Real Gromov-Witten Theory in All Genera and Real Enumerative Geometry: Construction

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Abstract

We construct positive-genus analogues of Welschinger's invariants for many real symplectic manifolds, including the odd-dimensional projective spaces and the renowned quintic threefold. In some cases, our invariants provide lower bounds for counts of real positive-genus curves in real algebraic varieties. Our approach to the orientability problem is based entirely on the topology of real bundle pairs over symmetric surfaces; the previous attempts involved direct computations for the determinant lines of Fredholm operators over bordered surfaces. We use the notion of real orientation introduced in this paper to obtain isomorphisms of real bundle pairs over families of symmetric surfaces and then apply the determinant functor to these isomorphisms. This allows us to endow the uncompactified moduli spaces of real maps from symmetric surfaces of all topological types with natural orientations and to verify that they extend across the codimension-one boundaries of these spaces, thus implementing a far-reaching proposal from C.-C. Liu's thesis for a fully fledged real Gromov-Witten theory. The second and third parts of this work concern applications: they describe important properties of our orientations on the moduli spaces, establish some connections with real enumerative geometry, provide the relevant equivariant localization data for projective spaces, and obtain vanishing results in the spirit of Walcher's predictions.

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1 Introduction

The theory of J-holomorphic maps plays prominent roles in symplectic topology, algebraic geometry, and string theory. The foundational work of [23, 29, 35, 27, 9] has established the theory of (closed) Gromov-Witten invariants, i.e. counts of J-holomorphic maps from closed Riemann surfaces to symplectic manifolds. In contrast, the theory of real Gromov-Witten invariants, i.e. counts of J-holomorphic maps from symmetric Riemann surfaces commuting with the involutions on the domain and the target, is still in early stages of development, especially in positive genera. The two main obstacles to defining real Gromov-Witten invariants are the potential non-orientability of the moduli space of real J-holomorphic maps and the existence of real codimension-one boundary strata.

In this paper, we introduce the notion of real orientation on a real symplectic $2n$-manifold (X, ω, ϕ); see Definitions 1.1 and 1.2. We overcome the first obstacle by showing that a real orientation induces orientations on the uncompactified moduli spaces of real maps for *all* genera of and for *all* types of involutions σ on the domain if n is odd; see Theorem 1.3. We then show that these orientations do not change across the codimension-one boundary strata after they are reversed for half of the involution types in each genus. This allows us to overcome the second obstacle by gluing the moduli spaces for different types of involutions along their common boundaries; this realizes an aspiration going back to [28]. We thus obtain real Gromov-Witten invariants of arbitrary genus for many real symplectic manifolds; see Theorems 1.4 and 1.5. Many projective complete intersections, including the quintic threefold which plays a central role in Gromov-Witten theory and string theory, are among these manifolds; see Proposition 2.1. These invariants can be used to obtain lower bounds for counts of real positive-genus curves in real algebraic varieties; see Proposition 2.5. For example, we find that there are at least 4 real genus 1 degree 6 irreducible curves passing through a generic collection of 6 pairs of conjugate points in $\mathbb{P}^3$.

1.1 Terminology and setup

An involution on a smooth manifold X is a diffeomorphism $\phi\colon X\longrightarrow X$ such that $\phi\circ\phi=\mathrm{id}_X$. Let

$$X^\phi=\big\{x\in X:\ \phi(x)=x\big\}$$

denote the fixed locus. An anti-symplectic involution ϕ on a symplectic manifold (X,ω) is an involution $\phi\colon X\longrightarrow X$ such that $\phi^*\omega=-\omega$. For example, the maps

$$\begin{aligned}\tau_n\colon \mathbb{P}^{n-1}&\longrightarrow\mathbb{P}^{n-1}, & [Z_1,\ldots,Z_n]&\longrightarrow[\overline{Z}_1,\ldots,\overline{Z}_n],\\ \eta_{2m}\colon \mathbb{P}^{2m-1}&\longrightarrow\mathbb{P}^{2m-1}, & [Z_1,Z_2,\ldots,Z_{2m-1},Z_{2m}]&\longrightarrow\big[-\overline{Z}_2,\overline{Z}_1,\ldots,-\overline{Z}_{2m},\overline{Z}_{2m-1}\big],\end{aligned}$$

are anti-symplectic involutions with respect to the standard Fubini-Study symplectic forms ω_n on $\mathbb{P}^{n-1}$ and ω_{2m} on $\mathbb{P}^{2m-1}$, respectively. If

$$k\geqslant 0,\qquad \mathbf{a}\equiv(a_1,\ldots,a_k)\in(\mathbb{Z}^+)^k\,,$$

and $X_{n;\mathbf{a}}\subset\mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n, then $\tau_{n;\mathbf{a}}\equiv\tau_n|_{X_{n;\mathbf{a}}}$ is an anti-symplectic involution on $X_{n;\mathbf{a}}$ with respect to the symplectic form $\omega_{n;\mathbf{a}}=\omega_n|_{X_{n;\mathbf{a}}}$. Similarly, if $X_{2m;\mathbf{a}}\subset\mathbb{P}^{2m-1}$ is preserved by η_{2m}, then $\eta_{2m;\mathbf{a}}\equiv\eta_{2m}|_{X_{2m;\mathbf{a}}}$ is an anti-symplectic involution on $X_{2m;\mathbf{a}}$ with respect to the symplectic form $\omega_{2m;\mathbf{a}}=\omega_{2m}|_{X_{2m;\mathbf{a}}}$. A real symplectic manifold is a triple (X,ω,ϕ) consisting of a symplectic manifold (X,ω) and an anti-symplectic involution ϕ.

Let (X,ϕ) be a manifold with an involution. A conjugation on a complex vector bundle $V\longrightarrow X$ lifting an involution ϕ is a vector bundle homomorphism $\varphi\colon V\longrightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\varphi\colon V\longrightarrow\phi^*V$ covering id_X) such that the restriction of φ to each fiber is anti-complex linear and $\varphi\circ\varphi=\mathrm{id}_V$. A real bundle pair $(V,\varphi)\longrightarrow(X,\phi)$ consists of a complex vector bundle $V\longrightarrow X$ and a conjugation φ on V lifting ϕ. For example,

$$(TX,\mathrm{d}\phi)\longrightarrow(X,\phi)\qquad\text{and}\qquad(X\times\mathbb{C}^n,\phi\times\mathfrak{c})\longrightarrow(X,\phi),$$

where $\mathfrak{c}\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n$ is the standard conjugation on $\mathbb{C}^n$, are real bundle pairs. For any real bundle pair $(V,\varphi)\longrightarrow(X,\phi)$, we denote by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)=(\Lambda^{\mathrm{top}}_{\mathbb{C}}V,\Lambda^{\mathrm{top}}_{\mathbb{C}}\varphi)$$

the top exterior power of V over $\mathbb{C}$ with the induced conjugation. Direct sums, duals, and tensor products over $\mathbb{C}$ of real bundle pairs over (X,ϕ) are again real bundle pairs over (X,ϕ).

A symmetric surface (Σ,σ) is a closed connected oriented smooth surface Σ (manifold of real dimension 2) with an orientation-reversing involution σ. The fixed locus of σ is a disjoint union of circles. If in addition (X,ϕ) is a manifold with an involution, a real map

$$u\colon(\Sigma,\sigma)\longrightarrow(X,\phi)$$

is a smooth map $u\colon\Sigma\longrightarrow X$ such that $u\circ\sigma=\phi\circ u$. We denote the space of such maps by $\mathfrak{B}_g(X)^{\phi,\sigma}$, with g denoting the genus of the domain Σ of σ.

For a symplectic manifold (X,ω), we denote by $\mathcal{J}_\omega$ the space of ω-compatible almost complex structures on X. If ϕ is an anti-symplectic involution on (X,ω), let

$$\mathcal{J}_\omega^\phi=\big\{J\in\mathcal{J}_\omega:\ \phi^*J=-J\big\}. \tag{1.1}$$

For a genus g symmetric surface (Σ,σ), we similarly denote by $\mathcal{J}_\Sigma^\sigma$ the space of complex structures $\mathfrak{j}$ on Σ compatible with the orientation such that $\sigma^*\mathfrak{j}=-\mathfrak{j}$. For $J\in\mathcal{J}_\omega^\phi$, $\mathfrak{j}\in\mathcal{J}_\Sigma^\sigma$, and $u\in\mathfrak{B}_g(X)^{\phi,\sigma}$, let

$$\bar\partial_{J,\mathfrak{j}}u=\frac{1}{2}\big(\mathrm{d}u+J\circ\mathrm{d}u\circ\mathfrak{j}\big)$$

be the $\bar\partial_J$-operator on $\mathfrak{B}_g(X)^{\phi,\sigma}$.

Let $g,l\in\mathbb{Z}^{\geqslant0}$, (Σ,σ) be a genus g symmetric surface, $B\in H_2(X;\mathbb{Z})-0$, and $J\in\mathcal{J}_\omega^\phi$. Let $\Delta^{2l}\subset\Sigma^{2l}$ be the big diagonal, i.e. the subset of $2l$-tuples with at least two coordinates equal. Denote by

$$\begin{aligned}\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}=\big\{&(u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),\mathfrak{j})\in\mathfrak{B}_g(X)^{\phi,\sigma}\times(\Sigma^{2l}-\Delta^{2l})\times\mathcal{J}_\Sigma^\sigma:\\ &z_i^-=\sigma(z_i^+)\ \forall\,i=1,\ldots,l,\ u_*[\Sigma]_\mathbb{Z}=B,\ \bar\partial_{J,\mathfrak{j}}u=0\big\}/\sim\end{aligned}$$

the (uncompactified) moduli space of equivalence classes of degree B real J-holomorphic maps from (Σ,σ) to (X,ϕ) with l conjugate pairs of marked points. Two marked J-holomorphic (ϕ,σ)-real maps determine the same element of this moduli space if they differ by an orientation-preserving diffeomorphism of Σ commuting with σ. We denote by

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}\supset\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \tag{1.2}$$

Gromov's convergence compactification of $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ obtained by including stable real maps from nodal symmetric surfaces. The (virtually) codimension-one boundary strata of

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}-\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\subset\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

consist of real J-holomorphic maps from one-nodal symmetric surfaces to (X,ϕ). Each stratum is either a (virtual) hypersurface in $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ or a (virtual) boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of orientation-reversing involutions σ on Σ. Let

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi}=\bigsqcup_\sigma\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\quad\text{and}\quad\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}=\bigcup_\sigma\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma} \tag{1.3}$$

denote the (disjoint) union of the uncompactified real moduli spaces and the union of the compactified real moduli spaces, respectively, taken over all topological types of orientation-reversing involutions σ on Σ.

Similarly to Example 4.3, we denote by

$$\det\bar\partial_\mathbb{C}\longrightarrow\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$$

the determinant line bundle of the standard real Cauchy-Riemann operator with values in $(\mathbb{C},\mathfrak{c})$. This real line bundle is not orientable if X is a point and $g\geqslant1$. It is not needed to formulate the main immediately applicable results of this paper, Theorems 1.4 and 1.5 below, but is used in the overarching statement of Theorem 1.3.

1.2 Real orientations and real GW-invariants

We now introduce the notion of real orientation on a real symplectic manifold and state the main theorems of this paper.

Definition 1.1. A real symplectic manifold (X,ω,ϕ) is real-orientable if there exists a rank 1 real bundle pair $(L,\widetilde{\phi})$ over (X,ϕ) such that

$$w_2(TX^{\phi}) = w_1(L^{\widetilde{\phi}})^2 \qquad \text{and} \qquad \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi) \approx (L,\widetilde{\phi})^{\otimes 2}. \tag{1.4}$$

Definition 1.2. A real orientation on a real-orientable symplectic manifold (X,ω,ϕ) consists of

(RO1) a rank 1 real bundle pair $(L,\widetilde{\phi})$ over (X,ϕ) satisfying (1.4),

(RO2) a homotopy class $[\psi]$ of isomorphisms of real bundle pairs in (1.4), and

(RO3) a spin structure $\mathfrak{s}$ on the real vector bundle $TX^{\phi}\oplus 2(L^*)^{\widetilde{\phi}^*}$ over X^{ϕ} compatible with the orientation induced by (RO2).

Theorem 1.3. *Let (X,ω,ϕ) be a real-orientable $2n$-manifold, $g,l\in\mathbb{Z}^{\geqslant 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_{\omega}^{\phi}$. Then a real orientation on (X,ω,ϕ) determines an orientation on the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}\big)\otimes\big(\det\bar{\partial}_{\mathbb{C}}\big)^{\otimes(n+1)} \longrightarrow \overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}. \tag{1.5}$$

In particular, the real moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ is orientable if n is odd.

A homotopy class of isomorphisms as in (1.4) determines an orientation on TX^{ϕ} and thus on $TX^{\phi}\oplus 2(L^*)^{\widetilde{\phi}^*}$; see the paragraph after Definition 5.1. In particular, Theorem 1.3 does not apply to any real symplectic manifold (X,ω,ϕ) with unorientable Lagrangian X^{ϕ}. By the first assumption in (1.4), the real vector bundle $TX^{\phi}\oplus 2(L^*)^{\widetilde{\phi}^*}$ over X^{ϕ} admits a spin structure. Since $2(L^*)^{\widetilde{\phi}^*}\approx L^*|_{X^{\phi}}$, a real orientation on (X,ω,ϕ) includes a relative spin structure on $X^{\phi}\subset X$ in the sense of [10, Definition 8.1.2].

The moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ is not smooth in general. Its tangent bundle in (1.5) should be viewed in the usual moduli-theoretic (or virtual) sense, i.e. as the index of suitably defined linearization of the $\bar{\partial}_J$-operator (which includes deformations of the complex structure $\mathfrak{j}$ on Σ). The first statement of Theorem 1.3 and its proof also apply to Kuranishi charts for $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ and the tangent spaces of the moduli spaces of real (J,ν)-maps for generic local ϕ-invariant deformations ν of [36]. A Kuranishi structure for $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ is obtained by carrying out the constructions of [27, 9] in a ϕ-invariant manner; see [37, Section 7] and [11, Appendix]. Since the (virtual) boundary of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ is empty, Theorem 1.3 implies that this moduli space carries a virtual fundamental class in some cases and thus gives rise to real GW-invariants in arbitrary genus.

Theorem 1.4. *Let (X,ω,ϕ) be a compact real-orientable $2n$-manifold with $n\notin 2\mathbb{Z}$, $g,l\in\mathbb{Z}^{\geqslant 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_{\omega}^{\phi}$. Then a real orientation on (X,ω,ϕ) endows the moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ with a virtual fundamental class and thus gives rise to genus g real GW-invariants of (X,ω,ϕ) that are independent of the choice of $J\in\mathcal{J}_{\omega}^{\phi}$.*

The resulting real GW-invariants of (X,ω,ϕ) in general depend on the choice of real orientation. This situation is analogous to the dependence on the choice of relative spin structure often seen in open GW-theory.

A notion of semi-positive for a real symplectic manifold (X,ω,ϕ) is introduced in [50, Definition 1.2]. Monotone symplectic manifolds with an anti-symplectic involution, including all projective spaces with the standard involutions and real Fano hypersurfaces of dimension at least 3, are semi-positive. By [50, Theorem 3.3], the semi-positive property of [50, Definition 1.2] plays the same role in real GW-theory as the semi-positive property of [30, Definition 6.4.1] plays in "classical" GW-theory. In particular, the real analogues of the geometric perturbations of [36] introduced in [50, Section 3.1] suffice to define the invariants of Theorem 1.4 with constraints pulled back from the target and the Deligne-Mumford moduli space of real curves for a semi-positive real symplectic manifold (X,ω,ϕ) endowed with a real orientation. In these cases, the virtual tangent space of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ appearing in (1.5) can be replaced by the actual tangent space of the moduli space of simple real (J,ν)-holomorphic maps from smooth and one-nodal symmetric surfaces of genus g. The invariance of the resulting counts of such maps can then be established by following along a path of auxiliary data; it can pass only through one-nodal degenerations.

Theorem 1.4 yields counts of real curves with conjugate pairs of insertions only. By the last statement of [13, Theorem 6.5], the orientability of the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l;k}$ of real genus g curves with l conjugate pairs of marked points and k real marked points does not capture the orientability of the analogous moduli space $\overline{\mathfrak{M}}_{g,l;k}(X,B;J)^\phi$ of real maps whenever $k>0$. Theorem 1.3 remains valid for such moduli spaces outside of certain "bad" codimension-one strata. However, these strata are avoided by generic one-parameter families of real maps in certain cases; Theorem 1.3 then yields counts of real curves with conjugate pairs of insertions and real point insertions.

Theorem 1.5. *Let (X,ω,ϕ) be a compact real-orientable 6-manifold such that $\langle c_1(X),B\rangle\in 4\mathbb{Z}$ for all $B\in H_2(X;\mathbb{Z})$ with $\phi_*B=-B$. For all*

$$B\in H_2(X;\mathbb{Z}),\quad \mu_1,\ldots,\mu_l\in H^6(X;\mathbb{Q})\cup H^2(X;\mathbb{Q}),\quad \textit{and}\quad k\in\mathbb{Z}^{\geqslant 0},$$

a real orientation on (X,ω,ϕ) determines a signed count

$$\big\langle \mu_1,\ldots,\mu_l;\mathrm{pt}^k\big\rangle_{1,B}^{\phi}\in\mathbb{Q}$$

of real J-holomorphic genus 1 degree B curves which is independent of the choice of $J\in\mathcal{J}_\omega^\phi$.

The $n=0$ case of Theorem 1.3 is essentially Proposition 6.1 which describes the orientability of the Deligne-Mumford moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ of genus g symmetric surfaces with l conjugate pairs of marked points. If $n\in 2\mathbb{Z}$ and $g+l\geqslant 2$, Theorem 1.3 implies that a real orientation on (X,ω,ϕ) induces an orientation on the real line bundle

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi\big)\otimes\mathfrak{f}^*\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}(T\mathbb{R}\overline{\mathcal{M}}_{g,l})\big)\longrightarrow\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi, \tag{1.6}$$

where $\mathfrak{f}$ is the forgetful morphism (3.2). This orientation can be used to construct GW-invariants of (X,ω,ϕ) with classes twisted by the orientation system of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$, as done in [13] in the $g=0$ case.

