finitely many points, we can assume that there exist non-empty open subsets $U\subset\Sigma_v$ and $W'\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ such that

$$W=W'\sqcup\widetilde{\sigma}_{\mathbb{R}}(W'), \quad \iota_v^\bullet(\Sigma_v)\cap W'=\iota_v^\bullet(U), \quad \iota_v^\bullet(\Sigma_v)\cap\widetilde{\sigma}_{\mathbb{R}}(W')=\varnothing. \tag{4.11}$$

The reasoning in the proof of [19, Proposition 3.2.1] applied in the setting of [24, Proposition 3.2] provides $\nu''\in\Gamma^{0,1}_{g,l;k}(X;J)$ such that

$$\mathrm{supp}(\nu'')\subset W'\times X \qquad\text{and}\qquad \int_{\Sigma_v}\big(\{\iota_v^\bullet|_{\Sigma_v}\times u_v\}^*\nu''\big)\wedge\eta\neq 0\,. \tag{4.12}$$

We define $\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^\phi$ by

$$\nu'|_{W'\times X}=\nu'', \quad \nu'|_{\widetilde{\sigma}_{\mathbb{R}}(W')\times X}=\big\{\widetilde{\sigma}_{\mathbb{R}}\times\phi\big\}^*\nu'', \quad \nu'|_{(\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}-W)\times X}=0\,. \tag{4.13}$$

By the last assumption in (4.11), ν' pairs non-trivially with η. $\square$

Lemma 4.2. *Suppose $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet\cup\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$, $J\in\mathcal{J}_\omega^\phi$, $\nu\in\Gamma_v^{0,1}(X;J)^\phi$, and $\mathbf{u}_v\in\widetilde{\mathfrak{M}}_v^*(J,\nu)$ is as in (4.1). If $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$, let $W\subset X$ be a ϕ-invariant open subset intersecting $u_v(\Sigma_v)$. If $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$, let*

$$\iota_v\colon\Sigma_v\longrightarrow\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k} \qquad\textit{and}\qquad W\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$$

be a normalization of a real irreducible component of a fiber of (2.10) and a $\widetilde{\sigma}_{\mathbb{R}}$-invariant open subset intersecting $\iota_v(\Sigma_v)$, respectively. Then $\widehat{\Gamma}^{0,1}_{J,\nu;W}(\mathbf{u}_v)=\Gamma^{0,1}(\mathbf{u}_v)$.

Proof. With $\Gamma_J^{1,0}(\mathbf{u}_v)$ as in the proof of Lemma 4.1, define

$$\Gamma^{1,0}(\mathbf{u}_v)=\{\eta\in\Gamma_J^{1,0}(\mathbf{u}_v)\colon\ \mathrm{d}\phi\circ\eta=-\eta\circ\mathrm{d}\sigma\}.$$

Since the involution σ on Σ_v is orientation-reversing, the pairing of $(0,1)$- and $(1,0)$-forms on Σ_v of Lemma 4.1 restricts to a pairing between the invariant subspaces $\Gamma_J^{0,1}(\mathbf{u}_v)$ and $\Gamma^{1,0}(\mathbf{u}_v)$ and induces an isomorphism between the cokernel of $D^0_{J,\nu;\mathbf{u}_v}$ and the kernel of $D^*_{\mathbf{u}_v}$ on $\Gamma^{1,0}(\mathbf{u}_v)$ as before.

Let $\eta\in\ker D^*_{\mathbf{u}_v}-\{0\}$. The only properties of $D^*_{\mathbf{u}_v}$ relevant for our purposes are (P1) and

(P2) $\mathrm{d}\phi\circ\eta=-\eta\circ\mathrm{d}\sigma$.

We show that η pairs non-trivially with some element of $T_J\mathcal{J}_\omega^\phi$ if $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$ and of $\Gamma^{0,1}_{g,l;k}(X;J)^\phi$ if $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$.

Suppose $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$; thus, $\phi\circ u_v=u_v\circ\sigma$. Since u_v is simple, we can assume there exist non-empty open subsets $U\subset\Sigma_v$ and $W'\subset X$ satisfying the condition before (4.8) and the first two conditions in (4.8). Let $A'\in T_J\mathcal{J}_\omega$ be as in (4.9) and define $A\in T_J\mathcal{J}_\omega^\phi$ by (4.10). By (P2),

$$\big((A\circ \mathrm{d}u_v\circ\mathfrak{j})\wedge\eta\big)\big|_{\sigma(U)}=-\sigma^*\Big(\big((A\circ \mathrm{d}u_v\circ\mathfrak{j})\wedge\eta\big)\big|_U\Big). \tag{4.14}$$

We conclude that

$$\int_{\Sigma_v}\big(A\circ \mathrm{d}u_v\circ\mathfrak{j}\big)\wedge\eta=\int_U\big(A\circ \mathrm{d}u_v\circ\mathfrak{j}\big)\wedge\eta+\int_{\sigma(U)}\big(A\circ \mathrm{d}u_v\circ\mathfrak{j}\big)\wedge\eta=2\int_U\big(A'\circ \mathrm{d}u_v\circ\mathfrak{j}\big)\wedge\eta\neq0, \tag{4.15}$$

i.e. A pairs non-trivially with η.