1.3 Previous results and acknowledgments

Invariant signed counts of real genus 0 curves with point constraints in real symplectic 4-manifolds and in many real symplectic 6-manifolds are defined in [39, 40]. An approach to interpreting these counts in the style of Gromov-Witten theory, i.e. as counts of parametrizations of such curves, is presented in [4, 37]. Signed counts of real genus 0 curves with conjugate pairs of arbitrary (not necessarily point) constraints in arbitrary dimensions are defined in [13]. All of these invariants involve morphisms from $\mathbb{P}^1$ with the standard involution $\tau \equiv \tau_2$ only and are constructed under the assumption that the fixed circle cannot shrink in a limit; thus, only the degenerations of type (H3) in Section 3.2 are relevant in this case. This assumption is dropped in [7] by combining counts of $(\mathbb{P}^1, \tau)$-morphisms with counts of $(\mathbb{P}^1, \eta)$-morphisms for the fixed-point-free involution $\eta \equiv \eta_2$ on $\mathbb{P}^1$ and thus also considering the degenerations of type (E). As the degenerations of types (H1) and (H2) do not appear in genus 0, [7] thus implements the genus 0 case of an aspiration raised in [28] and elucidated in [34, Section 1.5]. The target manifolds considered in [7] are real-orientable in the sense of Definition 1.1 and have spin fixed locus.

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2 Examples, properties, and applications

We begin this section with examples of distinct collections of real-orientable symplectic manifolds. We then describe a number of properties of the real GW-invariants of Theorems 1.4 and 1.5, including connections with real enumerative geometry and compatibility with key morphisms of GW-theory. With the exception of Proposition 2.3 and Corollaries 2.6 and 2.7, the claims below are established in [17, 18].

Proposition 2.1. *Let $m, n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^{\geqslant 0}$, and $\mathbf{a} \equiv (a_1, \ldots, a_k) \in (\mathbb{Z}^+)^k$.*

(1) If $X_{n;\mathbf{a}} \subset \mathbb{P}^{n-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by τ_n,

$$\sum_{i=1}^{k} a_i \equiv n \mod 2, \qquad \text{and} \qquad \sum_{i=1}^{k} a_i^2 \equiv \sum_{i=1}^{k} a_i \mod 4,$$

then $(X_{n;\mathbf{a}}, \omega_{n;\mathbf{a}}, \tau_{n;\mathbf{a}})$ is a real-orientable symplectic manifold.

(2) If $X_{2m;\mathbf{a}} \subset \mathbb{P}^{2m-1}$ is a complete intersection of multi-degree $\mathbf{a}$ preserved by η_{2m} and

$$a_1 + \ldots + a_k \equiv 2m \mod 4,$$

then $(X_{2m;\mathbf{a}}, \omega_{2m;\mathbf{a}}, \eta_{2m;\mathbf{a}})$ is a real-orientable symplectic manifold.

Proposition 2.2. *Let (X, ω, ϕ) be a real symplectic manifold with $w_2(X^\phi) = 0$. If*

(1) $H_1(X; \mathbb{Q}) = 0$ and $c_1(X) = 2(\mu - \phi^*\mu)$ for some $\mu \in H^2(X; \mathbb{Z})$ or

(2) X is compact Kahler, ϕ is anti-holomorphic, and $\mathcal{K}_X = 2([D]+[\overline{\phi_* D}])$ for some divisor D on X,

then (X,ω,ϕ) is a real-orientable symplectic manifold.

Both of these propositions are established in [18]. The first one is obtained by explicitly constructing suitable rank 1 real bundle pairs $(L,\widetilde{\phi})$, while the second follows easily from the proof of [7, Proposition 1.5].

We recall that GW-invariants involving insertions only from the target X are called primitive. Such GW-invariants are related to counts of J-holomorphic *curves* in X passing through a corresponding collection of constraints (i.e. of representatives for the Poincare duals of the insertions used). In contrast, GW-invariants also involving ψ-classes, i.e. the Chern classes of the universal tangent line bundles at the marked points, are called descendant. The next vanishing result extends [16, Theorem 2.5]. Since the proof of the latter applies, we refer the reader to [16].

Proposition 2.3. *Let (X,ω,ϕ) be a compact real-orientable $2n$-manifold with $n\notin 2\mathbb{Z}$ and $g\in\mathbb{Z}^{\geqslant 0}$. The primary genus g real GW-invariants of (X,ω,ϕ) with conjugate pairs of constraints that include an insertion $\mu\in H^*(X;\mathbb{Q})$ such that $\phi^*\mu=\mu$ vanish.*

The genus g real GW-invariants of $\mathbb{P}^{2n-1}$ with conjugate pairs of constraints can be computed using the virtual equivariant localization theorem of [21]. In the $g=1$ case, all torus fixed loci are contained in the smooth locus of the moduli space and the classical equivariant localization theorem of [2] suffices. The relevant fixed loci data, which we describe in [18] based on the properties of the orientations of Theorem 1.3 obtained in [17], is consistent with [38, (3.22)]. We also obtain the two types of cancellations of contributions from some fixed loci predicted in [38, Sections 3.2,3.3]. We use this data to obtain the following qualitative observations in [18]; they extend [7, Theorem 1.10] from the $g=0$ case and [8, Theorem 7.2] from the $g=1$ case (the latter assuming that genus 1 real GW-invariants can be defined).

Proposition 2.4. *The genus g degree d real GW-invariants of $(\mathbb{P}^{2n-1},\omega_{2n},\tau_{2n})$ and $(\mathbb{P}^{4n-1},\omega_{4n},\eta_{4n})$ with only conjugate pairs of insertions vanish if $d-g\in 2\mathbb{Z}$. The genus g real GW-invariants of $(\mathbb{P}^{4n-1},\omega_{4n},\tau_{4n})$ and $(\mathbb{P}^{4n-1},\omega_{4n},\eta_{4n})$ with only conjugate pairs of insertions differ by the factor of $(-1)^{g-1}$.*

The primary genus g real GW-invariants arising from Theorem 1.4 are in general combinations of counts of real curves of genus g and counts of real curves of lower genera and/or of lower degree (lower symplectic energy). In light of [44, Theorems 1A,1B] and [48, Theorem 1.5], it seems plausible that the former can be extracted from these GW-invariants to directly provide lower bounds for enumerative counts of real curves in good situations. This would typically involve delicate obstruction analysis. However, the situation is fairly simple if $g=1$ and $n=3$.

Proposition 2.5. *Let (X,ω,ϕ) be a compact real-orientable 6-manifold and $J\in\mathcal{J}_\omega^\phi$ be an almost complex structure which is genus 1 regular in the sense of [45, Definition 1.4]. The primary genus 1 real GW-invariants of (X,ω,ϕ) are then equal to the corresponding signed counts of real J-holomorphic curves and thus provide lower bounds for the number of real genus 1 irreducible curves in (X,J,ϕ).*

Since the standard complex structure J_0 on $\mathbb{P}^3$ is genus 1 regular, the genus 1 real GW-invariants of $(\mathbb{P}^3,\omega_4,\tau_4)$ and $(\mathbb{P}^3,\omega_4,\eta_4)$ are lower bounds for the enumerative counts of such curves in $(\mathbb{P}^3,J_0,\tau_4)$

and $(\mathbb{P}^3, J_0, \eta_4)$, respectively. The claim of Proposition 2.5 is particularly evident in the case of real invariants of $(\mathbb{P}^3, J_0, \tau_4)$ and $(\mathbb{P}^3, J_0, \eta_4)$. The only lower-genus contributions for the genus 1 GW-invariants of 6-dimensional symplectic manifolds can come from the genus 0 curves. If J is genus 1 regular, such contributions arise from the stratum of the moduli space consisting of morphisms with contracted genus 1 domain and a single effective bubble. In the case of real morphisms, the node of the domain of such a map would have to be real. There are no such morphisms in the case of $(\mathbb{P}^3, J_0, \eta_4)$ because the real locus of $(\mathbb{P}^3, \eta_4)$ is empty. In the case of $(\mathbb{P}^3, J_0, \tau_4)$, the genus 0 contribution to the genus 1 real GW-invariant is a multiple of the genus 0 real GW-invariant with the same insertions. The genus 0 real GW-invariants of $(\mathbb{P}^3, J_0, \tau_4)$ are known to vanish in the even degrees; see [40, Remark 2.4(2)] and [7, Theorem 1.10]. However, the substance of Proposition 2.5 is that the genus 0 real enumerative counts do not contribute to the genus 1 real GW-invariants in all of the cases under consideration; this is shown in [18]. The situation in higher genus is described in [32].

From the equivariant localization data in [18], we find that the genus 1 degree d real GW-invariant of $\mathbb{P}^3$ with d pairs of conjugate point insertions is 0 for $d=2$, -1 for $d=4$, and -4 for $d=6$. The $d=2$ number is as expected, since there are no connected degree 2 curves of any kind passing through two generic pairs of conjugate points in $\mathbb{P}^3$. The $d=4$ number is also not surprising, since there is only one genus 1 degree 4 curve passing through 8 generic points in $\mathbb{P}^3$; see the first three paragraphs of [26, Section 1]. By [12], the genus 0 and genus 1 degree 6 GW-invariants of $\mathbb{P}^3$ with 12 point insertions are 2576 and 1496/3, respectively. By [46, Theorem 1.1], this implies that the number of genus 1 degree 6 curves passing through 12 generic points in $\mathbb{P}^3$ is 2860. Our signed count of -4 for the real genus 1 degree 6 curves through 6 pairs of conjugate points in $\mathbb{P}^3$ is thus consistent with the complex count and provides a non-trivial lower bound for the number of real genus 1 degree 6 curves with 6 pairs of conjugate point insertions. Complete computations of the $d=2,4$ numbers and of the $d=6$ number appear in [18] and [19, 33], respectively.

In all cases, the lower-genus contributions to the primary genus g real GW-invariants arise from real curves passing through corresponding constraints. If $n=3$, $\langle c_1(X), B\rangle \neq 0$, and the almost complex structure $J\in\mathcal{J}_\omega^\phi$ is sufficiently regular, all such contributions arise from curves of the same degree. Since the real enumerative counts are of the same parity as the complex enumerative counts, Propositions 2.3, 2.4, and 2.5 yield the following observations concerning the *complex* enumerative invariants

$$\mathrm{E}_{g,B}\big(\mu_1, \phi^*\mu_1, \ldots, \mu_l, \phi^*\mu_l\big) \in \mathbb{Z} \tag{2.1}$$

with $\mu_i \in H^*(X;\mathbb{Z})$ that count genus g degree B J-holomorphic curves passing through generic representatives of the Poincare duals of μ_i.

Corollary 2.6. *Let (X,ω,ϕ) be a real-orientable 6-manifold, $g,l\in\mathbb{Z}^{\geqslant 0}$, and $B\in H_2(X;\mathbb{Z})$ with $\langle c_1(X), B\rangle \neq 0$. If $\phi^*\mu_i=\mu_i$ for some $i=1,\ldots,l$ and $J\in\mathcal{J}_\omega^\phi$ is sufficiently regular, then the number (2.1) is even.*

Corollary 2.7. *Let $g,l,d\in\mathbb{Z}^{\geqslant 0}$ with $d\geqslant 2g-1$. If either $\mu_i\in H^4(\mathbb{P}^3;\mathbb{Z})$ for some $i=1,\ldots,l$ or $g=0,1$ and $g-d\in 2\mathbb{Z}$, then the genus g degree d enumerative invariants of $\mathbb{P}^3$ of the form (2.1) are even.*

The real GW-invariants arising from Theorems 1.4 and 1.5 are compatible with standard morphisms of GW-theory, such as the morphisms forgetting pairs of conjugate marked points and the node-identifying immersions (2.3) below. By construction, the orientations on the real line bundles (1.5) induced by a fixed real orientation on (X,ω,ϕ) are preserved by the morphisms forgetting pairs of conjugate marked points (the fibers of these morphisms are canonically oriented). If $n\notin 2\mathbb{Z}$, this implies that the orientations on the moduli spaces of real morphisms induced by a fixed real orientation on (X,ω,ϕ) are preserved by the forgetful morphisms. If $n\in 2\mathbb{Z}$, the orientations on the real line bundles (1.6) induced by a fixed real orientation on (X,ω,ϕ) are preserved by the forgetful morphisms. In both cases, the orientations are compatible with the standard node-identifying immersions (2.3) below; see Proposition 2.8. This in turn implies that a uniform system of these orientations is determined by a choice of orientation of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,2}^{\tau}\approx[0,\infty]$, where $\tau\equiv\tau_2$ is the standard conjugation on $\mathbb{P}^1$, and a real orientation on (X,ω,ϕ). If $g\notin 2\mathbb{Z}$, this also implies that the real GW-invariants of $(\mathbb{P}^{2n-1},\omega_{2n},\tau_{2n})$ and $(\mathbb{P}^{4n-1},\omega_{4n},\eta_{4n})$ are independent of the choice of real orientation.

Let (X,ω,ϕ), l, B, and J be as in Theorem 1.3 and $g\in\mathbb{Z}$. We denote by $\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}$ the moduli space of stable real degree B morphisms from possibly disconnected nodal symmetric surfaces of Euler characteristic $2(1-g)$ with l pairs of conjugate marked points. For each $i=1,\ldots,l$, let

$$\mathrm{ev}_i\colon \overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}\longrightarrow X, \qquad \big[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-)\big]\longrightarrow u(z_i^+),$$

be the evaluation at the first point in the i-th pair of conjugate points. Let

$$\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}=\big\{[\mathbf{u}]\in\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}\colon\ \mathrm{ev}_{l-1}([\mathbf{u}])=\mathrm{ev}_l([\mathbf{u}])\big\}.$$

The short exact sequence

$$0\longrightarrow T\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}\longrightarrow T\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}|_{\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}}\longrightarrow \mathrm{ev}_l^*TX\longrightarrow 0$$

induces an isomorphism

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}|_{\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}}\big)\approx\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}\big)\otimes\mathrm{ev}_l^*\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}(TX)\big) \tag{2.2}$$

of real line bundles over $\overline{\mathfrak{M}}_{g,l}^{\prime\bullet}(X,B;J)^{\phi}$.

The identification of the last two pairs of conjugate marked points induces an immersion

$$\iota\colon \overline{\mathfrak{M}}_{g-2,l+2}^{\prime\bullet}(X,B;J)^{\phi}\longrightarrow\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}\,. \tag{2.3}$$

This immersion takes the main stratum of the domain, i.e. the subspace consisting of real morphisms from smooth symmetric surfaces, to the subspace of the target consisting of real morphisms from symmetric surfaces with one pair of conjugate nodes. There is a canonical isomorphism

$$\mathcal{N}\iota\equiv\frac{\iota^*T\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}}{T\overline{\mathfrak{M}}_{g-2,l+2}^{\prime\bullet}(X,B;J)^{\phi}}\approx\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$$

of the normal bundle of ι with the tensor product of the universal tangent line bundles for the first points in the last two conjugate pairs. It induces an isomorphism

$$\iota^*\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g,l}^{\bullet}(X,B;J)^{\phi}\big)\big)\approx\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\overline{\mathfrak{M}}_{g-2,l+2}^{\prime\bullet}(X,B;J)^{\phi}\big)\otimes\Lambda_{\mathbb{R}}^2\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big) \tag{2.4}$$

of real line bundles over $\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}$. Along with (2.2) with (g,l) replaced by $(g-2,l+2)$, it determines an isomorphism

$$\begin{aligned}\Lambda^{\text{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}|_{\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}}\big)\otimes\Lambda^{2}_{\mathbb{R}}\big(\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}\big)\\ \approx\iota^{*}\big(\Lambda^{\text{top}}_{\mathbb{R}}\big(T\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B;J)^{\phi}\big)\big)\otimes\text{ev}^{*}_{l+1}\big(\Lambda^{\text{top}}_{\mathbb{R}}(TX)\big)\end{aligned} \tag{2.5}$$

of real line bundles over $\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B;J)^{\phi}$.

Proposition 2.8. *Let (X,ω,ϕ), g,l, B, and J be as in Theorem 1.3 with $n\notin 2\mathbb{Z}$. The isomorphism (2.5) is orientation-reversing with respect to the orientations on the moduli spaces determined by a real orientation on (X,ω,ϕ) and the canonical orientations on $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ and TX.*

This proposition is established in [17]. Its substance is that the orientations on $\overline{\mathfrak{M}}'^{\bullet}_{g-2,l+2}(X,B,J)^{\phi}$ induced from the orientations of $\overline{\mathfrak{M}}^{\bullet}_{g-2,l+2}(X,B,J)^{\phi}$ and $\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B,J)^{\phi}$ via the isomorphisms (2.2) and (2.4) are opposite. This unfortunate reversal of orientations under the immersion (2.3) can be fixed by multiplying the orientation on $\overline{\mathfrak{M}}^{\bullet}_{g,l}(X,B,J)^{\phi}$ described at the end of Section 3.2 by $(-1)^{\lfloor g/2\rfloor+1}$, for example. Along with the sign flip at the end of Section 3, this would change the canonical orientation on $\mathfrak{M}^{\bullet}_{g,l}(X,B,J)^{\phi,\sigma}$ constructed in the proof of Corollary 5.10 by $(-1)^{\lfloor g/2\rfloor+|\sigma|_0}$, where $|\sigma|_0$ is the number of topological components of the fixed locus of (Σ,σ). This sign change would make the real genus 1 degree d GW-invariant of $(\mathbb{P}^3,\omega_4,\tau_4)$ with d pairs of conjugate point constraints to be 0 for $d=2$, 1 for $d=4$, and 4 for $d=6$. In particular, it would make the $d=4$ number congruent to its complex analogue modulo 4; this is the case for Welschinger's (genus 0) invariants for many target spaces. However, this property fails for the $(g,d)=(1,5)$ numbers (the real enumerative invariant is 0, while its complex analogue is 42).

We note that the statement of Proposition 2.8 is invariant under interchanging the points within the last two conjugate pairs simultaneously (this corresponds to reordering the nodes of a nodal map). This interchange reverses the orientation of the last factor on the left-hand side of (2.5), because the complex rank of $\mathcal{L}_{l+1}\otimes_{\mathbb{C}}\mathcal{L}_{l+2}$ is 1, and the orientation of the last factor on the right-hand side of (2.5), because the complex rank of TX is odd.

If $n\in 2\mathbb{Z}$ and $g+l\geqslant 2$, the comparison (2.5) should be made with the tangent bundles of the moduli spaces twisted as in (1.6). The proof of Proposition 2.8 appearing in [17] still applies, but leads to the opposite conclusion; see [17, Remark 1.3].

3 Outline of the main proofs

The origins of real GW-theory go back to [28], where the spaces (1.3) are topologized by adapting the description of Gromov's topology in [27] via versal families of deformations of abstract complex curves to the real setting. This demonstrates that the codimension 1 boundaries of the spaces in (1.2) form hypersurfaces inside the full moduli space (1.3) and thus reduces the problem of constructing a real GW-theory for a real symplectic manifold (X,ω,ϕ) to showing that

(A) the uncompactified moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ are orientable for all types of orientation-reversing involutions σ on a genus g symmetric surface, and

(B) an orientation of $\mathfrak{M}_{g,l}(X,B;J)^{\phi}$ extends across the (virtually) codimension-one strata of the compact moduli space $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$.

In this paper, we achieve both objectives for real-orientable $2n$-manifolds with $n\notin 2\mathbb{Z}$.

Let $g,l\in\mathbb{Z}^{\geqslant 0}$ with $g+l\geqslant 2$. Denote by $\mathcal{M}_{g,l}^{\sigma}$ the Deligne-Mumford moduli space of σ-compatible complex structures on a genus g symmetric surface (Σ,σ) with l conjugate pairs of marked points and by

$$\overline{\mathcal{M}}_{g,l}^{\sigma}\supset\mathcal{M}_{g,l}^{\sigma}$$

its compactification obtained by including stable nodal symmetric surfaces. The codimension-one boundary strata of $\overline{\mathcal{M}}_{g,l}^{\sigma}-\mathcal{M}_{g,l}^{\sigma}$ consist of real one-nodal symmetric surfaces. Each stratum is either a hypersurface in $\overline{\mathcal{M}}_{g,l}^{\sigma}$ or is a boundary of the spaces $\overline{\mathcal{M}}_{g,l}^{\sigma}$ for precisely two topological types of orientation-reversing involutions σ on Σ. Let

$$\mathbb{R}\mathcal{M}_{g,l}=\bigcup_{\sigma}\mathcal{M}_{g,l}^{\sigma}\qquad\text{and}\qquad\mathbb{R}\overline{\mathcal{M}}_{g,l}=\bigcup_{\sigma}\overline{\mathcal{M}}_{g,l}^{\sigma}$$

denote the (disjoint) union of the uncompactified real Deligne-Mumford moduli spaces and the union of the compactified real Deligne-Mumford moduli spaces, respectively, taken over all topological types of orientation-reversing involutions σ on Σ. The moduli space $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ is not orientable if $g\in\mathbb{Z}^{+}$. One of the two main steps in the proof of Theorem 1.3 is Proposition 6.1; it implies that the real line bundle

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T\mathbb{R}\overline{\mathcal{M}}_{g,l}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big)\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l}\tag{3.1}$$

has a canonical orientation.

With g,l as above, let

$$\mathfrak{f}\colon\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l}\tag{3.2}$$

denote the forgetful morphism. For each $[\mathbf{u}]\in\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ with stable domain, it induces a canonical isomorphism

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{[\mathbf{u}]}\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}\big)\approx\big(\det D_{(TX;\mathrm{d}\phi);\mathbf{u}}\big)\otimes\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(T_{\mathfrak{f}(\mathbf{u})}\overline{\mathcal{M}}_{g,l}^{\sigma}\big),\tag{3.3}$$

where $\det D_{(TX;\mathrm{d}\phi);\mathbf{u}}$ is the determinant of the linearization $D_{(TX;\mathrm{d}\phi);\mathbf{u}}$ of the real $\bar\partial_J$-operator at $\mathbf{u}$; see Section 4.3. The orientability of the last factor in (3.3) as $\mathbf{u}$ varies is indicated by the previous paragraph. We study the orientability of the first factor on the right-hand side of (3.3) via the relative determinant of $D_{(TX;\mathrm{d}\phi);\mathbf{u}}$,

$$\widehat{\det}\,D_{(TX;\mathrm{d}\phi);\mathbf{u}}\equiv\big(\det D_{(TX;\mathrm{d}\phi);\mathbf{u}}\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n},\tag{3.4}$$

where $2n=\dim X$ and $\det\bar\partial_{\Sigma;\mathbb{C}}$ is the standard real Cauchy-Riemann (or CR-) operator on the domain (Σ,σ) of $\mathbf{u}$ with values in $(\mathbb{C},\mathfrak{c})$. An orientation on (3.4) determines a correspondence between the orientations on $\det D_{(TX;\mathrm{d}\phi);\mathbf{u}}$ and on the determinant $\det n\bar\partial_{\Sigma;\mathbb{C}}$ of the standard real $\bar\partial$-operator on the trivial rank n real bundle $(\Sigma\times\mathbb{C}^n,\sigma\times\mathfrak{c})$ over (Σ,σ). On the other hand, orientations on $\widehat{\det}\,D_{(TX;\mathrm{d}\phi);u}$ are naturally related to the topology of real bundles pairs over (Σ,σ). In particular, the second main step in the proof of Theorem 1.3 is Proposition 5.2; it implies that a real orientation on (X,ω,ϕ) determines an orientation on (3.4) which varies continuously with $\mathbf{u}$. Combined with the canonical orientation of (3.1) and the canonical isomorphism of (3.3), the latter orientation determines an orientation on the line bundle (1.4).

3.1 The orientability problem

The typical approaches to the orientability problem in real GW-theory, i.e. (A) on page 11, involve computing the signs of the actions of appropriate real diffeomorphisms on determinant lines of real CR-operators over some coverings of $\mathfrak{M}_g(X,B;J)^{\phi,\sigma}$ arising from bordered surfaces. These approaches work as long as all relevant diffeomorphisms are homotopically fairly simple and in particular preserve a bordered surface in Σ that doubles to Σ or map it to its conjugate half. This is the case if the fixed locus $\Sigma^\sigma\subset\Sigma$ of the involution σ is separating; a good understanding of the orientability of the moduli spaces $\mathfrak{M}_g(X,B;J)^{\phi,\sigma}$ in such cases is obtained in [37, 11, 13, 6, 14, 15]. This is also the case for any involution σ of genus $g=0,1$. In particular, the restriction of Theorem 1.3 to $\mathfrak{M}_g(X,B;J)^{\phi,\sigma}$ for the genus 1 involutions σ is essentially [14, Theorem 1.2]; a less general version of [14, Theorem 1.2] is [8, Theorem 1.1]. However, understanding the orientability in the bordered case is not sufficient beyond genus 1, due to the presence of real diffeomorphisms of (Σ,σ) not preserving any half of Σ; see Example 4.1. The subtle effect of such diffeomorphisms on the orientability is hard to determine.

In contrast to [37, 11], in [15] we allowed the complex structure on a bordered domain to vary and considered diffeomorphisms interchanging the boundary components and their lifts to automorphisms of real bundle pairs. We discovered that they often act with the same signs on

(A1) a natural cover of $\mathcal{M}_g^\sigma$ and the determinant line bundle for the trivial rank 1 real bundle pair over it;

(A2) the determinants of real CR-operators on the square of a rank 1 real bundle pair with orientable real part and on the trivial rank 1 real bundle pair;

(A3) the determinants of real CR-operators on an odd-rank real bundle pair and its top exterior power;

see [15, Propositions 2.5,4.1,4.2]. In this paper, we show that these *analytic* statements are in fact underpinned by the *topological* statement of Proposition 5.2 concerning canonical homotopy classes of isomorphisms between real bundle pairs over a symmetric surface (Σ,σ). As we work on the more elemental, topological level of real bundle pairs, we do not compute the signs of any automorphisms, as is done in the bordered surfaces approach. We instead obtain isomorphisms of real bundle pairs over (families of) symmetric surfaces and apply the determinant functor to these isomorphisms (Corollaries 5.7 and 6.6). In contrast to the bordered surfaces approach, this works for all type of involutions on the domain and in flat families of (possibly) nodal curves.

Proposition 5.9, which appears to be of its own interest, endows the restriction of the line bundle (3.1) to each topological component $\mathcal{M}_{g,l}^\sigma$ of $\mathbb{R}\mathcal{M}_{g,l}$ with a canonical orientation and thus explains (A1). This canonical orientation over an element $[\mathcal{C}]$ of $\mathcal{M}_{g,l}^\sigma$ is obtained by tensoring canonical orientations on four lines:

(1) the orientation on the tensor product of the top exterior powers of the left and middle terms in (5.22) induced by the Kodaira-Spencer (or KS) isomorphism,

(2) the orientation on the tensor product of the top exterior powers of the middle term in (5.22) and of the right-hand side in (5.23) induced by the Dolbeault Isomorphism and Serre Duality,

(3) the orientation on (5.25) induced by the short exact sequence (5.24) and the specified orientations of (5.21),

(4) the orientation on the fiber of the line bundle (5.27) over $[\mathcal{C}]$ determined by Corollaries 5.6 and 5.7.

We combine the canonical orientation on the restriction of the line bundle (3.1) to the main stratum $\mathbb{R}\mathcal{M}_{g,l}$, the orientation on the relative determinants (3.4) induced by the real orientation on (X,ω,ϕ), and the isomorphism (3.3) to establish the restriction of Theorem 1.3 to the uncompactified moduli space $\mathfrak{M}_{g,l}(X,B;J)^\phi$; see Corollary 5.10.

3.2 The codimension-one boundary problem

Once the orientability problem (A) is resolved, one can study the codimension-one boundary problem, i.e. (B) on page 11. It then asks whether it is possible to choose an orientation on the subspace

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \subset \overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$$

for each topological type of orientation-reversing involutions σ on a genus g symmetric surface so that the resulting orientations do not change across the (virtually) codimension-one strata of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$. These strata are (virtual) hypersurfaces inside of the full moduli space and consist of morphisms from one-nodal symmetric surfaces to (X,ϕ).

As described in [28, Section 3], there are four distinct types of one-nodal symmetric surfaces (Σ,x_{12},σ):

(E) x_{12} is an isolated real node, i.e. x_{12} is an isolated point of the fixed locus $\Sigma^\sigma\subset\Sigma$;

(H) x_{12} is a non-isolated real node and

- (H1) the topological component Σ^σ_{12} of Σ^σ containing x_{12} is algebraically irreducible (the normalization $\widetilde{\Sigma}^{\widetilde{\sigma}}_{12}$ of Σ^σ_{12} is connected);
- (H2) the topological component Σ^σ_{12} of Σ^σ containing x_{12} is algebraically reducible, but Σ is algebraically irreducible (the normalization $\widetilde{\Sigma}^{\widetilde{\sigma}}_{12}$ of Σ_{12} is disconnected, but the normalization $\widetilde{\Sigma}$ of Σ is connected);
- (H3) Σ is algebraically reducible (the normalization $\widetilde{\Sigma}$ of Σ is disconnected).

In [8, Section 3], the above types are called (II), (IC1), (IC2), and (ID), respectively. In the genus 0 case, the degenerations (E) and (H3) are known as codimension-one sphere bubbling and disk bubbling, respectively; the degenerations (H1) and (H2) cannot occur in the genus 0 case.

The transitions between smooth symmetric surfaces across the four types of one-nodal symmetric surfaces are illustrated in [28, Figures 12-15]. A transition through a degeneration (H3) does not change the topological type of the involution. Thus, each stratum of morphisms from a one-nodal symmetric surface of type (H3) to (X,ϕ) is a hypersurface inside of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for some genus g involution σ. This transition does not play a material role in the approach of [39, 40], which is based on counting real genus 0 curves, rather than their parametrizations. In the approach of [4, 37], which is based on counting morphisms from disks as halves of morphisms from $(\mathbb{P}^1,\tau)$,

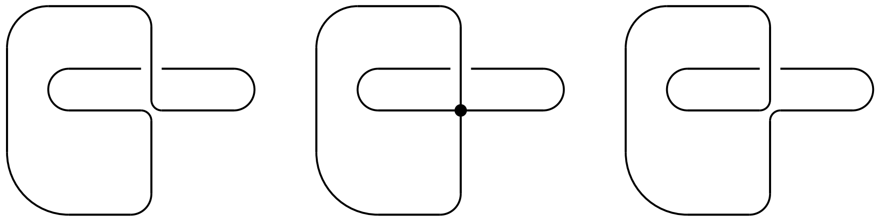

Figure 1: The real locus transition through (H2) and (H3) degenerations

the degeneration (H3) appears as a codimension-one boundary consisting of morphisms from two disks. This boundary is glued to itself in [4, 37] by the involution which corresponds to flipping one of the disks; this involution is orientation-reversing under suitable assumptions and so the orientation on the main stratum extends across the resulting hypersurfaces. A perspective that combines the hypersurface viewpoint of [39, 40] with the parametrizations setting of [4, 37] appears in [13]. It fits naturally with the approach of this paper to studying transitions through all four degeneration types.

A transition through a degeneration (E) changes the number $|\sigma|_0$ of topological components (circles) of the fixed locus $\Sigma^\sigma \subset \Sigma$ by one. In the terminology of Section 4.1, such a transition can be described as collapsing a standard boundary component of a bordered half-surface (corresponding to a component of Σ^σ) and then replacing it with a crosscap. In particular, each stratum of morphisms from a one-nodal symmetric surface of type (E) to (X,ϕ) is a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of genus g involutions σ. In the genus 0 case, the analysis of orientations necessary for the gluing of the two spaces along their common boundary is carried out in [7, Section 3].

A transition through a degeneration (H1) also changes the number $|\sigma|_0$ by one, but through a more complicated process. Such a transition transforms two components of Σ^σ into one and creates an additional crosscap "near" the node of the one-nodal surface (Σ, x_{12}, σ). Each stratum of morphisms from a one-nodal symmetric surface of type (H1) to (X,ϕ) is a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of genus g involutions σ. A degeneration (H1) cannot occur in genus 0, but does occur in genus 1 and higher; see the last diagram in [8, Figure 2].

A transition through a degeneration (H2) does not change the number of topological components of Σ^σ, but cuts one of them into two arcs and re-joins the arcs in the opposite way. The trans-

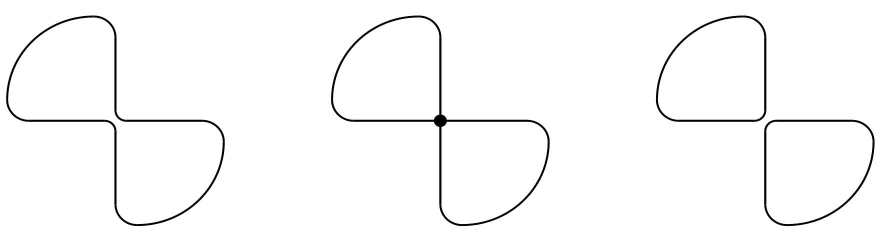

Figure 2: The real locus transition through an (H1) degeneration

formation of the real locus is the same as in the (H3) case, but an (H2) transition also inserts or removes two crosscaps. This transition may or may not change the topological type of the involution σ. If the fixed locus of (Σ, x_{12}, σ) is separating, then this transition changes the topological type of σ and each stratum of morphisms from (Σ, x_{12}, σ) to (X, ϕ) is a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi,\sigma}$ for precisely two topological types of genus g involutions σ. If the fixed locus of (Σ, x_{12}, σ) is non-separating, then this transition does not change the topological type of σ and each stratum of morphisms from (Σ, x_{12}, σ) to (X, ϕ) is a hypersurface inside of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi,\sigma}$ for some genus g involution σ. A degeneration (H2) cannot occur in genus 0 or 1, but does occur in genus 2 and higher.

The transitions (H1) and (H2) do not preserve any bordered surface in Σ that doubles to Σ, in contrast to the transition (E); the transition (H3) does not preserve any bordered surface in Σ that doubles to Σ either, but its nature is fairly simple. As in the case of (A) discussed in Section 3.1, this makes the issue (B) difficult to study using the standard approaches to orienting the determinant lines of real CR-operators even when issue (A) is resolved; see [8, Conjectures 1.3,6.3]. We approach (B) by studying isomorphisms between real bundle pairs, but this time over one-nodal symmetric surfaces (Σ, x_{12}, σ). As in the smooth case, this circumvents a direct computation of the signs of any automorphisms of the determinant lines of real CR-operators.

Corollary 5.10 uses a real orientation on (X, ω, ϕ) to endow the fiber of the line bundle (1.5) over each element $[\mathbf{u}]$ of $\mathfrak{M}_{g,l}(X, B; J)^{\phi}$ with an orientation. The latter is obtained by combining the orientation on the relative determinant (3.4) of the linearization of the $\bar{\partial}_J$-operator at $\mathbf{u}$ induced by the real orientation on (X, ω, ϕ) and the canonical orientation on the fiber of the line bundle (3.1) over $\mathfrak{f}(\mathbf{u})$ via the isomorphism (3.3). The real orientation on (X, ω, ϕ) specifies a homotopy class of isomorphisms (5.5) with $(V, \varphi) = u^*(TX, \mathrm{d}\phi)$. The latter determines an orientation on the relative determinant (3.4). An isomorphism in the specified homotopy class over a one-nodal symmetric surface (Σ, x_{12}, σ) extends to an isomorphism in the specified homotopy class for each nearby smooth symmetric surface. Therefore, so does the induced orientation on the relative determinant (3.4); see Corollary 6.7. This means that the induced orientation of the line bundle formed by the relative determinants (3.4) does not change across any of the codimension-one strata of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi}$.

The situation with the canonical orientation on the restriction of the line bundle (3.1) to $\mathbb{R}\mathcal{M}_{g,l}$ provided by Proposition 5.9 on page 32 is very different. This is partly indicated by the statement of Proposition 6.1, but the actual situation is even more delicate. This canonical orientation constructed in Proposition 5.9 is the tensor product of the four orientations listed at the end of Section 3.1. The line bundles on which these orientations are defined naturally extend across the codimension-one boundary strata of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The behavior of the four orientations across these strata of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$ is described in the proof of Proposition 6.1 at the end of Section 6.3. The orientations (2) and (3) in Section 3.1 do not change across any of codimension-one strata. The orientation (1), determined by the KS isomorphism (5.22) for smooth symmetric surfaces, changes across all codimension-one boundary strata; see Lemma 6.17. The orientation (4) over a smooth symmetric surface is induced by Corollary 5.7 from the canonical real orientation of Corollary 5.6 with $L = T^*\Sigma$. The analogue of L for a one-nodal symmetric surface (Σ, x_{12}, σ) is played by the restriction of the line bundle $\widehat{\mathcal{T}}$ of Lemma 6.8. The restriction of its real part to the singular component of the fixed locus is orientable for the degenerations of types (E) and (H1) and is not orientable for the degenerations of types (H2) and (H3); see Lemma 6.13. In the latter cases, the

orientation/parity of	induced by	(E)/(H1)	(H2)/(H3)
$\Lambda^{\mathrm{top}}_{\mathbb{R}}(T_\Sigma\mathcal{M}^\sigma_{g,l})\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}(\check{H}^1(\Sigma;T\Sigma))^\sigma)$	KS isomorphism (5.22)	$-$	$-$
$(\det\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes 2}})\otimes(\det\bar\partial_{\Sigma;\mathbb{C}})$	Corollaries 5.6 and 5.7	$+$	$-$
$\lvert\sigma\rvert_0$	N/A	$-$	$+$

Table 1: The extendability of the canonical orientations and of the parity of the number of components of Σ^σ across the codimension-one strata: $+$ extends, $-$ flips. All other canonical orientations factoring into the orientation of the line bundle (1.5) extend across all codimension-one strata.

orientation (4) for (Σ, x_{12}, σ) depends on the orientation of the fixed locus; see Corollary 5.6. For the degenerations of types (H2) and (H3), no orientation of the singular component of the fixed locus extends to nearby smooth symmetric surfaces (because (H2) and (H3) involve cutting a fixed circle into two arcs and re-joining them in the opposite way). Therefore, the orientation (4) changes in the transitions (H2) and (H3) and does not in the transitions (E) and (H1); see Corollary 6.16.