Let $v\in\mathrm{V}_{\mathbb{R}}(\overline{\gamma})$. We can assume that there exist non-empty open subsets $U\subset\Sigma_v$ and $W'\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ satisfying the first two conditions in (4.11) with $\iota_v^\bullet$ replaced by ι_v. Let $\nu''\in\Gamma^{0,1}_{g,l;k}(X;J)$ be as in (4.12) and define $\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^\phi$ by (4.13). By (P2), (4.14) and (4.15) hold with $A\circ\mathrm{d}u_v\circ\mathfrak{j}$ replaced by $\{\iota_v\times u_v\}^*\nu'$. □

4.2 Universal moduli spaces

Let $\mathrm{E}_{\mathbb{C}}(\gamma),\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma),\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma)\subset\mathrm{E}(\gamma)$ be as defined in Section 3.2. Choose $\mathrm{E}_+(\gamma)\subset\mathrm{E}_{\mathbb{C}}(\gamma)$ so that

$$\mathrm{E}_{\mathbb{C}}(\gamma)=\mathrm{E}_+(\gamma)\sqcup\sigma\big(\mathrm{E}_+(\gamma)\big). \tag{4.16}$$

For each $e\in\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma)$, choose a flag $f_e\in e$.

For $e\in\mathrm{E}_+(\gamma)$ and $e\in\mathrm{E}_{\mathbb{R}}(\gamma)$, let

$$\Delta_{\gamma;e}\subset X_{\gamma;e}\equiv\prod_{f\in e}X \qquad\text{and}\qquad \Delta^\phi_{\gamma;e}\subset X^\phi_{\gamma;e}\equiv\prod_{f\in e}X^\phi$$

be the respective diagonals. Define

$$\Delta_\gamma\equiv\prod_{e\in\mathrm{E}_+(\gamma)}\Delta_{\gamma;e}\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma)}X^\phi\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma)}\Delta^\phi_{\gamma;e}\subset X_\gamma\equiv\prod_{e\in\mathrm{E}_+(\gamma)}X_{\gamma;e}\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma)}X\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma)}X^\phi_{\gamma;e}\,. \tag{4.17}$$

The evaluation maps ev_f induce a map

$$\mathsf{ev}_\gamma\equiv\prod_{e\in\mathrm{E}_+(\gamma)}\prod_{f\in e}\mathsf{ev}_f\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma)}\mathsf{ev}_{f_e}\times\prod_{e\in\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma)}\prod_{f\in e}\mathsf{ev}_f:\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}^*_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^{\bullet,*}_v\longrightarrow X_\gamma \tag{4.18}$$

Define

$$\begin{aligned}\mathfrak{B}^\star_{\gamma,\varpi;\rho}&=\Big\{\mathbf{u}\in\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}^*_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^{\bullet,*}_v:\,\mathsf{ev}_\gamma(\mathbf{u})\in\Delta_\gamma\Big\},\\ \mathfrak{B}^*_{\gamma,\varpi;\rho}&=\Big\{\mathbf{u}\in\mathfrak{B}^\star_{\gamma,\varpi;\rho}:\,u_{v_1}(\Sigma_{v_1})\neq u_{v_2}(\Sigma_{v_2})\ \forall\,v_1,v_2\in\aleph(\gamma,\overline{\gamma})_\bullet,\,v_1\neq v_2\Big\}\,.\end{aligned} \tag{4.19}$$

An element of $\mathfrak{B}^{\star}_{\gamma,\varpi;\rho}$ corresponds to a tuple

$$\mathbf{u} \equiv \big(u\colon \Sigma \longrightarrow X, (z_f)_{f\in S_{l;k}}, \sigma, \mathfrak{j}\big) \equiv \big((\mathbf{u}_v)_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)}, (\mathbf{u}^{\bullet}_v)_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)}\big) \equiv \big(\mathbf{u}_v\big)_{v\in \mathrm{Ver}}, \tag{4.20}$$

where $(\Sigma, (z_f)_{f\in S_{l;k}}, \sigma, \mathfrak{j})$ is a fiber of (3.25) and u is a smooth (ϕ,σ)-real map such that

$$u_*[\Sigma_v] = \mathfrak{d}(v) \in H_2(X;\mathbb{Z}) \quad \forall\, v\in \mathrm{Ver}\,, \qquad u|_{\Sigma_v} = \mathrm{const} \quad \forall\, v\in \aleph(\gamma,\overline{\gamma})_0.$$

For a tuple $\mathbf{u}$ as above, let

$$\Gamma_0(\mathbf{u}) = \bigoplus_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)} \Gamma_0(\mathbf{u}_v) \oplus \bigoplus_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)} \Gamma_0(\mathbf{u}^{\bullet}_v). \tag{4.21}$$

If in addition $J\in \mathcal{J}^{\phi}_{\omega}$, let

$$\Gamma^{0,1}_J(\mathbf{u}) = \bigoplus_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)} \Gamma^{0,1}_J(\mathbf{u}_v) \oplus \bigoplus_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)} \Gamma^{0,1}_J(\mathbf{u}^{\bullet}_v). \tag{4.22}$$

Denote by

$$\mathfrak{F} \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X) \times \mathfrak{B}^{*}_{\gamma,\varpi;\rho} \tag{4.23}$$

the bundle with fibers $\mathfrak{F}_{(J,\nu;\mathbf{u})} = \Gamma^{0,1}_J(\mathbf{u})$. We define a section of this bundle

$$\bar\partial\colon \mathcal{H}^{\omega,\phi}_{g,l;k}(X) \times \mathfrak{B}^{*}_{\gamma,\varpi;\rho} \longrightarrow \mathfrak{F} \qquad \text{by} \quad \bar\partial(J,\nu;\mathbf{u})\big|_z = \bar\partial_{J,\mathfrak{j}} u\big|_z - \nu_{\gamma,\varpi;\rho}\big(z,u(z)\big) \ \ \forall\, z\in\Sigma\,. \tag{4.24}$$

The zero set of this section

$$\mathfrak{U}\widetilde{\mathfrak{M}}^{*}_{\gamma,\varpi;\rho}(J,\nu) \equiv \big\{(J,\nu;\mathbf{u})\in \mathcal{H}^{\omega,\phi}_{g,l;k}(X) \times \mathfrak{B}^{*}_{\gamma,\varpi;\rho} \colon \bar\partial(J,\nu;\mathbf{u}) = 0\big\} \tag{4.25}$$

is the universal moduli space. The preimage of a pair (J,ν) in $\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$ under the projection

$$\pi\colon \mathfrak{U}\widetilde{\mathfrak{M}}^{*}_{\gamma,\varpi;\rho}(J,\nu) \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X) \tag{4.26}$$

is the second subspace in (3.41).

For the purposes of applying the Sard-Smale Theorem [28, (1.3)], we complete

- the map components of the spaces $\mathfrak{B}_v$ and $\mathfrak{B}^{\bullet}_v$ and the spaces (4.21) in the L^p_2 Sobolev norm for some $p>2$ (fixed),
- the parameter space (3.3) in the C^m Hölder norm for some $m\geqslant 2$ (to be chosen later), and
- the spaces (4.22) in the L^p_1 Sobolev norm.

We denote the completed spaces and the induced spaces in (4.4), (4.7), (4.19)-(4.23), and (4.25) as before.

By the assumption $p>2$ and the Sobolev Embedding Theorem [19, Theorem B.1.12], the map component of any element of the completed spaces $\mathfrak{B}_v$ and $\mathfrak{B}^{\bullet}_v$ is a C^1-map. By the reasoning at the top of [19, p47], these completions are smooth separable Banach manifolds. By the reasoning at the bottom of [19, p49], the (completed) parameter space $\mathcal{H}^{\omega,\phi}_{g,l;k}(X)$ is a smooth separable Banach manifold. By the reasoning at the bottom of [19, p50], (4.24) is the restriction of a C^{m-2} section of C^{m-2} Banach bundle (4.23) over the product of the spaces $\mathfrak{B}_v$ and $\mathfrak{B}^{\bullet}_v$.

Proposition 4.3. *Let p and m be as above. Then*

$$\mathfrak{U}\widetilde{\mathfrak{M}}^*_{\gamma,\varpi;\rho}(J,\nu) \subset \mathcal{H}^{\omega,\phi}_{g,l;k}(X) \times \prod_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)} \mathfrak{B}_v \times \prod_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)} \mathfrak{B}^\bullet_v \tag{4.27}$$

is a separable C^{m-2} Banach submanifold and the projection (4.26) is a C^{m-2} Fredholm map of index

$$\mathrm{ind}_{\mathbb{R}}\pi = \dim_{g',l;k}(B') - |\gamma| + n\,\ell\big(\gamma,\aleph(\gamma,\overline{\gamma})_0\big) + \dim_{\mathbb{R}} G_{\gamma;\rho}\,. \tag{4.28}$$

Lemma 4.4. *Let p and m be as in Proposition 4.3. Then*

$$\mathfrak{B}^*_{\gamma,\varpi;\rho} \subset \prod_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)} \mathfrak{B}_v \times \prod_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)} \mathfrak{B}^\bullet_v$$

is a separable Banach submanifold of codimension

$$\mathrm{codim}_{\mathbb{R}} \mathfrak{B}^*_{\gamma,\varpi;\rho} = n\Big(|\gamma| - \ell\big(\gamma,\aleph(\gamma,\overline{\gamma})_0\big)\Big)\,. \tag{4.29}$$

Proof. With $\mathscr{E}_\gamma(\mathscr{V})$ as defined in (3.29), let

$$\mathrm{E}_+(\gamma,\overline{\gamma}) = \mathrm{E}_+(\gamma) - \mathscr{E}_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big), \quad \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma}) = \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma) - \mathscr{E}_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big),$$
$$\mathrm{E}_{\mathbb{R}}(\gamma,\overline{\gamma}) = \mathrm{E}_{\mathbb{R}}(\gamma) - \mathscr{E}_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big).$$

Define

$$\Delta_{\gamma,\overline{\gamma}} \equiv \prod_{e\in \mathrm{E}_+(\gamma,\overline{\gamma})} \Delta_{\gamma;e} \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})} X^\phi \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})} \Delta^\phi_{\gamma;e} \subset X_{\gamma,\overline{\gamma}} \equiv \prod_{e\in \mathrm{E}_+(\gamma,\overline{\gamma})} X_{\gamma;e} \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})} X \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})} X^\phi_{\gamma;e}\,,$$
$$\mathsf{ev}_{\gamma,\overline{\gamma}} \equiv \prod_{e\in \mathrm{E}_+(\gamma,\overline{\gamma})} \prod_{f\in e} \mathsf{ev}_f \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})} \mathsf{ev}_{f_e} \times \prod_{e\in \mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})} \prod_{f\in e} \mathsf{ev}_f \colon \prod_{v\in \mathrm{V}_{\mathbb{R}}(\gamma)} \mathfrak{B}^*_v \times \prod_{\{v,\sigma(v)\}\subset \mathrm{V}_{\mathbb{C}}(\gamma)} \mathfrak{B}^{\bullet,*}_v \longrightarrow X_{\gamma,\overline{\gamma}}\,.$$

For each element $\mathbf{u} \in \mathfrak{B}^\star_{\gamma,\varpi;\rho}$, let

$$L_{\gamma,\overline{\gamma};\mathbf{u}} \equiv \bigoplus_{e\in \mathrm{E}_+(\gamma,\overline{\gamma})} \bigoplus_{f\in e} L_f \oplus \bigoplus_{e\in \mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})} L_{f_e} \oplus \bigoplus_{e\in \mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})} \bigoplus_{f\in e} L_f \colon$$
$$\bigoplus_{v\in \aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0} \Gamma_0(\mathbf{u}_v) \oplus \bigoplus_{v\in \aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0^c} \Gamma(\mathbf{u}_v) \oplus \bigoplus_{\{v,\sigma(v)\}\subset \aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0} \Gamma_0(\mathbf{u}^\bullet_v) \oplus \bigoplus_{\{v,\sigma(v)\}\subset \aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0^c} \Gamma(\mathbf{u}^\bullet_v) \longrightarrow T_{\mathsf{ev}_{\gamma,\overline{\gamma}}(\mathbf{u})} X_{\gamma,\overline{\gamma}}\,.$$