The key points of the previous paragraph are summarized in Table 1. They imply that the canonical orientation on the restriction of the line bundle (3.1) to $\mathbb{R}\mathcal{M}_{g,l}$ provided by Proposition 5.9 does not change in the transitions (H2) and (H3) and changes in the transitions (E) and (H1). These transitions have the same effect on the parity of the number $|\sigma|_0$ of connected components of the fixed locus Σ^σ of (Σ,σ). Thus, the canonical orientation on the restriction of (3.1) to $\mathbb{R}\mathcal{M}_{g,l}$ multiplied by $(-1)^{g+|\sigma|_0+1}$ over each topological component $\mathcal{M}^\sigma_{g,l}$ of $\mathbb{R}\mathcal{M}_{g,l}$ extends over all of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The same considerations apply to the orientation on the restriction of the line bundle (1.5) to $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ provided by Corollary 5.10. If $n\notin 2\mathbb{Z}$, this sign modification leaves the orientations of the moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ for separating involutions σ unchanged.

4 Notation and review

In this section, we set up the notation and terminology used throughout Sections 5 and 6. We recall some facts about symmetric surfaces, associated half-surfaces, their moduli spaces, real Cauchy-Riemann operators, and their determinant line bundles.

4.1 Symmetric surfaces and half-surfaces

Let (Σ,σ) be a genus g symmetric surface. We denote by $|\sigma|_0\in\mathbb{Z}^{\geqslant 0}$ the number of connected components of Σ^σ; each of them is a circle. Let $\langle\sigma\rangle=0$ if the quotient Σ/σ is orientable, i.e. $\Sigma-\Sigma^\sigma$ is disconnected, and $\langle\sigma\rangle=1$ otherwise. There are $\left\lfloor\frac{3g+4}{2}\right\rfloor$ different topological types of orientation-reversing involutions σ on Σ classified by the triples $(g,|\sigma|_0,\langle\sigma\rangle)$; see [31, Corollary 1.1].

An oriented symmetric half-surface (or simply oriented sh-surface) is a pair (Σ^b,c) consisting of an oriented bordered smooth surface Σ^b and an involution $c\colon\partial\Sigma^b\longrightarrow\partial\Sigma^b$ preserving each component and the orientation of $\partial\Sigma^b$. The restriction of c to a boundary component is either the identity or

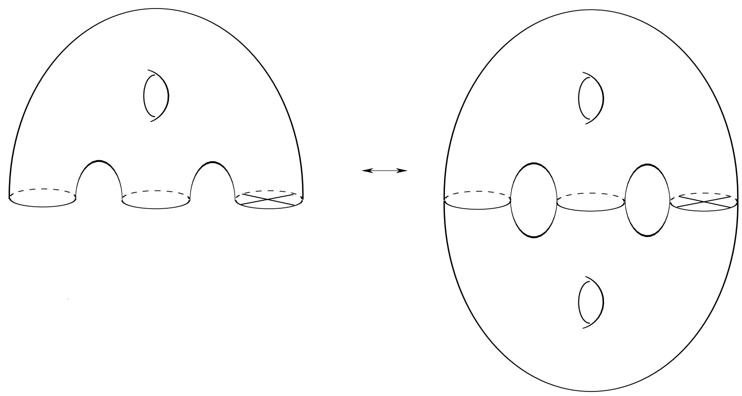

Figure 3: Doubling an oriented sh-surface

the antipodal map

$$\mathfrak{a}\colon S^1 \longrightarrow S^1, \qquad z \longrightarrow -z, \tag{4.1}$$

for a suitable identification of $(\partial\Sigma^b)_i$ with $S^1 \subset \mathbb{C}$; the latter type of boundary structure is called crosscap in the string theory literature. We define

$$c_i = c|_{(\partial\Sigma^b)_i}, \qquad |c_i| = \begin{cases} 0, & \text{if } c_i = \mathrm{id}; \\ 1, & \text{otherwise}; \end{cases} \qquad |c|_k = \big|\{(\partial\Sigma^b)_i \subset \Sigma^b \colon |c_i| = k\}\big| \quad k = 0, 1.$$

Thus, $|c|_0$ is the number of standard boundary components of $(\Sigma^b, \partial\Sigma^b)$ and $|c|_1$ is the number of crosscaps. Up to isomorphism, each oriented sh-surface (Σ^b, c) is determined by the genus g of Σ^b, the number $|c|_0$ of ordinary boundary components, and the number $|c|_1$ of crosscaps. We denote by $(\Sigma_{g,m_0,m_1}, c_{g,m_0,m_1})$ the genus g oriented sh-surface with $|c_{g,m_0,m_1}|_0 = m_0$ and $|c_{g,m_0,m_1}|_1 = m_1$.

An oriented sh-surface (Σ^b, c) of type (g, m_0, m_1) doubles to a symmetric surface (Σ, σ) of type

$$(g(\Sigma), |\sigma|_0, \langle\sigma\rangle) = \begin{cases} (2g+m_0+m_1-1, m_0, 0), & \text{if } m_1 = 0; \\ (2g+m_0+m_1-1, m_0, 1), & \text{if } m_1 \neq 0; \end{cases}$$

so that σ restricts to c on the cutting circles (the boundary of Σ^b); see [14, (1.6)] and Figure 3. Since this doubling construction covers all topological types of orientation-reversing involutions σ on Σ, for every symmetric surface (Σ, σ) there is an oriented sh-surface (Σ^b, c) which doubles to (Σ, σ). In general, the topological type of such an sh-surface is not unique. There is a topologically unique oriented sh-surface (Σ^b, c) doubling to a symmetric surface (Σ, σ) if $\langle\sigma\rangle = 0$, in which case (Σ^b, c) has no crosscaps, or $|\sigma|_0 \geqslant g(\Sigma) - 1$, in which case (Σ^b, c) is either of genus at most 1 and has no crosscaps or of genus 0 and has at most 2 crosscaps.

Denote by $\mathcal{D}_\sigma$ the group of orientation-preserving diffeomorphisms of Σ commuting with the involution σ. If (X, ϕ) is a smooth manifold with an involution, $l \in \mathbb{Z}^{\geqslant 0}$, and $B \in H_2(X; \mathbb{Z})$, let

$$\mathfrak{B}_{g,l}(X, B)^{\phi,\sigma} \subset \mathfrak{B}_g(X)^{\phi,\sigma} \times \Sigma^{2l}$$

denote the space of real maps $u:(\Sigma,\sigma)\longrightarrow(X,\phi)$ with $u_*[\Sigma]_{\mathbb{Z}}=B$ and l pairs of conjugate non-real marked distinct points. We define

$$\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}=\big(\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma}\times\mathcal{J}_\Sigma^\sigma\big)/\mathcal{D}_\sigma.$$

The action of $\mathcal{D}_\sigma$ on $\mathcal{J}_\Sigma$ given by $h\cdot\mathfrak{j}=h^*\mathfrak{j}$ preserves $\mathcal{J}_\Sigma^\sigma$; thus, the above quotient is well-defined. If $J\in\mathcal{J}_\omega^\phi$, the moduli space of marked real J-holomorphic maps in the class $B\in H_2(X;\mathbb{Z})$ is the subspace

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}=\big\{[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),\mathfrak{j}]\in\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}:\ \bar\partial_{J,\mathfrak{j}}u=0\big\},$$

where $\bar\partial_{J,\mathfrak{j}}$ is the usual Cauchy-Riemann operator with respect to the complex structures J on X and $\mathfrak{j}$ on Σ. If $g+l\geqslant 2$,

$$\mathcal{M}_{g,l}^\sigma\equiv\mathfrak{M}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}\equiv\mathcal{H}_{g,l}(\mathrm{pt},0)^{\mathrm{id},\sigma}$$

is the moduli space of marked symmetric domains. There is a natural forgetful morphism

$$\mathfrak{f}:\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}\longrightarrow\mathcal{M}_{g,l}^\sigma\,;$$

it drops the map component u from each element of the domain.

The following example shows that the orientability of a moduli space of symmetric half-surfaces does not imply the orientability of the corresponding moduli space of symmetric surfaces. It indicates the subtle effect of diffeomorphisms of a symmetric surface (Σ,σ) not preserving any half-surface Σ^b and the difficulties arising in the standard approaches to the orientability problem (A) on page 11 in positive genus.

Example 4.1. Let Σ^b be an sh-surface of genus 2 with one boundary component and non-trivial involution, as in the left diagram of Figure 4. Its double is a symmetric surface (Σ,σ) of genus 4 without a fixed locus, as in the middle diagram of Figure 4. The moduli space $\mathcal{M}_{\Sigma^b}^c$ of sh-surfaces Σ^b is orientable by [14, Lemma 6.1] and [15, Lemma 2.1]. The natural automorphisms of $\mathcal{M}_{\Sigma^b}^c$ associated with real orientation-reversing diffeomorphisms of Σ^b are orientation-preserving by [14, Lemma 6.1] and [15, Corollary 2.3]. On the double Σ of Σ^b, these diffeomorphisms correspond to flipping the surface across the crosscap. The real moduli space $\mathcal{M}_4^\sigma$ parametrizing such symmetric surfaces Σ is not orientable for the following reason. By [31, Theorem 1.2], every representative of a point in $\mathcal{M}_4^\sigma$ has 5 invariant circles which separate the surface, as in the right diagram of Figure 4. There is a real diffeomorphism h which fixes 3 of these circles and interchanges the other 2. By [15, Corollary 2.2], the mapping torus of h defines a loop in $\mathcal{M}_4^\sigma$ which pairs non-trivially with the first Stiefel-Whitney class of the moduli space.

4.2 Gromov's convergence topology

Let $\mathcal{C}\equiv(\Sigma,z_1,\ldots,z_l,\mathfrak{j})$ be a compact nodal marked Riemann surface. A flat family of deformations of $\mathcal{C}$ is a tuple

$$\big(\pi:\mathcal{U}\longrightarrow\Delta,s_1:\Delta\longrightarrow\mathcal{U},\ldots,s_l:\Delta\longrightarrow\mathcal{U}\big),\tag{4.2}$$

where $\mathcal{U}$ is a complex manifold, $\Delta\subset\mathbb{C}^N$ is a ball around 0, and $\pi,s_1,\ldots,s_l$ are holomorphic maps such that

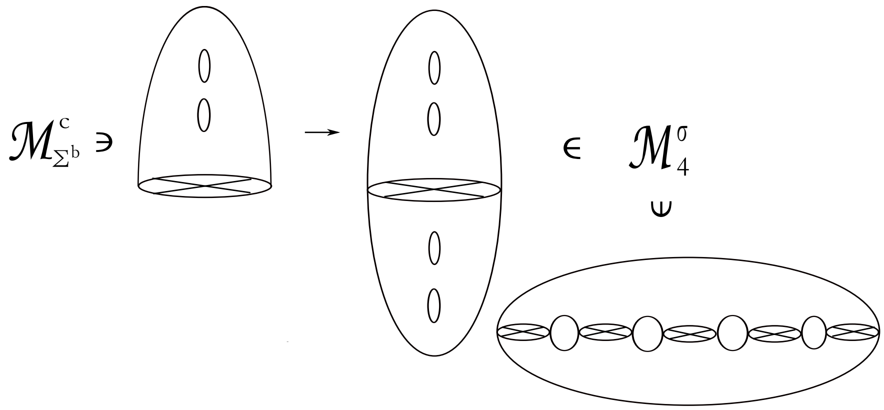

Figure 4: Orientability of crosscaps vs. real moduli spaces

- $\Sigma_{\mathbf{t}} \equiv \pi^{-1}(\mathbf{t})$ is a (possibly nodal) Riemann surface for each $\mathbf{t} \in \Delta$ and π is a submersion outside of the nodes of the fibers of π,
- for every $\mathbf{t}^* \equiv (t_1^*, \ldots, t_N^*) \in \Delta$ and every node $z^* \in \Sigma_{\mathbf{t}^*}$, there exist $i \in \{1, \ldots, N\}$ with $t_i^* = 0$, neighborhoods $\Delta_{\mathbf{t}^*}$ of $\mathbf{t}^*$ in Δ and $\mathcal{U}_{z^*}$ of z^* in $\mathcal{U}$, and a holomorphic identification
$$\Psi: \mathcal{U}_{z^*} \longrightarrow \big\{\big((t_1, \ldots, t_N), x, y\big) \in \Delta_{\mathbf{t}^*} \times \mathbb{C}^2 : xy = t_i\big\}$$
such that the composition of Ψ with the projection to $\Delta_{\mathbf{t}^*}$ equals $\pi|_{\mathcal{U}_{z^*}}$,
- $\pi \circ s_i = \mathrm{id}_\Delta$ and $s_i(\mathbf{t}) \neq s_j(\mathbf{t})$ for all $\mathbf{t} \in \Delta$ and $i, j = 1, \ldots, l$ with $i \neq j$,
- $(\Sigma_0, s_1(0), \ldots, s_l(0)) = \mathcal{C}$.

Let $\mathcal{C} \equiv (\Sigma, \sigma, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j})$ be a nodal marked symmetric Riemann surface. A flat family of deformations of $\mathcal{C}$ is a tuple
$$\big(\pi: \mathcal{U} \longrightarrow \Delta, \widetilde{\mathfrak{c}}: \mathcal{U} \longrightarrow \mathcal{U}, s_1: \Delta \longrightarrow \mathcal{U}, \ldots, s_l: \Delta \longrightarrow \mathcal{U}\big)$$
such that $(\pi, s_1, \widetilde{\mathfrak{c}} \circ s_1, \ldots, s_l, \widetilde{\mathfrak{c}} \circ s_l)$ is a flat family of deformations of $(\Sigma, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j})$ and $\widetilde{\mathfrak{c}}$ is an anti-holomorphic involution on $\mathcal{U}$ lifting the standard involution $\mathfrak{c}$ on Δ and restricting to σ over $\Sigma = \pi^{-1}(0)$. In such a case, let $\sigma_{\mathbf{t}} = \widetilde{\mathfrak{c}}|_{\Sigma_{\mathbf{t}}}$ for each parameter $\mathbf{t}$ in $\Delta_{\mathbb{R}} \equiv \Delta \cap \mathbb{R}^N$.

For any nodal surface Σ, we denote by $\Sigma^* \subset \Sigma$ the subset of its smooth points. Suppose $\pi: \mathcal{U} \longrightarrow \Delta$ is a flat family of deformations of $(\Sigma, \mathfrak{j})$. There then exist a neighborhood Δ' of 0 in Δ and a continuous collapsing map
$$q: \mathcal{U}\big|_{\Delta'} \longrightarrow \Sigma$$
so that the preimage of each node of Σ under the restriction of q to $\Sigma_{\mathbf{t}}$ with $\mathbf{t} \in \Delta'$ is either a node of $\Sigma_{\mathbf{t}}$ or an embedded circle and the map
$$\pi \times q: q^{-1}(\Sigma^*) \longrightarrow \Delta' \times \Sigma^*$$
is a diffeomorphism. For each $\mathbf{t} \in \Delta'$, let
$$\psi_{\mathbf{t}}: \Sigma^* \longrightarrow q^{-1}(\Sigma^*) \cap \Sigma_{\mathbf{t}} \tag{4.3}$$

be the restriction of its inverse to $\mathbf{t}\times\Sigma^*$. If $\mathbf{t}_r\in\Delta$ is a sequence converging to $0\in\Delta$ and $u_r:\Sigma_{\mathbf{t}_r}\longrightarrow X$ is a sequence of continuous maps that are smooth on $\Sigma^*_{\mathbf{t}_r}$, we say that the sequence u_r converges to a smooth map $u:\Sigma^*\longrightarrow X$ u.c.s. (uniformly on compact subsets) if the sequence of maps

$$u_r\circ\psi_{\mathbf{t}_r}:\Sigma^*\longrightarrow X$$

converges to u uniformly in the C^∞-topology on compact subsets of Σ^*. This notion is independent of the choices of Δ' and trivialization of $\pi|_{q^{-1}(\Sigma^*)}$.

For a Riemannian metric g on X and an L^p_1-map $u:\Sigma\longrightarrow X$, for some $p>2$, let

$$E_g(u)\equiv\frac{1}{2}\int_\Sigma|\mathrm{d}u|_g^2\in\mathbb{R}^{\geqslant 0}$$

denote the energy of u; this notion is independent of the choice of $\mathfrak{j}$-compatible metric on Σ.

Definition 4.2 (Gromov's Convergence)**.** Suppose (X,ϕ) is a manifold with an involution, g is a Riemannian metric on X, and J_r is an almost complex structure on X for every $r\in\mathbb{Z}^{\geqslant 0}\sqcup\{\infty\}$. A sequence $(\mathcal{C}_r,\sigma_r,u_r)$ of ϕ-real J_r-holomorphic maps with l conjugate pairs of marked points converges to a ϕ-real J_∞-holomorphic map $(\mathcal{C}_\infty,\sigma_\infty,u_\infty)$ with l conjugate pairs of marked points if $E_g(u_r)\longrightarrow E_g(u_\infty)$ as $r\longrightarrow\infty$ and there exist

(a) a flat family $(\pi,\widetilde{\mathfrak{c}},s_1,\ldots,s_l)$ of deformations of $(\mathcal{C}_\infty,\sigma_\infty)$ as above,

(b) a sequence $\mathbf{t}_r\in\Delta_{\mathbb{R}}$ converging to $0\in\Delta$, and

(c) equivalences $h_r:(\Sigma_{\mathbf{t}_r},\sigma_r)\longrightarrow(\mathcal{C}_r,\sigma_{\mathbf{t}_r})$

such that $u_r\circ h_r$ converges to $u_\infty|_{\Sigma^*_\infty}$ u.c.s.

Suppose (X,ϕ), g, and J_r are as in Definition 4.2, X is compact, and the sequence J_r converges to J_∞ with respect to the C^2-topology. Gromov's Compactness Theorem for J-holomorphic maps, arising from [23], then implies that every sequence of stable ϕ-real J_r-holomorphic maps u_r with l conjugate pairs of marked points so that $\liminf E_g(u_i)$ is finite contains a subsequence that converges in the sense of Definition 4.2 to some stable ϕ-real J_∞-holomorphic map $(\mathcal{C}_\infty,u_\infty)$ with l conjugate pairs of marked points. By the compactness of Σ_∞, this notion of convergence is independent of the choice of metric g on X.