For $v\in \aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0$, the map component u_v of every $\mathbf{u}_v \in \mathfrak{B}^*_v$ is constant. For $v\in \aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0$, the map components u_v and $u_{\sigma(v)}$ of every $\mathbf{u}^\bullet_v \in \mathfrak{B}^{\bullet,*}_v$ are constant. Thus,

$$\mathfrak{B}^0_{\gamma,\varpi} \equiv \Big\{\mathbf{u} \in \prod_{v\in \aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0} \mathfrak{B}^*_v \times \prod_{\{v,\sigma(v)\}\subset \aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0} \mathfrak{B}^{\bullet,*}_v \colon \mathsf{ev}_f(u_{\varepsilon(f)}) = \mathsf{ev}_{f'}(u_{\varepsilon(f')})\ \forall\,\{f,f'\} \in \mathscr{E}_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big)\Big\}$$
$$\subset \prod_{v\in \aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0} \mathfrak{B}^*_v \times \prod_{\{v,\sigma(v)\}\subset \aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0} \mathfrak{B}^{\bullet,*}_v$$

is a smooth submanifold of dimension $n|\pi_0(\gamma,\aleph(\gamma,\overline{\gamma})_0)|$ and thus of codimension

$$\mathrm{codim}_{\mathbb{R}} \mathfrak{B}^0_{\gamma,\varpi} = n\Big(\big|\mathscr{E}_\gamma\big(\aleph(\gamma,\overline{\gamma})_0\big)\big| - \ell\big(\gamma,\aleph(\gamma,\overline{\gamma})_0\big)\Big)\,. \tag{4.30}$$

By definition,

$$\mathfrak{B}^{\star}_{\gamma,\varpi;\rho} = \Big\{\mathbf{u}\in\mathfrak{B}^0_{\gamma,\varpi}\times\prod_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\mathfrak{B}^*_v\times\prod_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\mathfrak{B}^{\bullet,*}_v\colon \mathsf{ev}_{\gamma,\overline{\gamma}}(\mathbf{u})\in\Delta_{\gamma,\overline{\gamma}}\Big\}. \tag{4.31}$$

For each $v\in\mathrm{V}_{\mathbb{R}}(\gamma)$, $\mathfrak{B}^*_v$ is an open subset of $\mathfrak{B}_v$. For each $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$, $\mathfrak{B}^{\bullet,*}_v$ is an open subset of $\mathfrak{B}^{\bullet}_v$. Since $\mathfrak{B}^*_{\gamma,\varpi;\rho}$ is an open subset of $\mathfrak{B}^{\star}_{\gamma,\varpi;\rho}$, it is sufficient to show that

$$\mathfrak{B}^{\star}_{\gamma,\varpi;\rho}\subset\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}^*_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^{\bullet,*}_v$$

is a Banach submanifold of codimension (4.29). We show below that

$$T_{\mathsf{ev}_{\gamma,\overline{\gamma}}(\mathbf{u})}X_{\gamma,\overline{\gamma}} = \mathrm{Im}\,L_{\gamma,\overline{\gamma};\mathbf{u}} + T_{\mathsf{ev}_{\gamma,\overline{\gamma}}(\mathbf{u})}\Delta_{\gamma,\overline{\gamma}} \tag{4.32}$$

for every $\mathbf{u}\in\mathfrak{B}^{\star}_{\gamma,\varpi;\rho}$. Thus, the smooth map

$$\mathsf{ev}_{\gamma,\overline{\gamma}}\colon\mathfrak{B}^0_{\gamma,\varpi}\times\prod_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\mathfrak{B}^*_v\times\prod_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\mathfrak{B}^{\bullet,*}_v\longrightarrow X_{\gamma,\overline{\gamma}}$$

is transverse to the submanifold $\Delta_{\gamma,\overline{\gamma}}$. Along with (4.31), the Implicit Function for Banach manifolds, and (4.30), this implies the lemma.

For each edge e in $\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})$, $f_e\notin\aleph(\gamma,\overline{\gamma})_0$. For each edge e in $\mathrm{E}_{\mathbb{C}}(\gamma,\overline{\gamma})\cup\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})$, there exists a flag $f_e\in e$ such that $f_e\notin\aleph(\gamma,\overline{\gamma})_0$. For each $v\in\mathrm{Ver}$, let

$$\begin{gathered}\mathrm{Fl}^*_v=\varepsilon^{-1}(v)\cap\big\{f_e\colon e\in\mathrm{E}_+(\gamma,\overline{\gamma})\cup\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})\cup\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})\big\},\\ \mathrm{Fl}^*_{v;\mathbb{R}}=\mathrm{Fl}^*_v\cap S_{v;\mathbb{R}}(\gamma),\qquad \mathrm{Fl}^*_{v;\mathbb{C}}=\mathrm{Fl}^*_v\cap S_{v;\mathbb{C}}(\gamma).\end{gathered}$$

If $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$, $\mathrm{Fl}^*_{v;\mathbb{R}}=\varnothing$ and $\mathrm{Fl}^*_{v;\mathbb{C}}=\mathrm{Fl}^*_v$. By (4.16),

$$\sigma(\mathrm{Fl}^*_v)\cap\mathrm{Fl}^*_{\sigma(v)}=\mathrm{Fl}^*_{v;\mathbb{R}},\quad \big\{f_e\colon e\in\mathrm{E}_+(\gamma,\overline{\gamma})\cup\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})\cup\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})\big\}=\bigcup_{v\in\aleph(\gamma,\overline{\gamma})^c_0}\mathrm{Fl}^*_v. \tag{4.33}$$