4.3 Determinant lines of Fredholm operators

Let (V,φ) be a real bundle pair over a symmetric surface (Σ,σ). A real Cauchy-Riemann (or CR-) operator on (V,φ) is a linear map of the form

$$\begin{aligned} D=\bar\partial+A:\Gamma(\Sigma;V)^\varphi&\equiv\big\{\xi\in\Gamma(\Sigma;V):\ \xi\circ\sigma=\varphi\circ\xi\big\}\\ &\longrightarrow\Gamma^{0,1}_{\mathfrak{j}}(\Sigma;V)^\varphi\equiv\big\{\zeta\in\Gamma(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}V):\ \zeta\circ\mathrm{d}\sigma=\varphi\circ\zeta\big\},\end{aligned}\tag{4.4}$$

where $\bar\partial$ is the holomorphic $\bar\partial$-operator for some $\mathfrak{j}\in\mathcal{J}^\sigma_\Sigma$ and a holomorphic structure in V reversed by φ and

$$A\in\Gamma\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(V,(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}V)\big)^\varphi$$

is a zeroth-order deformation term. A real CR-operator on a real bundle pair is Fredholm in the appropriate completions. The space of completions of all real CR-operators on (V,φ) is contractible with respect to the operator norm.

If X, Y are Banach spaces and $D\colon X \longrightarrow Y$ is a Fredholm operator, let

$$\det D \equiv \Lambda^{\mathrm{top}}_{\mathbb{R}}(\ker D) \otimes \big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(\operatorname{cok} D)\big)^*$$

denote the determinant line of D. A continuous family of such Fredholm operators D_t over a topological space $\mathcal{H}$ determines a line bundle over $\mathcal{H}$, called the determinant line bundle of $\{D_t\}$ and denoted $\det D$; see [30, Section A.2] and [49]. Combined with the note at the end of the previous paragraph, this implies that there is a canonical homotopy class of isomorphisms between the determinants of any two CR-operators on a real bundle pair (V,φ); we thus denote any such determinant by $\det D_{(V,\varphi)}$.

An exact triple (short exact sequence) $\mathfrak{t}$ of Fredholm operators

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[d, "D'"] & X \arrow[r] \arrow[d, "D"] & X'' \arrow[r] \arrow[d, "D''"] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd} \tag{4.5}$$

determines a canonical isomorphism

$$\Psi_{\mathfrak{t}}\colon (\det D') \otimes (\det D'') \xrightarrow{\approx} \det D. \tag{4.6}$$

For a continuous family of exact triples of Fredholm operators, the isomorphisms (4.6) give rise to a canonical isomorphism between the determinant line bundles.

Let $(\Sigma_0, \sigma_0, \mathfrak{j}_0)$ be a (possibly nodal) symmetric Riemann surface and $(\pi\colon \mathcal{U} \longrightarrow \Delta, \widetilde{\mathfrak{c}}\colon \mathcal{U} \longrightarrow \mathcal{U})$ be a flat family of deformations of $(\Sigma_0, \sigma_0, \mathfrak{j}_0)$ as in Section 4.2. Suppose (V,φ) is a real bundle pair over $(\mathcal{U}, \widetilde{\mathfrak{c}})$, ∇ is a φ-compatible (complex-linear) connection in V, and

$$A \in \Gamma\big(\mathcal{U}; \mathrm{Hom}_{\mathbb{R}}(V, (T^*\mathcal{U}, J)^{0,1} \otimes_{\mathbb{C}} V)\big)^{\varphi}, \tag{4.7}$$

where J is the complex structure on $\mathcal{U}$. The restrictions of ∇ and A to each fiber $(\Sigma_{\mathbf{t}}, \sigma_{\mathbf{t}})$ of π with $\mathbf{t} \in \Delta_{\mathbb{R}}$ then determine a real CR-operator

$$D_{(V,\varphi);\mathbf{t}}\colon \Gamma\big(\Sigma_{\mathbf{t}}; V|_{\Sigma_{\mathbf{t}}}\big)^{\varphi} \longrightarrow \Gamma^{0,1}_{\mathfrak{j}_{\mathbf{t}}}\big(\Sigma_{\mathbf{t}}; V|_{\Sigma_{\mathbf{t}}}\big)^{\varphi} \tag{4.8}$$

on $(V,\varphi)|_{\Sigma_{\mathbf{t}}}$. Let

$$\pi_{(V,\varphi)}\colon \det D_{(V,\varphi)} \equiv \bigsqcup_{\mathbf{t} \in \Delta_{\mathbb{R}}} \big(\{\mathbf{t}\} \times \det D_{(V,\varphi);\mathbf{t}}\big) \longrightarrow \Delta_{\mathbb{R}}\,. \tag{4.9}$$

The set $\det D_{(V,\varphi)}$ carries natural topologies so that the projection $\pi_{(V,\varphi)}$ is a real line bundle; see Appendix A. The case of (4.9) with $\Delta \subset \mathbb{C}$ (and thus $\Delta_{\mathbb{R}}$ is an open subset of $\mathbb{R}$) and (Σ_0, σ_0) having only a conjugate pair of nodes underpins all orienting constructions in the open GW-theory and Fukaya category literature that follow [10, Section 8.1].

Good topologies on the total space of (4.9) arise directly from some of the analytic considerations of [27] combined with the algebraic conclusions of [25]. This implies that the resulting topologies satisfy analogues of all properties listed in [49, Section 2]. In particular, (D1) a homotopy class of continuous isomorphisms $\Psi:(V_1,\varphi_1)\longrightarrow(V_2,\varphi_2)$ of real bundle pairs over $(\mathcal{U},\widetilde{\mathfrak{c}})|_{\Delta_{\mathbb{R}}}$ determines a homotopy class of isomorphisms

$$\det D_\Psi:\det D_{(V_1,\varphi_1)}\longrightarrow\det D_{(V_2,\varphi_2)}$$

of line bundles over $\Delta_{\mathbb{R}}$;

(D2) the isomorphisms (4.6) determine a homotopy class of isomorphisms

$$\det\big(D_{(V_1,\varphi_1)\oplus(V_2,\varphi_2)}\big)\approx\big(\det D_{(V_1,\varphi_1)}\big)\otimes\big(\det D_{(V_2,\varphi_2)}\big)$$

of line bundles over $\Delta_{\mathbb{R}}$ for all real bundle pairs (V_1,φ_1) and (V_2,φ_2) over $(\mathcal{U},\widetilde{\mathfrak{c}})$.

These two properties correspond to the Naturality I and Direct Sum properties in [49, Section 2].

Families of real CR-operators often arise by pulling back data from a target manifold by smooth maps as follows. Suppose (X,J,ϕ) is an almost complex manifold with an anti-complex involution and (V,φ) is a real bundle pair over (X,ϕ). Let ∇ be a φ-compatible (complex-linear) connection in V and

$$A\in\Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(V,(T^*X,J)^{0,1}\otimes_{\mathbb{C}}V)\big)^{\varphi}.$$

For any real map $u:(\Sigma,\sigma)\longrightarrow(X,\phi)$ from a symmetric surface and $\mathfrak{j}\in\mathcal{J}_\Sigma^\sigma$, let ∇^u denote the induced connection in u^*V and

$$A_{\mathfrak{j};u}=A\circ\partial_{\mathfrak{j}}u\in\Gamma(\Sigma;\mathrm{Hom}_{\mathbb{R}}(u^*V,(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}u^*V))^{u^*\varphi}.$$

The homomorphisms

$$\bar\partial_u^\nabla=\frac{1}{2}(\nabla^u+\mathfrak{i}\circ\nabla^u\circ\mathfrak{j}),\ \ D_{(V,\varphi);u}\equiv\bar\partial_u^\nabla+A_{\mathfrak{j};u}:\Gamma(\Sigma;u^*V)^{u^*\varphi}\longrightarrow\Gamma^{0,1}_{\mathfrak{j}}(\Sigma;u^*V)^{u^*\varphi}$$

are real CR-operators on $u^*(V,\varphi)\longrightarrow(\Sigma,\sigma)$ that form families of real CR-operators over families of maps.

For $g,l\in\mathbb{Z}^{\geqslant0}$ and $B\in H_2(X;\mathbb{Z})$, let

$$\det D_{(V,\varphi)}\longrightarrow\mathfrak{B}_{g,l}(X,B)^{\phi,\sigma}\times\mathcal{J}_\Sigma^\sigma$$

denote the determinant line bundle of the family of the CR-operators $D_{(V,\varphi);(u,\mathfrak{j})}$ constructed as above. This line bundle descends to an orbi-bundle

$$\det D_{(V,\varphi)}\longrightarrow\mathcal{H}_{g,l}(X,B)^{\phi,\sigma};$$

it is a line bundle over the open subspace of the base consisting of marked maps with no non-trivial automorphisms.

Let $(\pi,\widetilde{\mathfrak{c}},s_1,\ldots,s_l)$ be a flat family of deformations as in Section 4.2. A smooth real map $F:\mathcal{U}\longrightarrow X$ pulls back the connection ∇ and the zeroth-order deformation term A on (V,φ) above to a connection ∇^F and a zeroth-order deformation term A^F on $F^*(V,\varphi)$. The latter in turn determine a line bundle

$$\pi_{F^*(V,\varphi)}:\det D_{F^*(V,\varphi)}\longrightarrow\Delta_{\mathbb{R}}$$

as in (4.9), which we call $\det D_{(V,\varphi)}$ when there is no ambiguity.

Example 4.3. Let $g,l\in\mathbb{Z}^{\geqslant 0}$ with $g+l\geqslant 2$. The pair $(V,\varphi)\equiv(\mathbb{C},\mathfrak{c})$ is a real bundle pair over $(\mathrm{pt},\mathrm{id})$. The induced families of the operators $\bar\partial_{\mathbb{C};u}\equiv D_{(\mathbb{C},\mathfrak{c});u}$ over flat families of stable real genus g curves with l conjugate pairs of marked points define a line bundle

$$\det\bar\partial_{\mathbb{C}}\longrightarrow\mathbb{R}\overline{\mathcal{M}}_{g,l}\,.$$

If (X,ϕ) is an almost complex manifold with anti-complex involution ϕ and

$$(V,\varphi)=(X\times\mathbb{C},\phi\times\mathfrak{c})\longrightarrow(X,\phi),$$

then there is a canonical isomorphism

$$\det D_{(\mathbb{C},\mathfrak{c})}\approx\mathfrak{f}^*\big(\det\bar\partial_{\mathbb{C}}\big)$$

of line bundles over $\mathcal{H}_{g,l}(X,B)^{\phi,\sigma}$.

5 Real orientations on real bundle pairs

The main stepping stone in our proof of Theorem 1.3 for the uncompactified moduli space

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi}\subset\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$$

is Proposition 5.2 below. By Corollary 5.7 of this proposition, a real orientation on a rank n real bundle pair (V,φ) over a symmetric surface (Σ,σ) determines an orientation on the relative determinant

$$\widehat{\det}\,D\equiv\big(\det D\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes n}\tag{5.1}$$

for every real CR-operator D on (V,φ), where $\bar\partial_{\Sigma;\mathbb{C}}$ is the standard real CR-operator on (Σ,σ) with values in $(\mathbb{C},\mathfrak{c})$.

Definition 5.1. Let (X,ϕ) be a topological space with an involution and (V,φ) be a real bundle pair over (X,ϕ). A real orientation on (V,φ) consists of

(RO1) a rank 1 real bundle pair $(L,\widetilde\phi)$ over (X,ϕ) such that

$$w_2(V^{\varphi})=w_1(L^{\widetilde\phi})^2\qquad\text{and}\qquad\Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi)\approx(L,\widetilde\phi)^{\otimes 2},\tag{5.2}$$

(RO2) a homotopy class $[\psi]$ of isomorphisms of real bundle pairs in (5.2), and

(RO3) a spin structure $\mathfrak{s}$ on the real vector bundle $V^{\varphi}\oplus 2(L^*)^{\widetilde\phi^*}$ over X^{ϕ} compatible with the orientation induced by (RO2).

An isomorphism Θ in (5.2) restricts to an isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}V^{\varphi}\approx(L^{\widetilde\phi})^{\otimes 2}\tag{5.3}$$

of real line bundles over X^{ϕ}. Since the vector bundles $(L^{\widetilde\phi})^{\otimes 2}$ and $2(L^*)^{\widetilde\phi^*}$ are canonically oriented, Θ determines orientations on V^{φ} and $V^{\varphi}\oplus 2(L^*)^{\widetilde\phi^*}$. We will call them the orientations determined by (RO2) if Θ lies in the chosen homotopy class. An isomorphism Θ in (5.2) also induces an isomorphism

$$\begin{aligned}\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big) &\approx \Lambda_{\mathbb{C}}^{\mathrm{top}}(V,\varphi)\otimes(L^*,\widetilde{\phi}^*)^{\otimes 2}\\ &\approx (L,\widetilde{\phi})^{\otimes 2}\otimes(L^*,\widetilde{\phi}^*)^{\otimes 2}\approx \big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big),\end{aligned} \tag{5.4}$$

where the last isomorphism is the canonical pairing. We will call the homotopy class of isomorphisms (5.4) induced by the isomorphisms Θ in (RO2) the homotopy class determined by (RO2).

Proposition 5.2. *Suppose (Σ,σ) is a symmetric surface and (V,φ) is a rank n real bundle pair over (Σ,σ). A real orientation on (V,φ) as in Definition 5.1 determines a homotopy class of isomorphisms*

$$\Psi\colon \big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\approx\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big) \tag{5.5}$$

of real bundle pairs over (Σ,σ). An isomorphism Ψ belongs to this homotopy class if and only if the restriction of Ψ to the real locus induces the chosen spin structure (RO3) and the isomorphism

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi\colon \Lambda_{\mathbb{C}}^{\mathrm{top}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\longrightarrow\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)=\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big) \tag{5.6}$$

lies in the homotopy class determined by (RO2).

This proposition is proved in Section 5.2 after some topological preliminaries concerning symmetric functions on symmetric surfaces are established in Section 5.1. Proposition 5.2 is applied to the orientability problem (A) on page 11 in Section 5.3.

5.1 Homotopies of functions from symmetric surfaces

Let (X,ϕ) be a topological space with an involution. For any Lie group G with a natural conjugation, such as $\mathbb{C}^*$, $\mathrm{SL}_n\mathbb{C}$, or $\mathrm{GL}_n\mathbb{C}$, denote by $\mathcal{C}(X,\phi;G)$ the topological group of continuous maps $f\colon X\longrightarrow G$ such that $f(\phi(x))=\overline{f(x)}$ for all $x\in X$. The restrictions of such functions to the fixed locus $X^\phi\subset X$ take values in the real locus of G, i.e. $\mathbb{R}^*$, $\mathrm{SL}_n\mathbb{R}$, and $\mathrm{GL}_n\mathbb{R}$, in the three examples.

Lemma 5.3. *Let (Σ,σ) be a symmetric surface with fixed components $\Sigma_1^\sigma,\ldots,\Sigma_m^\sigma$ and $n\in\mathbb{Z}^+$. For every $i=1,\ldots,m$ and continuous map $\psi\colon\Sigma_i^\sigma\longrightarrow\mathrm{GL}_n\mathbb{R}$, there exists $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{GL}_n\mathbb{C})$ such that $f|_{\Sigma_i^\sigma}=\psi$ and f is the identity outside of an arbitrarily small neighborhood of Σ_i^σ. The same statement holds with $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$ replaced by $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$, respectively.*

Proof. Let $S^1\times(-2,2)\longrightarrow\Sigma$ be a parametrization of a neighborhood U of Σ_i^σ such that $S^1\times 0$ corresponds to Σ_i^σ and

$$\sigma(\theta,t)=(\theta,-t)\qquad\forall\ (\theta,t)\in S^1\times(-2,2).$$

Since the inclusion $\mathrm{GL}_n\mathbb{R}\longrightarrow\mathrm{GL}_n\mathbb{C}$ induces trivial homomorphisms from π_1 of either component of $\mathrm{GL}_n\mathbb{R}$ to $\pi_1(\mathrm{GL}_n\mathbb{C})$, we can homotope ψ to the identity-valued constant map through maps $h_t\colon S^1\longrightarrow\mathrm{GL}_n\mathbb{C}$. We define f on U by

$$f(\theta,t)=\begin{cases}h_t(\theta), & \text{if } t\in[0,1];\\ I_n, & \text{if } t\in[1,2);\\ \overline{h_{-t}(\theta)}, & \text{if } t\in(-2,0];\end{cases}$$

and extend it as the identity-valued constant map over $\Sigma-U$. The same argument applies with $\mathrm{GL}_n\mathbb{R}$ and $\mathrm{GL}_n\mathbb{C}$ replaced by $\mathrm{SL}_n\mathbb{R}$ and $\mathrm{SL}_n\mathbb{C}$, respectively. ☐

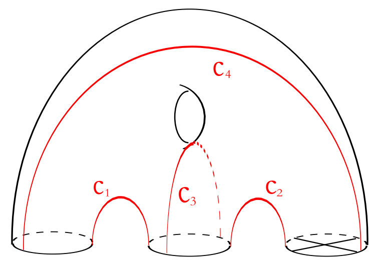

Figure 5: The paths $C_1,\ldots,C_4$ cut Σ^b to a disk.

Lemma 5.4. *Suppose* (Σ,σ) *is a symmetric surface,* $n\in\mathbb{Z}^+$*, and* $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$*. If*

$$f|_{\Sigma^\sigma}\colon \Sigma^\sigma \longrightarrow \mathrm{SL}_n\mathbb{R}$$

is homotopic to a constant map, then f *is homotopic to the constant map* Id *through maps* $f_t\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$*.*

Proof. Let (Σ^b,c) be an oriented sh-surface which doubles to (Σ,σ). By assumption,

$$f|_{(\partial\Sigma^b)_i}\colon (\partial\Sigma^b)_i \longrightarrow \mathrm{SL}_n\mathbb{C}$$

is homotopic to Id through maps $f_t\in\mathcal{C}((\partial\Sigma^b)_i,c;\mathrm{SL}_n\mathbb{C})$ on each boundary component $(\partial\Sigma^b)_i$ of Σ^b with $|c_i|=0$. Since $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$, this is also the case for $f|_{(\partial\Sigma^b)_i}$ for each boundary component $(\partial\Sigma^b)_i$ of Σ^b with $|c_i|=1$; see [7, Lemma 2.4].