Let $\mathbf{u}\in\mathfrak{B}^{\star}_{\gamma,\varpi;\rho}$ and

$$\pi_{\gamma,\overline{\gamma}}\colon T_{\mathsf{ev}_{\gamma,\overline{\gamma}}(\mathbf{u})}X_{\gamma,\overline{\gamma}}\longrightarrow\bigoplus_{e\in\mathrm{E}_+(\gamma,\overline{\gamma})}T_{\mathsf{ev}_{f_e}(u_{\varepsilon(f_e)})}X\oplus\bigoplus_{e\in\mathrm{E}_{\mathbb{R}\mathbb{C}}(\gamma,\overline{\gamma})}T_{\mathsf{ev}_{f_e}(u_{\varepsilon(f_e)})}X\oplus\bigoplus_{e\in\mathrm{E}_{\mathbb{R}\mathbb{R}}(\gamma,\overline{\gamma})}T_{\mathsf{ev}_{f_e}(u_{\varepsilon(f_e)})}X^{\phi}$$

be the projection on the components indexed by the flags of the form f_e. By the first statement in (4.33), the homomorphisms

$$\bigoplus_{f\in\mathrm{Fl}^*_{v;\mathbb{C}}}L_f\oplus\bigoplus_{f\in\mathrm{Fl}^*_{v;\mathbb{R}}}L_f\colon\Gamma(\mathbf{u}_v)\equiv\Gamma(u_v)^{\phi,\sigma}\longrightarrow\bigoplus_{f\in\mathrm{Fl}^*_{v;\mathbb{C}}}T_{u_v(z_f)}X\ \oplus\ \bigoplus_{f\in\mathrm{Fl}^*_{v;\mathbb{R}}}T_{u_v(z_f)}X^{\phi}$$

with $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0$ and the homomorphisms

$$\bigoplus_{f\in\mathrm{Fl}^*_v}L_f\colon\Gamma(\mathbf{u}_v)\equiv\Gamma\big(u_v\sqcup u_{\sigma(v)}\big)^{\phi,\sigma}\longrightarrow\bigoplus_{f\in\mathrm{Fl}^*_v}T_{u_v(z_f)}X$$

with $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0$ are surjective. Thus, the restriction of $\pi_{\gamma,\overline{\gamma}}$ to the first subspace on the right-hand side of (4.32) is surjective as well. This in turn implies (4.32). $\square$

Proof of Proposition 4.3. Let

$$\mathfrak{F}_v \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}_v, \quad v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0, \quad \text{and} \quad \mathfrak{F}^\bullet_v \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}^\bullet_v, \quad v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0, \tag{4.34}$$

be the bundles with fibers $\Gamma^{0,1}_J(\mathbf{u}_v)$ and $\Gamma^{0,1}_J(\mathbf{u}^\bullet_v)$, respectively. Denote by

$$\begin{aligned}\pi_v\colon \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^\bullet_v \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}_v, \quad v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0,\\ \pi^\bullet_v\colon \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^\bullet_v \longrightarrow \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}^\bullet_v, \quad v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0,\end{aligned}$$

the projection maps. Define sections $\bar\partial_v$ and $\bar\partial^\bullet_v$ of (4.34) by (4.24) with $(\mathbf{u},\Sigma)$ replaced by $(\mathbf{u}_v,\Sigma_v)$ and $(\mathbf{u}_v\sqcup\mathbf{u}_{\sigma(v)},\Sigma_v\sqcup\Sigma_{\sigma(v)})$, respectively.

The section $\bar\partial$ in (4.24) is the restriction of the section

$$\begin{aligned}\bar\partial\equiv \bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\pi^*_v\bar\partial_v\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0}\pi^\bullet_v{}^*\bar\partial^\bullet_v\colon\\ \mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\prod_{v\in\mathrm{V}_{\mathbb{R}}(\gamma)}\mathfrak{B}_v\times\prod_{\{v,\sigma(v)\}\subset\mathrm{V}_{\mathbb{C}}(\gamma)}\mathfrak{B}^\bullet_v\longrightarrow\bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\pi^*_v\mathfrak{F}_v\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0}\pi^\bullet_v{}^*\mathfrak{F}^\bullet_v\end{aligned}\tag{4.35}$$

to the base of the bundle (4.23). By Lemma 4.4, the latter is a separable Banach submanifold of the base of (4.35). We show that the bundle section (4.24) is transverse to the zero set. Fix an element $(J,\nu;\mathbf{u})$ of its zero set (4.25) with $\mathbf{u}$ as in (4.20).

For each $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0$, let

$$\begin{gathered}D^0_{J,\nu;\mathbf{u}_v}\bar\partial\colon T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}_v),\\ D^0_{J,\nu;\mathbf{u}_v}\bar\partial(A,\nu';\xi)=D^0_{J,\nu_{\gamma,\varpi;\rho};\mathbf{u}_v}\xi+\frac{1}{2}A\circ\mathrm{d}u_v\circ\mathfrak{j}-\{q_{\gamma,\varpi;v}\times u_v\}^*\nu',\end{gathered}$$

with the last term above defined to be 0 if $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_\bullet$. We note that

$$\begin{aligned}A\circ\mathrm{d}u_v\circ\mathfrak{j}=0 &\qquad \text{if}\quad u_v(\Sigma_v)\cap\mathrm{supp}(A)=\varnothing,\\ \{q_{\gamma,\varpi;v}\times u_v\}^*\nu'=0 &\qquad \text{if}\quad \big(q_{\gamma,\varpi;v}(\Sigma_v)\times X\big)\cap\mathrm{supp}(\nu')=\varnothing.\end{aligned}\tag{4.36}$$

For each $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0$, let

$$\begin{gathered}D^0_{J,\nu;\mathbf{u}^\bullet_v}\bar\partial\colon T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u}^\bullet_v)\longrightarrow\Gamma^{0,1}_J(\mathbf{u}^\bullet_v),\\ D^0_{J,\nu;\mathbf{u}^\bullet_v}\bar\partial(A,\nu';\xi)=D^0_{J,\nu;\mathbf{u}^\bullet_v}\xi+\frac{1}{2}\big(A\circ\mathrm{d}u_v\circ\mathfrak{j},A\circ\mathrm{d}u_{\sigma(v)}\circ\mathfrak{j}\big)-\big\{q^\bullet_{\gamma,\varpi_{\gamma,\varpi;\rho};v}\times(u_v\sqcup u_{\sigma(v)})\big\}^*\nu',\end{gathered}$$

with the last term above defined to be 0 if $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_\bullet$. We note that

$$\begin{aligned}\big(A\circ\mathrm{d}u_v\circ\mathfrak{j},A\circ\mathrm{d}u_{\sigma(v)}\circ\mathfrak{j}\big)=0 &\qquad \text{if}\quad u_v(\Sigma_v)\cap\mathrm{supp}(A)=\varnothing,\\ \{q^\bullet_{\gamma,\varpi;v}\times u_v\}^*\nu'=0 &\qquad \text{if}\quad \big(q^\bullet_{\gamma,\varpi;v}(\Sigma_v)\times X\big)\cap\mathrm{supp}(\nu')=\varnothing.\end{aligned}\tag{4.37}$$

The homomorphisms $D^0_{J,\nu;\mathbf{u}_v}\bar\partial$ and $D^0_{J,\nu;\mathbf{u}^\bullet_v}\bar\partial$ are the restrictions of the linearizations of $\bar\partial_v$ at $(J,\nu;\mathbf{u}_v)$ and of $\bar\partial^\bullet_v$ at $(J,\nu;\mathbf{u}^\bullet_v)$ to

$$\begin{aligned} T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u}_v) &\subset T_{(J,\nu;\mathbf{u}_v)}\big(\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}_v\big) \qquad \text{and}\\ T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u}^\bullet_v) &\subset T_{(J,\nu;\mathbf{u}^\bullet_v)}\big(\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\times\mathfrak{B}^\bullet_v\big), \end{aligned}$$

respectively. Since $\Gamma_0(\mathbf{u})\subset T_{\mathbf{u}}\mathfrak{B}^*_{\gamma,\varpi;\rho}$, it is sufficient to show that the homomorphism

$$\begin{split} \bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\pi_v^*D^0_{J,\nu;\mathbf{u}_v}\bar\partial\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0}\pi^{\bullet\,*}_vD^0_{J,\nu;\mathbf{u}^\bullet_v}\bar\partial:\\ T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u})\longrightarrow\bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0}\Gamma^{0,1}_J(\mathbf{u}_v)\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0}\Gamma^{0,1}_J(\mathbf{u}^\bullet_v) \end{split} \tag{4.38}$$

is surjective.

Since $\mathbf{u}\in\mathfrak{B}^*_{\gamma,\varpi;\rho}$, the subsets $u_v(\Sigma_v)\subset X$ with $v\in\aleph(\gamma,\overline{\gamma})_\bullet$ are distinct. There thus exist ϕ-invariant open subsets $W_v\subset X$ with $v\in\aleph(\gamma,\overline{\gamma})_\bullet$ such that

$$u_v(\Sigma_v)\cap W_v\neq\varnothing \qquad \forall\,v\in\aleph(\gamma,\overline{\gamma})_\bullet, \tag{4.39}$$
$$u_{v_1}(\Sigma_{v_1})\cap W_{v_2}=\varnothing,\quad W_{v_1}\cap W_{v_2}=\varnothing \qquad \forall\,v_1,v_2\in\aleph(\gamma,\overline{\gamma})_\bullet,\ v_1\neq v_2,\sigma(v_2). \tag{4.40}$$

For $v\in\mathrm{V}_{\mathbb{C}}(\overline{\gamma})$, let $q_{\gamma,\varpi;v}=q^\bullet_{\gamma,\varpi;v}|_{\Sigma_v}$. The subsets $q_{\gamma,\varpi;v}(\Sigma_v)\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ with $v\in\mathrm{V}(\overline{\gamma})$ are also distinct. There thus exist $\widetilde\sigma_{\mathbb{R}}$-invariant open subsets $W_v\subset\mathbb{R}\widetilde{\mathcal{U}}_{g,l;k}$ with $v\in\mathrm{V}(\overline{\gamma})$ such that

$$q_{\gamma,\varpi;v}(\Sigma_v)\cap W_v\neq\varnothing \qquad \forall\,v\in\mathrm{V}(\overline{\gamma}), \tag{4.41}$$
$$q_{\gamma,\varpi;v_1}(\Sigma_{v_1})\cap W_{v_2}=\varnothing,\quad W_{v_1}\cap W_{v_2}=\varnothing \qquad \forall\,v_1,v_2\in\mathrm{V}(\overline{\gamma}),\ v_1\neq v_2,\sigma(v_2). \tag{4.42}$$

Define

$$\begin{aligned} T_v\mathcal{H} &= \big\{A\in T_J\mathcal{J}^\phi_\omega\colon \mathrm{supp}(A)\subset W_v\big\} && \forall\,v\in\aleph(\gamma,\overline{\gamma})_\bullet,\\ T_v\mathcal{H} &= \big\{\nu'\in\Gamma^{0,1}_{g,l;k}(X;J)^\phi\colon \mathrm{supp}(\nu')\subset W_v\times X\big\} && \forall\,v\in\mathrm{V}(\overline{\gamma})\,. \end{aligned}$$

By Lemmas 4.1 and 4.2, (4.39), and (4.41),

$$\begin{aligned} D^0_{J,\nu;\mathbf{u}_v}\bar\partial\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}_v)\big) &= \Gamma^{0,1}_J(\mathbf{u}_v) && \forall\ v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0,\\ D^0_{J,\nu;\mathbf{u}^\bullet_v}\bar\partial\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}^\bullet_v)\big) &= \Gamma^{0,1}_J(\mathbf{u}^\bullet_v) && \forall\ v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0\,. \end{aligned} \tag{4.43}$$