A homotopy f_t as above extends over Σ^b as follows. Suppose $f_0=f|_{\partial\Sigma^b}$ and $f_1=\mathrm{Id}$. Let $\mathbb{I}=[0,1]$ and $(\partial\Sigma^b)\times\mathbb{I}\longrightarrow U$ be a parametrization of a (closed) neighborhood U of $\partial\Sigma^b\subset\Sigma^b$ with coordinates (w,s). Define

$$G_t\colon \Sigma^b \longrightarrow \mathrm{SL}_n\mathbb{C} \qquad \text{by} \quad G_t(z) = \begin{cases} f_{(1-s)t}(w)\cdot f^{-1}(w), & \text{if } z=(w,s)\in U\approx(\partial\Sigma^b)\times\mathbb{I};\\ I_n, & \text{if } z\in\Sigma^b-U. \end{cases}$$

Since $G_t(w,1)=I_n$ for all t, this map is continuous. Moreover, $G_0(z)=I_n$ for all $z\in\Sigma^b$ and

$$G_t(w,0) = f_t(w)\cdot f^{-1}(w)$$

is a homotopy between Id and f^{-1}. Thus, $H_t=G_t\cdot f$ is a homotopy over Σ^b extending f_t.

By the previous paragraph, we may assume that f is the constant map Id on $\partial\Sigma^b$. Choose embedded non-intersecting paths $\{C_i\}$ in Σ^b with endpoints on $\partial\Sigma^b$ which cut Σ^b into a disk D^2; see Figure 5. The restriction of f to each C_i defines an element of

$$\pi_1\big(\mathrm{SL}_n\mathbb{C},I_n\big)\approx\pi_1\big(\mathrm{SU}_n,I_n\big)=0.$$

Thus, we can homotope f to Id over C_i while keeping it fixed at the endpoints. Similarly to the previous paragraph, this homotopy extends over Σ^b without changing f over $\partial\Sigma^b$ or over C_j for any $C_j \neq C_i$. Thus, we may assume that f is the constant map Id over the boundary of D^2. Since

$$\pi_2\big(\mathrm{SL}_n\mathbb{C}, I_n\big) \approx \pi_2\big(\mathrm{SU}_n, I_n\big) = 0,$$

the map $f\colon (D^2, S^1) \longrightarrow (\mathrm{SL}_n\mathbb{C}, I_n)$ can be homotoped to Id as a relative map. Doubling such a homotopy f_t by the requirement that $f_t(\sigma(z)) = \overline{f_t(z)}$ for all $z \in \Sigma$, we obtain the desired homotopy from f to Id over all of Σ. □

Corollary 5.5. *Let (Σ,σ) be a symmetric surface and*

$$\Phi, \Psi\colon (V,\varphi) \longrightarrow \big(\Sigma\times\mathbb{C}^n, \sigma\times\mathfrak{c}\big)$$

be isomorphisms of real bundle pairs over (Σ,σ). If the isomorphisms

$$\begin{gathered} \Phi|_{V^\varphi}, \Psi|_{V^\varphi}\colon V^\varphi \longrightarrow \Sigma\times\mathbb{R}^n, \\ \Lambda^{\mathrm{top}}_{\mathbb{C}}\Phi, \Lambda^n_{\mathbb{C}}\Psi\colon \Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi) \longrightarrow \Lambda^{\mathrm{top}}_{\mathbb{C}}\big(\Sigma\times\mathbb{C}^n, \sigma\times\mathfrak{c}\big) = \big(\Sigma\times\mathbb{C}, \sigma\times\mathfrak{c}\big) \end{gathered} \tag{5.7}$$

are homotopic, then so are the isomorphisms Φ and Ψ.

Proof. Let $f \in \mathcal{C}(\Sigma,\sigma;\mathbb{C}^*)$ be given by

$$\Lambda^{\mathrm{top}}_{\mathbb{C}}\Phi = f\,\Lambda^n_{\mathbb{C}}\Psi\colon \Lambda^{\mathrm{top}}_{\mathbb{C}}(V,\varphi) \longrightarrow \big(\Sigma\times\mathbb{C}, \sigma\times\mathfrak{c}\big).$$

Since the second pair of isomorphisms in (5.7) are homotopic, there exists a path $f_t \in \mathcal{C}(\Sigma,\sigma;\mathbb{C}^*)$ such that $f_0 = 1$ and $f_1 = f$. Let

$$\Psi_{f_t}\colon (V,\varphi) \longrightarrow \big(\Sigma\times\mathbb{C}^n, \sigma\times\mathfrak{c}\big)$$

be the composition of Ψ with the real bundle map

$$\big(\Sigma\times\mathbb{C}^n, \sigma\times\mathfrak{c}\big) \longrightarrow \big(\Sigma\times\mathbb{C}^n, \sigma\times\mathfrak{c}\big), \quad (z,v) \longrightarrow \begin{pmatrix} f_t(z) & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix} v. \tag{5.8}$$

Thus, $\Psi_f = \Psi_{f_1}$ is homotopic to Ψ and $\Lambda^{\mathrm{top}}_{\mathbb{C}}\Phi = \Lambda^{\mathrm{top}}_{\mathbb{C}}\Psi_f$.

Let $F \in \mathcal{C}(\Sigma,\sigma;\mathrm{GL}_n\mathbb{C})$ be given by

$$\Psi(v) = \big\{\mathrm{id}\times F(\pi(v))\big\}\big(\Phi(v)\big) \qquad \forall\ v \in V,$$

where $\pi : V \longrightarrow \Sigma$ is the projection map. By the previous paragraph, we can assume that $F \in \mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$. Since the first pair of isomorphisms in (5.7) are homotopic,

$$F|_{\Sigma^\sigma_i}\colon \Sigma^\sigma_i \longrightarrow \mathrm{SL}_n\mathbb{R}$$

is homotopically trivial for every component $\Sigma^\sigma_i \subset \Sigma^\sigma$ of the fixed locus. By Lemma 5.4, F is thus homotopic to the constant map Id through elements $F_t \in \mathcal{C}(\Sigma,\sigma;\mathrm{SL}_n\mathbb{C})$. This establishes the claim. □

5.2 Isomorphisms induced by real orientations

We now apply Lemma 5.3 and Corollary 5.5 to establish Proposition 5.2. We then deduce some corollaries from this proposition.

Proof of Proposition 5.2. Let $\Sigma_1^\sigma, \ldots \Sigma_m^\sigma \subset \Sigma^\sigma$ be the connected components of the fixed locus. Since $c_1(V\oplus 2L^*)=0$ and the vector bundle $V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*}$ is orientable, an isomorphism Ψ as in (5.5) exists; see [3, Propositions 4.1,4.2]. For each $i=1,\ldots,m$, choose $\psi_i\colon \Sigma_i^\sigma \longrightarrow \mathrm{GL}_{n+2}\mathbb{R}$ so that the composition of the restriction of Ψ to $(V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*})|_{\Sigma_i^\sigma}$ with the isomorphism

$$\Sigma_i^\sigma\times\mathbb{R}^{n+2}\longrightarrow\Sigma_i^\sigma\times\mathbb{R}^{n+2}, \qquad (z,v)\longrightarrow\big(z,\psi_i(z)v\big),$$

induces the chosen orientation and spin structure on $(V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*})|_{\Sigma_i^\sigma}$. Let $f_i:\Sigma\longrightarrow\mathrm{GL}_{n+2}\mathbb{C}$ be a continuous map as in Lemma 5.3 corresponding to (i,ψ_i). The composition of the original isomorphism Ψ with the real map

$$\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)\longrightarrow\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big), \qquad (z,v)\longrightarrow\big(z,f_1(z)\ldots f_m(z)v\big),$$

is again an isomorphism of real bundle pairs as in (5.5).

By the previous paragraph, there exists an isomorphism Ψ as in (5.5) that induces the chosen orientation and spin structure on $V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*}$. It determines an isomorphism

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\approx\Lambda_{\mathbb{C}}^{\mathrm{top}}\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)=\big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big)$$

and thus an isomorphism $\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi$ as in (5.6). If ψ is the isomorphism in (5.6) determined by an isomorphism in (5.2) from the chosen homotopy class (RO2), then

$$\psi=f\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi \tag{5.9}$$

for some $f\in\mathcal{C}(\Sigma,\sigma;\mathbb{C}^*)$. Let

$$\Psi_f\colon\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\approx\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)$$

be defined as in (5.8). By (5.9), $\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi_f=\psi$. Since Ψ and ψ induce the same orientations on $V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*}$, $f|_{\Sigma^\sigma}>0$. Thus, Ψ_f induces the same orientation and spin structure on $V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*}$ as Ψ.

We conclude that there exists an isomorphism Ψ as in (5.5) inducing the chosen orientation and spin structure on $V^\varphi\oplus 2(L^*)^{\widetilde{\phi}^*}$ so that the isomorphism $\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi$ lies in the homotopy class of the isomorphisms (5.4) determined by (RO2). By Corollary 5.5, any two such isomorphisms are homotopic. $\square$

Corollary 5.6. *Suppose (Σ,σ) is a symmetric surface and $(L,\widetilde{\phi})\longrightarrow(\Sigma,\sigma)$ is a rank 1 real bundle pair. If $L^{\widetilde{\phi}}\longrightarrow\Sigma^\sigma$ is orientable, there exists a canonical homotopy class of isomorphisms*

$$\big(L^{\otimes 2}\oplus 2L^*,\widetilde{\phi}^{\otimes 2}\oplus 2\widetilde{\phi}^*\big)\approx\big(\Sigma\times\mathbb{C}^3,\sigma\times\mathfrak{c}\big) \tag{5.10}$$

of real bundle pairs over (Σ,σ). In general, an orientation of each component Σ_i^σ of Σ^σ such that $L^{\widetilde{\phi}}|_{\Sigma_i^\sigma}$ is non-orientable determines a canonical homotopy class of isomorphisms (5.10); changing an orientation of such a component Σ_i^σ changes the induced spin structure, but not the orientation, of the real part of LHS in (5.10) over Σ_i^σ.

Proof. The line bundle $(L^{\widetilde{\phi}})^{\otimes 2}$ is canonically oriented and thus has a canonical homotopy class of trivializations. We apply Proposition 5.2 with $(V,\varphi)=(L,\widetilde{\phi})^{\otimes 2}$. There is then a canonical choice of isomorphism in (5.2). It induces the canonical orientations on the real parts of $2(L^*,\widetilde{\phi}^*)$ or of LHS in (5.10). If $L^{\widetilde{\phi}}$ is orientable, an orientation on $L^{\widetilde{\phi}}$ determines a homotopy class of trivializations of the real part of LHS in (5.10). The resulting spin structure is independent of the choice of the orientation.

If the restriction of $L^{\widetilde{\phi}}$ to a component $\Sigma_i^\sigma \approx \mathbb{RP}^1$ of the fixed locus $\Sigma^\sigma \subset \Sigma$ is not orientable, then $(L^*)^{\widetilde{\phi}^*}|_{\Sigma_i^\sigma}$ is isomorphic to the tautological line bundle

$$\gamma \equiv \big\{\big(\ell,(x,y)\big)\in\mathbb{RP}^1\times\mathbb{R}^2\colon\ (x,y)\in\ell\subset\mathbb{R}^2\big\} \longrightarrow \mathbb{RP}^1 .$$

Combining this isomorphism with the trivialization

$$\gamma\oplus\gamma \longrightarrow \mathbb{RP}^1\times\mathbb{R}^2, \qquad \big(\ell,(x_1,y_1),(x_2,y_2)\big) \longrightarrow \big(\ell,(x_1-y_2,x_2+y_1)\big), \tag{5.11}$$

we obtain an isomorphism

$$2(L^*)^{\widetilde{\phi}^*} \longrightarrow \mathbb{RP}^1\times\mathbb{R}^2. \tag{5.12}$$

It induces the canonical orientation on the domain. The homotopy class of the isomorphism (5.12) does not depend on the choice of isomorphism of $(L^*)^{\widetilde{\phi}^*}|_{\Sigma_i^\sigma}$ with γ, once an identification of Σ_i^σ with $\mathbb{RP}^1$ is fixed. However, it does depend on the orientation class of this identification even after stabilization by the trivial line bundle, as shown in the next paragraph.

A bundle isomorphism $\gamma\longrightarrow\gamma$ covering an orientation-reversing map $\mathbb{RP}^1\longrightarrow\mathbb{RP}^1$ is given by

$$\gamma \longrightarrow \gamma, \qquad \big([u,v],(x,y)\big) \longrightarrow \big([u,-v],(x,-y)\big).$$

The composition of this isomorphism with the isomorphism (5.11) is the isomorphism

$$\gamma\oplus\gamma \longrightarrow \mathbb{RP}^1\times\mathbb{R}^2, \qquad \big(\ell,(x_1,y_1),(x_2,y_2)\big) \longrightarrow \big(\ell,(x_1+y_2,x_2-y_1)\big). \tag{5.13}$$

Under the standard identification of $\mathbb{R}^2$ with $\mathbb{C}$, $\mathbb{RP}^1$ can be parametrized as

$$S^1 \longrightarrow \mathbb{RP}^1, \qquad \mathrm{e}^{\mathrm{i}\theta} \longrightarrow \big[\mathrm{e}^{\mathrm{i}\theta/2}\big].$$

Under this identification, the isomorphisms (5.11) and (5.13) are given by

$$\begin{aligned}
\big(\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta/2}, b\mathrm{e}^{\mathrm{i}\theta/2}\big) &\longrightarrow \big(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta/2}(a+\mathrm{i}b)\big) \qquad &\forall\ a,b\in\mathbb{R},\\
\big(\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{\mathrm{i}\theta/2}, b\mathrm{e}^{\mathrm{i}\theta/2}\big) &\longrightarrow \big(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta/2}(a+\mathrm{i}b)\big) \qquad &\forall\ a,b\in\mathbb{R},
\end{aligned}$$

respectively. They differ by the map

$$S^1 \longrightarrow \mathrm{GL}_2\mathbb{R}, \qquad \mathrm{e}^{\mathrm{i}\theta} \longrightarrow \mathrm{e}^{-\mathrm{i}\theta}.$$

Since this map generates $\pi_1(\mathrm{GL}_2\mathbb{R})$, the trivializations of $\gamma\oplus\gamma$ in (5.11) and (5.13) are not homotopy equivalent, even after stabilization by the trivial line bundle. $\square$

Corollary 5.7. *Suppose (Σ,σ) is a symmetric surface and $D=D_{(V,\varphi)}$ is a real CR-operator on a rank n real bundle pair (V,φ) over (Σ,σ). Then a real orientation on (V,φ) as in Definition 5.1 induces an orientation on the relative determinant $\widehat{\det}\,D$ of D in (5.1). Changing a real orientation on (V,φ) by changing the spin structure $\mathfrak{s}$ in (RO3) over one component Σ^σ_i of Σ^σ reverses the orientation on $\widehat{\det}\,D$.*

Proof. Let $((L,\widetilde{\phi}),[\psi],\mathfrak{s})$ be a real orientation on (V,φ). By (4.6), there is a canonical homotopy class of isomorphisms

$$\det D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)} \approx \big(\det D_{(V,\varphi)}\big)\otimes\big(\det D_{(L^*,\widetilde{\phi}^*)}\big)^{\otimes 2}$$

of real lines, where the subscripts indicate the real bundle pair associated with the corresponding real CR-operator. Since the last factor above is canonically oriented, so is the line

$$\big(\det D_{(V,\varphi)}\big)\otimes\big(\det D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\big)\,. \tag{5.14}$$

By Proposition 5.2, the real orientation on (V,φ) determines a homotopy class of isomorphisms

$$\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big)\approx\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big).$$

By (4.6), the latter in turn determines an orientation on the line

$$\widehat{\det} D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\equiv\big(\det D_{(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*)}\big)\otimes\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes(n+2)}\,.$$

Combining this with the canonical orientation of the line (5.14), we obtain an orientation on $\widehat{\det}\,D$.

Let $\mathfrak{s}_{\mathrm{can}}$ denote the canonical spin structure on $\Sigma\times\mathbb{R}^{n+2}$. By Proposition 5.2, the identity automorphism of $\Lambda^{\mathrm{top}}_{\mathbb{C}}(\Sigma\times\mathbb{C}^{n+2})$ and a spin structure on $\Sigma^\sigma\times\mathbb{R}^{n+2}$ determine a homotopy class of isomorphisms

$$\Psi\colon\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)\longrightarrow\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big) \tag{5.15}$$

of real bundle pairs over (Σ,σ). The latter in turn determines a homotopy class of isomorphisms

$$\det D_\Psi\colon\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes(n+2)}=\det\!\big((n+2)\bar\partial_{\Sigma;\mathbb{C}}\big)\longrightarrow\det\!\big((n+2)\bar\partial_{\Sigma;\mathbb{C}}\big)=\big(\det\bar\partial_{\Sigma;\mathbb{C}}\big)^{\otimes(n+2)}\,. \tag{5.16}$$

For the purposes of the last claim of this corollary, it is sufficient to check that the last isomorphisms are orientation-reversing for the spin structure $\mathfrak{s}_i$ on $\Sigma^\sigma\times\mathbb{R}^{n+2}$ which differs from $\mathfrak{s}_{\mathrm{can}}$ on precisely one component Σ^σ_i of Σ^σ.

By Lemma 5.3, we can assume that the map Ψ in (5.15) is the identity outside of a tubular neighborhood $U\subset\Sigma$ of Σ^σ_i with $\overline{U}\subset\Sigma$ disjoint from $\Sigma^\sigma-\Sigma^\sigma_i$. Pinching each of the two components of the boundary $\partial\overline{U}'$ of a slightly larger tubular neighborhood U', we obtain a nodal symmetric surface (Σ_0,σ_0) consisting of $(\mathbb{P}^1,\tau)$ and a smooth symmetric, possibly disconnected, surface (Σ',σ') which share a pair of conjugate points. We can choose a flat family

$$\big(\pi\colon\mathcal{U}\longrightarrow\Delta,\widetilde{\mathfrak{c}}\colon\mathcal{U}\longrightarrow\mathcal{U}\big)$$

of deformations of (Σ_0,σ_0) as in Section 4.2 with $\Delta\subset\mathbb{C}^2$ and $(\Sigma_{\mathbf{t}^*},\sigma_{\mathbf{t}^*})=(\Sigma,\sigma)$ for some $\mathbf{t}^*\in\Delta_{\mathbb{R}}$ and a quotient map

$$q\colon\Delta\times\Sigma_{\mathbf{t}^*}\longrightarrow\mathcal{U}$$

intertwining the involutions $\mathfrak{c}\times\sigma$ and $\tilde{\mathfrak{c}}$ so that q is a diffeomorphism outside of $\Delta\times\partial\overline{U}'$ and the map

$$\Sigma_{\mathbf{t}^*} \longrightarrow \Sigma_{\mathbf{t}^*}, \qquad z \longrightarrow q(\mathbf{t}^*, z),$$

is the identity.

Since the isomorphism Ψ in (5.15) is the identity over $\Sigma - U \supset \partial\overline{U}'$, Ψ induces an isomorphism

$$\widetilde{\Psi}\colon \big(\mathcal{U}\times\mathbb{C}^{n+2}, \tilde{\mathfrak{c}}\times\mathfrak{c}\big) \longrightarrow \big(\mathcal{U}\times\mathbb{C}^{n+2}, \tilde{\mathfrak{c}}\times\mathfrak{c}\big)$$

which restricts to Ψ over $(\Sigma_{\mathbf{t}^*}, \sigma_{\mathbf{t}^*})$ and to the identity over $q(\Delta\times(\Sigma - U)) \supset \Sigma'$. Denote by

$$\Psi_0\colon \big(\Sigma_0\times\mathbb{C}^{n+2}, \sigma_0\times\mathfrak{c}\big) \longrightarrow \big(\Sigma_0\times\mathbb{C}^{n+2}, \sigma_0\times\mathfrak{c}\big)$$

the restriction of $\widetilde{\Psi}$ over (Σ_0, σ_0) and by

$$\det D_{\Psi_0}\colon \det\!\big((n+2)\bar{\partial}_{\Sigma_0;\mathbb{C}}\big) \longrightarrow \det\!\big((n+2)\bar{\partial}_{\Sigma_0;\mathbb{C}}\big) \tag{5.17}$$

the homotopy class of isomorphisms induced by Ψ_0.

By (D1) on page 23, $\widetilde{\Psi}$ determines a homotopy class of isomorphisms

$$\det D_{\widetilde{\Psi}}\colon \det\!\big((n+2)\bar{\partial}_{\mathcal{U}|_{\Delta_{\mathbb{R}}};\mathbb{C}}\big) \longrightarrow \det\!\big((n+2)\bar{\partial}_{\mathcal{U}|_{\Delta_{\mathbb{R}}};\mathbb{C}}\big)$$

of determinant line bundles over $\Delta_{\mathbb{R}}$. Since $\det D_{\widetilde{\Psi}}$ restricts to $\det D_{\Psi}$ over $(\Sigma_{\mathbf{t}^*}, \sigma_{\mathbf{t}^*})$ and to $\det D_{\Psi_0}$ over (Σ_0, σ_0), the isomorphisms (5.16) are orientation-reversing if and only if the isomorphisms $\det D_{\Psi_0}$ are orientation-reversing. The latter correspond to the tensor products of isomorphisms $\det D_{\Psi_0|_{\Sigma'}}$ for (Σ', σ') and $\det D_{\Psi_0|_{\mathbb{P}^1}}$ for $(\mathbb{P}^1, \tau)$. The isomorphisms $\det D_{\Psi_0|_{\Sigma'}}$ are the identity. Since the isomorphisms Ψ_0 reverse the spin structure on the fixed locus $q(\Sigma_i^\sigma)$ of $(\mathbb{P}^1, \tau)$, the isomorphisms $\det D_{\Psi_0|_{\mathbb{P}^1}}$ are orientation-reversing; see [10, Proposition 8.1.7]. We conclude that the isomorphisms (5.17) and thus (5.16) are orientation-reversing. $\square$

Corollary 5.8. *Suppose (X, J, ϕ) is an almost complex manifold with an anti-complex involution and (V, φ) is a rank n real bundle pair over (X, ϕ). Let $B \in H_2(X;\mathbb{Z})$, $g, l \in \mathbb{Z}^{\geqslant 0}$, and (Σ, σ) be a genus g symmetric surface. Then a real orientation on (V, φ) as in Definition 5.1 induces an orientation on the line bundle*

$$\widehat{\det}\, D_{(V,\varphi)} \equiv \big(\det D_{(V,\varphi)}\big) \otimes \big(\det \bar{\partial}_{\mathbb{C}}\big)^{\otimes n} \longrightarrow \mathcal{H}_{g,l}(X, B)^{\phi,\sigma}\,. \tag{5.18}$$

Proof. By Corollary 5.7 applied with the real bundle pairs $u^*(V, \varphi)$ and $u^*(L, \widetilde{\phi})$ over (Σ, σ), a real orientation on (V, φ) determines an orientation on the fiber of the line bundle

$$\big(\det D_{(V,\varphi)}\big) \otimes \big(\det \bar{\partial}_{\mathbb{C}}\big)^{\otimes n} \longrightarrow \mathfrak{B}_g(X, B)^{\phi,\sigma} \times \mathcal{J}_\Sigma^\sigma$$

over each point $(u, \mathfrak{j})$ which varies continuously with $(u, \mathfrak{j})$. Since the resulting orientation on this line bundle is completely determined by the chosen real orientation on (V, φ) via the isomorphisms (5.5), it descends to the quotient (5.18). $\square$

5.3 The orientability of uncompactified moduli spaces

We will now apply Proposition 5.2 to study the orientability of the uncompactified real moduli spaces in Theorem 1.3. We first consider the case $X = \mathrm{pt}$ and then use it to establish the restriction of Theorem 1.3 to the main stratum $\mathfrak{M}_{g,l}(X,B;J)^\phi$ of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$.

Proposition 5.9. *Let $g,l \in \mathbb{Z}^{\geqslant 0}$ be such that $g+l \geqslant 2$. For every genus g type σ of orientation-reversing involutions, the line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathcal{M}^\sigma_{g,l}\big) \otimes \big(\det \bar\partial_{\mathbb{C}}\big) \longrightarrow \mathcal{M}^\sigma_{g,l} \tag{5.19}$$

is canonically oriented. The interchanges of pairs of conjugate points and the forgetful morphisms preserve this orientation; the interchange of the points within a conjugate pair reverses this orientation.

Proof. The cardinality of the automorphism group is an upper semi-continuous function on the compact moduli space $\overline{\mathcal{M}}^\sigma_{g,l}$. Thus, there exists $l(g) \in \mathbb{Z}^+$ so that for every $l \geqslant l(g)$ every element

$$[\mathcal{C}] \equiv \big[\Sigma, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j}\big] \in \mathcal{M}^\sigma_{g,l}$$

has no automorphisms. We first establish the proposition under the assumption that $l \geqslant l(g)$.

Let $\mathcal{T} \longrightarrow \mathcal{U}^\sigma_{g,l}$ denote the vertical tangent bundle over the universal curve for $\mathcal{M}^\sigma_{g,l}$. For $\mathcal{C}$ as above, let

$$T\mathcal{C} = T\Sigma\big(-z_1^+ - z_1^- - \ldots - z_l^+ - z_l^-\big) \quad \text{and} \quad T^*\mathcal{C} = T^*\Sigma\big(z_1^+ + z_1^- + \ldots + z_l^+ + z_l^-\big), \tag{5.20}$$

be the twisted holomorphic line bundles associated to the sheaves of holomorphic tangent vector fields vanishing at the marked points and of meromorphic one-forms with at most simple poles at the marked points and holomorphic everywhere else. We construct these line bundles using holomorphic identifications of small neighborhoods of z_i^+ and z_i^- interchanged by σ. The involutions $\mathrm{d}\sigma$ on $T\Sigma$ and $\mathrm{d}\sigma^*$ on $T^*\Sigma$ then induce involutions on $T\mathcal{C}$ and $T^*\mathcal{C}$; we denote the induced involutions also by $\mathrm{d}\sigma$ and $\mathrm{d}\sigma^*$.

Let $S\mathcal{C}^+$ and $S\mathcal{C}^-$ be the skyscraper sheaves over Σ given by

$$S\mathcal{C}^+ = T^*\Sigma|_{z_1^+ + \ldots + z_l^+}, \qquad S\mathcal{C}^- = T^*\Sigma|_{z_1^- + \ldots + z_l^-}.$$

The projection

$$\pi_1 \colon \check{H}^0(\Sigma; S\mathcal{C}^+ \oplus S\mathcal{C}^-)^\sigma = \big(\check{H}^0(\Sigma; S\mathcal{C}^+) \oplus \check{H}^0(\Sigma; S\mathcal{C}^-)\big)^\sigma \longrightarrow \check{H}^0(\Sigma; S\mathcal{C}^+) \tag{5.21}$$

is an isomorphism of real vector spaces. We orient $\check{H}^0(\Sigma; S\mathcal{C}^+ \oplus S\mathcal{C}^-)^\sigma$ and its dual via the isomorphism

$$\pi_1^* \colon \check{H}^0(\Sigma; S\mathcal{C}^+)^* = T_{z_1^+}\Sigma \oplus \ldots \oplus T_{z_l^+}\Sigma \longrightarrow \big(\check{H}^0(\Sigma; S\mathcal{C}^+ \oplus S\mathcal{C}^-)^\sigma\big)^*$$

from the complex orientations of $T_{z_1^+}\Sigma, \ldots, T_{z_l^+}\Sigma$.

The Kodaira-Spencer (or KS) map and the Dolbeault isomorphism provide canonical isomorphisms

$$T_{[\mathcal{C}]}\mathcal{M}^\sigma_{g,l} \approx \check{H}^1(\Sigma; T\mathcal{C})^\sigma \approx H^1(\Sigma; T\mathcal{C})^\sigma; \tag{5.22}$$

see [28, Section 3.1.2] and [22, p151]. By Serre Duality (or SD), there is a canonical isomorphism

$$H^1(\Sigma;T\mathcal{C}) \approx \big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)\big)^*;$$

see [22, p153]. Since σ is orientation-reversing, the real part of the SD pairing identifies the space of invariant sections on one side with the space of anti-invariant sections on the other; the latter is isomorphic to the space of invariant sections by multiplication by $\mathfrak{i}$. Thus, there is a canonical isomorphism

$$H^1(\Sigma;T\mathcal{C})^\sigma \approx \big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^\sigma\big)^*. \tag{5.23}$$

Since the degree of the holomorphic line bundle $T\mathcal{C}$ is negative,

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(H^0(\Sigma;T^*\mathcal{C}\otimes T^*\Sigma)^\sigma\big) = \det\bar\partial_{(T^*\mathcal{C},\mathrm{d}\sigma^*)\otimes(T^*\Sigma,\mathrm{d}\sigma^*)}.$$

The long exact sequence in cohomology for the sequence

$$0 \longrightarrow T^*\Sigma\otimes T^*\Sigma \longrightarrow T^*\mathcal{C}\otimes T^*\Sigma \longrightarrow S\mathcal{C}^+\oplus S\mathcal{C}^- \longrightarrow 0 \tag{5.24}$$

and the chosen orientation on $\check{H}^0(\Sigma;S\mathcal{C}^+\oplus S\mathcal{C}^-)^\sigma$ induce an orientation on the line

$$\det\bar\partial_{(T^*\mathcal{C},\mathrm{d}\sigma^*)\otimes(T^*\Sigma,\mathrm{d}\sigma^*)}\otimes\det\bar\partial_{(T^*\Sigma,\mathrm{d}\sigma^*)^{\otimes2}}\,. \tag{5.25}$$

Thus, the real line bundle

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathcal{M}^\sigma_{g,l}\big)\otimes\big(\det\bar\partial_{(\mathcal{T}^*,\mathrm{d}\sigma^*)^{\otimes2}}\big) \longrightarrow \mathcal{M}^\sigma_{g,l} \tag{5.26}$$

is canonically oriented.

By Corollary 5.6 applied with $(L,\widetilde\phi)=(T^*\Sigma,\mathrm{d}\sigma^*)$, there is a canonical homotopy class of isomorphisms

$$\big(T^*\Sigma^{\otimes2}\oplus2\,T\Sigma,(\mathrm{d}\sigma^*)^{\otimes2}\oplus2\mathrm{d}\sigma\big) \approx \big(\Sigma\times\mathbb{C}^3,\sigma\times\mathfrak{c}\big)$$

of real bundle pairs over (Σ,σ). Since the determinants of $\bar\partial$-operators on the real bundle pairs $2(T\Sigma,\mathrm{d}\sigma)$ and $2(\Sigma\times\mathbb{C}^2,\sigma\times\mathfrak{c})$ are canonically oriented, so is the line bundle

$$\big(\det\bar\partial_{(\mathcal{T}^*,\mathrm{d}\sigma^*)^{\otimes2}}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big) \longrightarrow \mathcal{M}^\sigma_{g,l}. \tag{5.27}$$

Combining this orientation with the canonical orientation for the line bundle (5.26), we obtain an orientation on the line bundle (5.19).

Since the interchanges of pairs of conjugate points and the forgetful morphisms preserve the orientation of (5.21), they also preserve the orientation on (5.19) constructed above. Since the interchange of the points within a conjugate pair reverses the orientation of (5.21), it also reverses the orientation on (5.19).

For $l<l(g)$, we orient the line bundle (5.19) by downward induction from the orientation of (5.19) with l replaced by $l+1$ and the orientation of the fibers of the forgetful morphism

$$\mathcal{M}^\sigma_{g,l+1} \longrightarrow \mathcal{M}^\sigma_{g,l} \tag{5.28}$$

obtained from the complex orientation of $T_{z^+_{l+1}}\Sigma$. If the fixed locus Σ^σ of (Σ,σ) is separating, the fibers of this morphism are disconnected and differ by the interchange of the points in the last conjugate pair of points. However, the induced orientation on (5.19) is still well-defined for the following reason. By Proposition 5.9 with l replaced by $l+1$, the interchange of the points within a conjugate pair reverses the orientation on the line bundle (5.19) with l replaced by $l+1$. In the case of the last conjugate pair of points, such an interchange also reverses the orientation of the fibers of (5.28). Thus, it has no effect on the induced orientation on (5.19). $\square$

Corollary 5.10. *Theorem 1.3 holds with $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ replaced by $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ for every genus g orientation-reversing involution σ.*

Proof. We first assume that $g+l\geqslant 2$ as in Proposition 5.9. The forgetful morphism $\mathfrak{f}$ induces a canonical isomorphism

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big(T\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\big) \approx \big(\det D_{(TX,\mathrm{d}\phi)}\big)\otimes \mathfrak{f}^*\big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(T\mathcal{M}^{\sigma}_{g,l})\big) \longrightarrow \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \qquad (5.29)$$

of real line bundles. By Corollary 5.8 applied with $(V,\varphi)=(TX,\mathrm{d}\phi)$, a real orientation on (X,ω,ϕ) determines an orientation on

$$\widehat{\det}\, D_{(TX,\mathrm{d}\phi)} \equiv \big(\det D_{(TX,\mathrm{d}\phi)}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big)^{\otimes n} \longrightarrow \mathcal{H}_{g,l}(X,B)^{\phi,\sigma}\,. \qquad (5.30)$$

Combining the canonical isomorphism (5.29), the canonical orientation of (5.19), and the orientation of (5.30) determined by the chosen real orientation on (X,ω,ϕ), we obtain an orientation on the line bundle (1.5) over $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$.

If $g+l<2$, we orient the line bundle (1.5) from the orientation of (1.5) with l replaced by $l+2$ and the orientation of the fibers of the forgetful morphism

$$\mathfrak{M}_{g,l+2}(X,B;J)^{\phi,\sigma} \longrightarrow \mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma} \qquad (5.31)$$

obtained from the complex orientations of $T_{z^+_{l+1}}\Sigma$ and $T_{z^+_{l+2}}\Sigma$. The induced orientation on (1.5) is still well-defined for the following reason. By Proposition 5.9, the interchange of the points within a conjugate pair reverses the orientation on the line bundle (5.19) with l replaced by $l+2$ and thus on the line bundle (1.5) with l replaced by $l+2$. In the case of the last two pairs of conjugate points, such an interchange also reverses the orientation of the fibers of (5.31). Thus, it has no effect on the induced orientation on (1.5). $\square$

Proposition 5.9 is also obtained in [6]; see Corollaires 1.2 and 1.1, Proposition 1.4, and Lemmes 1.3 and 1.4 in [6]. A version of Corollary 5.10 for certain covers of the uncompactified moduli spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ appears in [6] as well. The orientability of these covers is obtained in [6] in a subset of cases for which Corollary 5.10 implies the orientability of the spaces $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ themselves (while Theorem 1.3 also yields the orientability of their compactifications). For example, let $X_{n;\delta}\subset\mathbb{P}^{n-1}$ denote a hypersurface of degree $\delta\in\mathbb{Z}^+$ preserved by τ_n. Corollary 5.10 implies that $\mathfrak{M}_{g,l}(X_{n;\delta},B;J)^{\tau_{n;\delta},\sigma}$ is orientable if

$$\delta = 0,1 \mod 4 \qquad \text{and} \qquad \delta\equiv n \mod 2.$$

With the second condition strengthened to $\delta\equiv n \mod 4$, this conclusion is obtained in [6, Corollaire 2.4] under the additional assumption that Σ^{σ} is a single circle. If Σ^{σ} consists of more than one circle, [6, Corollaire 2.4] shows that this conclusion holds after pulling back to a cover of $\mathfrak{M}_{g,l}(X_{n;\delta},B;J)^{\phi,\sigma}$. The orientability of the compactified moduli spaces of real maps necessary for defining real GW-invariants is not considered in [6].

A canonical orientation on the real line $\widehat{\det} D$ in Corollary 5.7 under overlapping topological assumptions is obtained in [6] using a completely different approach. We obtain it as an immediate consequence of the existence of a canonical homotopy class of isomorphisms for the corresponding real bundle pairs. The argument of [6] is heavily analytic in nature and is based on explicit sign computations for certain automorphisms of determinant line bundles in [5]. In contrast, our proof is completely topological; the proofs of the two statements from [7] and [3] cited in the proofs of Lemma 5.4 and Proposition 5.2, respectively, are also topological and take up only a few pages in total. This approach allows us to study the extendability of the canonical orientations of Corollary 5.10 across the codimension-one boundary strata of the moduli spaces on the topological level of real bundle pairs; see Section 6.

6 Extensions over compactifications

In this section, we study the extendability of the canonical isomorphisms and orientations of Section 5 across paths passing through one-nodal symmetric surfaces. Proposition 6.1 below implies that the line bundle (3.1) is orientable. This is a key technical result needed to extend the proof of Corollary 5.10 to the compactified setting of Theorem 1.3. We deduce this proposition from the proof of Proposition 5.9 and the statements of Corollary 6.16 and Lemma 6.17.