By definition,

$$\begin{split} D^0_{J,\nu;\mathbf{u}_{v_1}}\bar\partial\big(\Gamma_0(\mathbf{u}_{v_1'})\big),D^0_{J,\nu;\mathbf{u}_{v_1}}\bar\partial\big(\Gamma_0(\mathbf{u}^\bullet_{v_2})\big),D^0_{J,\nu;\mathbf{u}^\bullet_{v_2}}\bar\partial\big(\Gamma_0(\mathbf{u}_{v_1})\big),D^0_{J,\nu;\mathbf{u}^\bullet_{v_2}}\bar\partial\big(\Gamma_0(\mathbf{u}^\bullet_{v_2'})\big)=\{0\},\\ \forall\ v_1,v_1'\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0,\ v_2,v_2'\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0,\ v_1\neq v_1',\ v_2\neq v_2',\sigma(v_2'). \end{split} \tag{4.44}$$

By (4.36), (4.37), and the first statements in (4.40) and (4.42),

$$\begin{split} D^0_{J,\nu;\mathbf{u}_{v_1}}\bar\partial\big(T_v\mathcal{H}\big),D^0_{J,\nu;\mathbf{u}^\bullet_{v_2}}\bar\partial\big(T_v\mathcal{H}\big)=\{0\}\\ \forall\ v_1\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})^c_0,\ v_2\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})^c_0,\ v\in\aleph(\gamma,\overline{\gamma})^c_0,\ v\neq v_1,v_2,\sigma(v_2),\ (v_i,v)\notin\mathrm{V}(\overline{\gamma})\times\aleph(\gamma,\overline{\gamma})_\bullet. \end{split} \tag{4.45}$$

By the last statements in (4.40) and (4.42),

$$\bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0^c}\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}_v)\big)\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0^c}\big(T_v\mathcal{H}\oplus\Gamma_0(\mathbf{u}_v^\bullet)\big)\subset T_{(J,\nu)}\mathcal{H}^{\omega,\phi}_{g,l;k}(X)\oplus\Gamma_0(\mathbf{u}). \tag{4.46}$$

By (4.43)-(4.46), the homomorphism (4.38) is surjective. This establishes the first claim of the proposition.

Let G_v for $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$ and $G_{\rho;v}$ for $v\in\mathrm{V}_{\mathbb{R}}(\gamma)$ with $|\varepsilon^{-1}(v)|\leqslant 2$ be as in Section 3.2. Denote by G_v for $v\in\mathrm{V}_{\mathbb{C}}(\gamma)$ with $|\varepsilon^{-1}(v)|\geqslant 3$ and $G_{\rho;v}$ for $v\in\mathrm{V}_{\mathbb{R}}(\gamma)$ with $|\varepsilon^{-1}(v)|\geqslant 3$ the trivial groups. Thus,

$$\begin{aligned}\dim_{\mathbb{R}}\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}&=3\big(\mathfrak{g}(v)-1\big)+\big|\varepsilon^{-1}(v)\big|+\dim_{\mathbb{R}}G_{\rho;v}\qquad\forall\,v\in\mathrm{V}_{\mathbb{R}}(\gamma),\\ \dim_{\mathbb{R}}\widetilde{\mathcal{M}}^\bullet_{\gamma;v}&=6\big(\mathfrak{g}(v)-1\big)+2\big|\varepsilon^{-1}(v)\big|+\dim_{\mathbb{R}}G_v\qquad\forall\,v\in\mathrm{V}_{\mathbb{C}}(\gamma).\end{aligned}$$

Since $\mathfrak{g}(\sigma(v))=\mathfrak{g}(v)$ and $|\varepsilon^{-1}(\sigma(v))|=|\varepsilon^{-1}(v)|$, it follows that

$$\begin{aligned}\sum_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0^c}\dim_{\mathbb{R}}\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}+\sum_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0^c}\dim_{\mathbb{R}}\widetilde{\mathcal{M}}^\bullet_{\gamma;v}&=\sum_{v\in\mathrm{Ver}}\Big(3\big(\mathfrak{g}(v)-1\big)+\big|\varepsilon^{-1}(v)\big|\Big)+\dim_{\mathbb{R}}G_{\gamma;\rho}\\ &=3(g'-1)+2l+k-|\gamma|+\dim_{\mathbb{R}}G_{\gamma;\rho};\end{aligned}\tag{4.47}$$

the second equality above follows from (3.15) with $\overline{\gamma}$ replaced by $\gamma\in\mathcal{A}^\phi_{g',l;k}(B')$.

Let $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0^c$. By the proof of [9, Proposition 3.6], the restriction

$$D_{J,\nu;\mathbf{u}_v}:\Gamma(\mathbf{u}_v)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\mathbf{u}_v)$$

of $D_{J,\nu;u_v}$ is a Fredholm operator of index

$$\mathrm{ind}_{\mathbb{R}}D_{J,\nu;\mathbf{u}_v}=\big\langle c_1(TX),\mathfrak{d}(v)\big\rangle+n\big(1-\mathfrak{g}(v)\big).$$

Thus, the restriction

$$\widetilde{D}_{J,\nu;\mathbf{u}_v}:T_{\mathbf{u}_v}\mathfrak{B}_v\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\mathbf{u}_v)$$

of the linearization of $\bar\partial_v$ at $(J,\nu_{\gamma,\varpi;\rho};\mathbf{u}_v)$ is a Fredholm operator of index

$$\mathrm{ind}_{\mathbb{R}}\widetilde{D}_{J,\nu;\mathbf{u}_v}=\big\langle c_1(TX),\mathfrak{d}(v)\big\rangle+n\big(1-\mathfrak{g}(v)\big)+\dim_{\mathbb{R}}\mathbb{R}\widetilde{\mathcal{M}}_{\gamma;v}\,. \tag{4.48}$$

This formula is also valid for $v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0$ if $\bar\partial_v$ is replaced by the section of the rank 0 bundle.

Let $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0^c$. By the reasoning at the beginning of the proof of Lemma 4.1 and [19, Theorem C.1.10], the restriction

$$D^\bullet_{J,\nu;\mathbf{u}_v}:\Gamma(\mathbf{u}_v^\bullet)\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\mathbf{u}_v^\bullet)$$

of $D^\bullet_{J,\nu;u_v}$ is a Fredholm operator of index

$$\mathrm{ind}_{\mathbb{R}}D^\bullet_{J,\nu;\mathbf{u}_v}=2\big\langle c_1(TX),\mathfrak{d}(v)\big\rangle+2n\big(1-\mathfrak{g}(v)\big).$$

Thus, the restriction

$$\widetilde{D}_{J,\nu;\mathbf{u}_v^\bullet}:T_{\mathbf{u}_v^\bullet}\mathfrak{B}_v^\bullet\longrightarrow\Gamma^{0,1}_{J,\mathfrak{j}}(\mathbf{u}_v^\bullet)$$

of the linearization of $\bar{\partial}_v^\bullet$ at $(J,\nu_{\gamma,\varpi;\rho};\mathbf{u}_v)$ is a Fredholm operator of index

$$\mathrm{ind}_{\mathbb{R}}\widetilde{D}_{J,\nu;\mathbf{u}_v^\bullet}=2\big\langle c_1(TX),\mathfrak{d}(v)\big\rangle+2n\big(1-\mathfrak{g}(v)\big)+\dim_{\mathbb{R}}\widetilde{\mathcal{M}}_{\gamma;v}^\bullet\,. \tag{4.49}$$

This formula is also valid for $v\in\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0$ if $\bar{\partial}_v^\bullet$ is replaced by the section of the rank 0 bundle.

By (4.47)-(4.49), (3.15) with $\overline{\gamma}$ replaced $\gamma\in\mathcal{A}_{g',l;k}^\phi(B')$, and (3.20) with B replaced by B', the restriction

$$\widetilde{D}_{J,\nu;\mathbf{u}}\colon\bigoplus_{v\in\aleph_{\mathbb{R}}(\gamma,\overline{\gamma})_0^c}T_{\mathbf{u}_v}\mathfrak{B}_v\oplus\bigoplus_{\{v,\sigma(v)\}\subset\aleph_{\mathbb{C}}(\gamma,\overline{\gamma})_0^c}T_{\mathbf{u}_v^\bullet}\mathfrak{B}_v^\bullet\longrightarrow\Gamma_{J,\mathfrak{j}}^{0,1}(\mathbf{u})$$

of the linearization of $\bar{\partial}$ at $(J,\nu;\mathbf{u})$ is a Fredholm operator of index

$$\begin{aligned}\mathrm{ind}_{\mathbb{R}}\widetilde{D}_{J,\nu;\mathbf{u}}&=\big\langle c_1(TX),B'\big\rangle+(n-3)(1-g')+2l+k+(n-1)|\gamma|+\dim_{\mathbb{R}}G_{\gamma;\rho}\\&=\dim_{g',l;k}(B')+(n-1)|\gamma|+\dim_{\mathbb{R}}G_{\gamma;\rho}\,.\end{aligned}$$

Along with Lemma 4.4, this implies that the restriction

$$\widetilde{D}'_{J,\nu;\mathbf{u}}\colon T_{\mathbf{u}}\mathfrak{B}_{\gamma,\varpi;\rho}^*\longrightarrow\Gamma_{J,\mathfrak{j}}^{0,1}(\mathbf{u}) \tag{4.50}$$

of the linearization of (4.24) at $(J,\nu;\mathbf{u})$ is a Fredholm operator of index (4.28). The last claim of the proposition follows from this conclusion by the reasoning at the beginning of the proof of [19, Theorem 3.1.6(ii)]. □

By the reasoning in the proof of [19, Theorem 3.1.6(ii)], (J,ν) is a regular value of π if and only if the operator (4.50) is surjective for every element $\mathbf{u}$ of the preimage

$$\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}^*(J,\nu)\equiv\pi^{-1}(J,\nu)\subset\mathfrak{U}\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}^*(J,\nu) \tag{4.51}$$

of (J,ν). Suppose $m-2>\mathrm{ind}_{\mathbb{R}}\pi$. By the Sard-Smale Theorem, the set $\widehat{\mathcal{H}}^m$ of regular values of π is then a Baire subset of second category in (the C^m-completion of) $\mathcal{H}_{g,l;k}^{\omega,\phi}(X)$ in the C^m-topology. Along with Taubes' argument in the proof of [19, Theorem 3.1.6(ii)], this implies that the subset

$$\widehat{\mathcal{H}}_{g,l;k}^{\omega,\phi}(X)\subset\mathcal{H}_{g,l;k}^{\omega,\phi}(X)$$

of smooth pairs (J,ν) so that the operator (4.50) is surjective for every element $\mathbf{u}$ of $\widetilde{\mathfrak{M}}_{\gamma,\varpi;\rho}^*(J,\nu)$ is Baire of second category in the space of all smooth pairs (J,ν) in the C^∞-topology. For every pair (J,ν) in this subset, the left-hand side in (4.51) is a smooth submanifold of the right-hand side of (4.27) of dimension (4.28). The group $G_{\gamma;\rho}^0$ acts smoothly and freely on this submanifold. The smooth structure on the quotient descends to a smooth structure on $\mathcal{Z}_{\gamma,\varpi}^*(J,\nu)$.

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