Proposition 6.1. *Let $g,l \in \mathbb{Z}^{\geqslant 0}$ be such that $g+l \geqslant 2$. The orientation on the restriction of the real line bundle (3.1) to $\mathbb{R}\mathcal{M}_{g,l}$ provided by Proposition 5.9 flips across the codimension-one boundary strata of types (E) and (H1) and extends across the codimension-one boundary strata of types (H2) and (H3).*

6.1 One-nodal symmetric surfaces

A one-nodal oriented surface Σ is a topological space obtained by identifying two distinct points of a closed oriented smooth surface $\widetilde{\Sigma}$, not necessarily connected. The surface $\widetilde{\Sigma}$ is called the normalization of Σ; it is unique up to a diffeomorphism preserving the two distinct points as a set. A one-nodal symmetric surface (Σ,σ) is a connected one-nodal surface Σ with an involution σ induced by an orientation-reversing involution $\widetilde{\sigma}$ on the normalization $\widetilde{\Sigma}$ of Σ. Throughout this section, we will denote the two distinguished points of $\widetilde{\Sigma}$ by x_1 and x_2 and their image in Σ, i.e. the node, by x_{12}. The four topological possibilities for the singular structure of (Σ,σ) are described by (E)-(H3) in Section 3.2. Note that

$$\widetilde{\sigma}(x_i) = \begin{cases} x_{3-i}, & \text{if } (\Sigma,\sigma) \text{ is of type (E);} \\ x_i, & \text{if } (\Sigma,\sigma) \text{ is of type (H).} \end{cases}$$

Let $\widetilde{\sigma}'(x_1)=x_2$ and $\widetilde{\sigma}'(x_2)=x_1$.

We begin by extending the main statements of Sections 5.1 and 5.2 to one-nodal symmetric surfaces. In particular, we observe that Proposition 5.2 extends to such surfaces. In [20], we show that Proposition 5.2 actually extends to all nodal symmetric surfaces.

Proposition 6.2. *The conclusion of Proposition 5.2 holds for one-nodal symmetric surfaces.*

Lemma 6.3. *The conclusion of Lemma 5.4 holds for one-nodal symmetric surfaces.*

Proof. Let $\widetilde{f} \in \mathcal{C}(\widetilde{\Sigma}, \widetilde{\sigma}; \mathrm{SL}_n\mathbb{C})$ be the function corresponding to $f \in \mathcal{C}(\Sigma, \sigma; \mathrm{SL}_n\mathbb{C})$. In particular, $\widetilde{f}(x_1) = \widetilde{f}(x_2)$.

Suppose (Σ, σ) is of type (E). Proceeding as in the proof of Lemma 5.4, choose Σ^b and U so that $x_1 \in \Sigma^b - U$, the cutting paths C_i so that $x_1 \notin C_i$, and the extensions of the homotopies of $\widetilde{f}$ from C_i to Σ^b so that they do not change $\widetilde{f}$ at x_1. Choose an embedded path γ in the disk D^2 in the last paragraph of the proof of Lemma 5.4 from x_1 to ∂D^2. Since $\widetilde{f}(x_1) \in \mathrm{SL}_n\mathbb{R}$ in this case, we can homotope $\widetilde{f}$ to Id over γ while keeping the values of $\widetilde{f}$ at x_1 and at the other endpoint in $\mathrm{SL}_n\mathbb{R}$ and at Id, respectively. Similarly to the second paragraph in the proof of Lemma 5.4, this homotopy extends over D^2 without changing $\widetilde{f}$ over ∂D^2 and thus descends to Σ^b. We then cut D^2 along γ into another disk and proceed as in the second half of the last paragraph in the proof of Lemma 5.4. The doubled homotopy then satisfies $\widetilde{f}_t(x_1) = \widetilde{f}_t(x_2)$ and so descends to Σ.

If (Σ, σ) is of type (H), then

$$\widetilde{f} \colon \bigcup_{|c_i|=0} (\partial \Sigma^b)_i \longrightarrow \mathrm{SL}_n\mathbb{R}$$

is homotopic to Id through maps $\widetilde{f}_t$ such that $\widetilde{f}_t(x_1) = \widetilde{f}_t(x_2)$. The remainder of the proof of Lemma 5.4 preserves this condition on the homotopy. $\square$

Corollary 6.4. *The conclusion of Corollary 5.5 holds for one-nodal symmetric surfaces.*

Proof. The first paragraph of the proof of Corollary 5.5 applies without any changes. The second paragraph applies with Lemma 5.4 replaced by Lemma 6.3. $\square$

Lemma 6.5. *Let $(V, \mathfrak{i})$ be a finite-dimensional complex vector space and $A, B : V \longrightarrow V$ be $\mathbb{C}$-antilinear isomorphisms such that $A^2, B^2 = \mathrm{Id}_V$. Then there exists a $\mathbb{C}$-linear isomorphism $\psi \colon V \longrightarrow V$ such that $\psi = A \circ \psi \circ B$. If*

$$\{\Lambda_{\mathbb{C}}^{\mathrm{top}} A\} \circ \{\Lambda_{\mathbb{C}}^{\mathrm{top}} B\} = \{\Lambda_{\mathbb{C}}^{\mathrm{top}} B\} \circ \{\Lambda_{\mathbb{C}}^{\mathrm{top}} A\} \colon \Lambda_{\mathbb{C}}^{\mathrm{top}} V \longrightarrow \Lambda_{\mathbb{C}}^{\mathrm{top}} V, \tag{6.1}$$

then ψ can be chosen so that $\Lambda_{\mathbb{C}}^{\mathrm{top}} \psi = \mathrm{Id}$.

Proof. Since $A^2, B^2 = \mathrm{Id}_V$, the isomorphisms A, B are diagonalizable with all eigenvalues ± 1. Since A, B are $\mathbb{C}$-antilinear, we can choose $\mathbb{C}$-bases $\{v_i\}$ and $\{w_i\}$ for V such that

$$A(v_i) = v_i, \quad A(\mathfrak{i}v_i) = -\mathfrak{i}v_i, \qquad B(w_i) = w_i, \quad B(\mathfrak{i}w_i) = -\mathfrak{i}w_i.$$

The $\mathbb{C}$-linear isomorphism $\psi \colon V \longrightarrow V$ defined by $\psi(w_i) = v_i$ then has the first desired property.

The automorphisms $\Lambda_{\mathbb{C}}^{\mathrm{top}} A$ and $\Lambda_{\mathbb{C}}^{\mathrm{top}} B$ are $\mathbb{C}$-antilinear and have one eigenvalue of $+1$ and one of -1. If (6.1) holds, the eigenspaces of $\Lambda_{\mathbb{C}}^{\mathrm{top}} A$ and $\Lambda_{\mathbb{C}}^{\mathrm{top}} B$ are the same and so

$$v_1 \wedge_{\mathbb{C}} \ldots \wedge_{\mathbb{C}} v_n = r \cdot w_1 \wedge_{\mathbb{C}} \ldots \wedge_{\mathbb{C}} w_n \in \Lambda_{\mathbb{C}}^{\mathrm{top}} V$$

for some $r \in \mathbb{R}^*$. Replacing w_1 by rw_1 in the previous paragraph, we obtain an isomorphism ψ that also satisfies the second property. $\square$

Proof of Proposition 6.2. Let $\widetilde{V},\widetilde{L}\longrightarrow\widetilde{\Sigma}$ be complex vector bundles and

$$\psi_1\colon \widetilde{V}\big|_{x_1}\longrightarrow \widetilde{V}\big|_{x_2} \qquad\text{and}\qquad \psi_2\colon \widetilde{L}\big|_{x_1}\longrightarrow \widetilde{L}\big|_{x_2}$$

be isomorphisms of complex vector spaces such that

$$V=\widetilde{V}/\sim,\quad v\sim\psi_1(v)\ \forall\,v\in\widetilde{V}\big|_{x_1}, \qquad\text{and}\qquad L=\widetilde{L}/\sim,\quad v\sim\psi_2(v)\ \forall\,v\in\widetilde{L}\big|_{x_1}.$$

Denote by $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ the lift of φ to $\widetilde{V}$ and the lift of $\widetilde{\phi}$ to $\widetilde{L}$, respectively. Define

$$\big(\widetilde{W},\widetilde{\varphi}_{12}\big)=\big(\widetilde{V}\oplus 2\widetilde{L}^*,\widetilde{\varphi}_1\oplus 2\widetilde{\varphi}_2^*\big), \qquad \psi_{12}=\psi_1\oplus 2(\psi_2^{-1})^*\colon \widetilde{W}\big|_{x_1}\longrightarrow \widetilde{W}\big|_{x_2}.$$

Thus, $(\widetilde{V},\widetilde{\varphi}_1)$ and $(\widetilde{L},\widetilde{\varphi}_2)$ are real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$ that descend to the real bundle pairs (V,φ) and $(L,\widetilde{\phi})$ over (Σ,σ). Furthermore,

$$\psi_{12}(v)=\begin{cases}\widetilde{\varphi}_{12}(\psi_{12}^{-1}(\widetilde{\varphi}_{12}(v))), & \text{if } (\Sigma,\sigma) \text{ is of type (E)};\\ \widetilde{\varphi}_{12}(\psi_{12}(\widetilde{\varphi}_{12}(v))), & \text{if } (\Sigma,\sigma) \text{ is of type (H)};\end{cases} \tag{6.2}$$

for all $v\in\widetilde{W}\big|_{x_1}$.

For any $f\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{GL}_{n+2}\mathbb{C})$, let

$$\widetilde{\Psi}_f\colon \big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big)\longrightarrow\big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big), \qquad \widetilde{\Psi}_f(z,v)=\big(z,f(z)v\big).$$

The choices (RO2) and (RO3) in Definition 5.1 for (Σ,σ) lift to $(\widetilde{\Sigma},\widetilde{\sigma})$. By Proposition 5.2, there thus exists an isomorphism

$$\widetilde{\Phi}\colon (\widetilde{W},\widetilde{\varphi}_{12})\longrightarrow\big(\widetilde{\Sigma}\times\mathbb{C}^{n+2},\widetilde{\sigma}\times\mathfrak{c}\big)$$

of real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$ that lies in the homotopy class determined by the lifted real orientation. By the proof of Proposition 5.2, $\widetilde{\Phi}$ can be chosen so that it induces the isomorphism in (5.6) over $(\widetilde{\Sigma},\widetilde{\sigma})$ determined by the lift of a given isomorphism in (5.2) over (Σ,σ). This implies that

$$\big\{\widetilde{\sigma}'\times\mathrm{Id}\big\}\circ\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{\Phi}\big\}=\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{\Phi}\big\}\circ\big\{\Lambda^{\mathrm{top}}_{\mathbb{C}}\psi_{12}\big\}\colon \Lambda^{\mathrm{top}}_{\mathbb{C}}\widetilde{W}|_{x_1}\longrightarrow\{x_2\}\times\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathbb{C}^{n+2}=\{x_2\}\times\mathbb{C}. \tag{6.3}$$

We show below that there exists $f\in\mathcal{C}(\Sigma,\sigma;\mathrm{SL}_{n+2}\mathbb{C})$ so that

$$\big\{\widetilde{\sigma}'\times\mathrm{Id}\big\}\circ\widetilde{\Psi}_f\circ\widetilde{\Phi}=\widetilde{\Psi}_f\circ\widetilde{\Phi}\circ\psi_{12}\colon \widetilde{W}|_{x_1}\longrightarrow\{x_2\}\times\mathbb{C}^{n+2}. \tag{6.4}$$

Thus, $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ descends to an isomorphism Ψ in (5.5) of real bundle pairs over (Σ,σ) that induces the isomorphism in (5.6) determined by a given isomorphism in (5.2). Furthermore, f can be chosen so that Ψ satisfies the spin structure requirement of Proposition 5.2. By Corollary 6.4, any two isomorphisms (5.5) satisfying the conditions at the end of Proposition 5.2 are homotopic.

Suppose (Σ,σ) is of type (E). By (6.2), the $\mathbb{C}$-antilinear isomorphisms

$$\mathrm{id}\times\mathfrak{c},\ \big\{\widetilde{\sigma}\times\mathfrak{c}\big\}\circ\widetilde{\Phi}\circ\psi_{12}\circ\widetilde{\Phi}^{-1}=\widetilde{\Phi}\circ\widetilde{\varphi}_{12}\circ\psi_{12}\circ\widetilde{\Phi}^{-1}\colon \{x_1\}\times\mathbb{C}^{n+2}\longrightarrow\{x_1\}\times\mathbb{C}^{n+2}$$

square to the identity. By (6.3), the top exterior powers of these automorphisms commute (both compositions are the identity). By Lemma 6.5, there thus exists $\psi\in\mathrm{SL}_{n+2}\mathbb{C}$ such that

$$\mathrm{id}\times\psi=\big\{\widetilde{\sigma}\times\mathfrak{c}\psi\mathfrak{c}\big\}\circ\widetilde{\Phi}\circ\psi_{12}\circ\widetilde{\Phi}^{-1}\colon \{x_1\}\times\mathbb{C}^{n+2}\longrightarrow\{x_1\}\times\mathbb{C}^{n+2}. \tag{6.5}$$

Since $\mathrm{SL}_{n+2}\mathbb{C}$ is connected, there exist $f\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{SL}_{n+2}\mathbb{C})$ and a neighborhood U of x_1 in $\widetilde{\Sigma}$ such that

$$f(z)=\begin{cases}\psi, & \text{if } z=x_1;\\ \mathrm{Id}, & \text{if } z\notin U\cup\widetilde{\sigma}(U);\end{cases}\qquad U\cap\widetilde{\sigma}(U)=\varnothing. \tag{6.6}$$

By (6.5) and (6.6), f satisfies (6.4). Since f restricts to the identity over $\widetilde{\Sigma}^{\widetilde{\sigma}}$, $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ induces the same orientation and spin structure over $\widetilde{\Sigma}^{\widetilde{\sigma}}-\{x_{12}\}$ as $\widetilde{\Phi}$. The orientation and spin conditions are automatically satisfied over x_{12}, since they are determined by the real part of the isomorphism (5.6).

If (Σ,σ) is of type (H), define $\psi\in\mathrm{GL}_{n+2}\mathbb{C}$ by

$$\mathrm{id}\times\psi=\big\{\widetilde{\sigma}'\times\mathrm{Id}\big\}\circ\widetilde{\Phi}\circ\psi_{12}\circ\widetilde{\Phi}^{-1}\colon\{x_1\}\times\mathbb{C}^{n+2}\longrightarrow\{x_1\}\times\mathbb{C}^{n+2}. \tag{6.7}$$

By (6.2) and (6.3), $\psi\circ\mathfrak{c}=\mathfrak{c}\circ\psi$ and $\det_{\mathbb{C}}\psi=1$, i.e. $\psi\in\mathrm{SL}_{n+2}\mathbb{R}$. If (Σ,σ) is of type (H2) or (H3), i.e. x_1 and x_2 lie on different topological components $\widetilde{\Sigma}_1^{\widetilde{\sigma}},\widetilde{\Sigma}_2^{\widetilde{\sigma}}$ of $\widetilde{\Sigma}^{\widetilde{\sigma}}$, let

$$\widetilde{\psi}\colon\widetilde{\Sigma}_1^{\widetilde{\sigma}}\longrightarrow\mathrm{SL}_{n+2}\mathbb{R} \tag{6.8}$$

be the constant function with value ψ. If (Σ,σ) is of type (H1), i.e. x_1 and x_2 lie on the same topological component $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ of $\widetilde{\Sigma}^{\widetilde{\sigma}}$, first choose (6.8) so that $\widetilde{\psi}(x_1)=\psi$ and $\widetilde{\psi}(x_2)=\mathrm{Id}$. Since $f=\widetilde{\psi}$ satisfies (6.4), $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ induces a trivialization of $V^{\varphi}\oplus 2(L^*)^{\widetilde{\phi}^*}$ over the image Σ_1^{σ} of $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ in Σ. This is also the case if $\widetilde{\psi}$ is replaced by $\widetilde{\psi}'\widetilde{\psi}$ for any $\widetilde{\psi}'$ as in (6.8) such that $\widetilde{\psi}'(x_1),\widetilde{\psi}'(x_2)=\mathrm{Id}$. Choose such $\widetilde{\psi}'$ so that the induced trivialization on each of the two loops in Σ_1^{σ} lies in the chosen spin structure; we then replace $\widetilde{\psi}$ with $\widetilde{\psi}'\widetilde{\psi}$. Returning to the general (H) case, choose $f\in\mathcal{C}(\widetilde{\Sigma},\widetilde{\sigma};\mathrm{SL}_{n+2}\mathbb{C})$ and a neighborhood U of $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ in $\widetilde{\Sigma}$ such that

$$f(z)=\begin{cases}\widetilde{\psi}, & \text{if } z\in\widetilde{\Sigma}_1^{\widetilde{\sigma}};\\ \mathrm{Id}, & \text{if } z\notin U;\end{cases}\qquad U\cap\big(\widetilde{\Sigma}^{\widetilde{\sigma}}-\widetilde{\Sigma}_1^{\widetilde{\sigma}}\big)=\varnothing; \tag{6.9}$$

this is possible by Lemma 5.3. By (6.7) and (6.9), $\widetilde{\Psi}$ satisfies (6.4). Since f restricts to the identity over $\widetilde{\Sigma}^{\widetilde{\sigma}}-\widetilde{\Sigma}_1^{\widetilde{\sigma}}$, $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ induces the same orientation and spin structure over $\widetilde{\Sigma}^{\widetilde{\sigma}}-\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ as $\widetilde{\Phi}$. If (Σ,σ) is of type (H2) or (H3), the latter is also the case over $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ because f is constant over $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$. If (Σ,σ) is of type (H1), the orientation and spin structure structure induced by $\widetilde{\Psi}_f\circ\widetilde{\Phi}$ over Σ_1^{σ} are those of the original real orientation by the choice of $\widetilde{\psi}$ above. □

Corollary 6.6. *The first conclusion of Corollary 5.7 holds for one-nodal symmetric surfaces.*

Proof. An orientation on the determinant line of a real CR-operator on a real bundle pair (V,φ) over a one-nodal symmetric surface (Σ,σ) is determined by

(1) an orientation on the determinant line of a real CR-operator on the corresponding real bundle pair $(\widetilde{V},\widetilde{\varphi})$ over $(\widetilde{\Sigma},\widetilde{\sigma})$ as in the proof of Proposition 6.2, and

(2) an orientation on the real vector space $V_{x_{12}}^{\varphi}$.

An isomorphism of real bundle pairs over (Σ,σ) as in (5.5) lifts to a similar isomorphism over $(\widetilde{\Sigma},\widetilde{\sigma})$ which respects all identifications on the lifted bundles. A real orientation on (V,φ) determines (2) and an isomorphism of real bundle pairs over (Σ,σ) as in (5.5); see Proposition 6.2. Thus, the claim follows from the proof of the first conclusion of Corollary 5.7. □

6.2 Smoothings of one-nodal symmetric surfaces

Let $\mathcal{C}\equiv(\Sigma,\sigma,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-))$ be a one-nodal marked symmetric Riemann surface and

$$(\pi:\mathcal{U}\longrightarrow\Delta,\widetilde{\mathfrak{c}}:\mathcal{U}\longrightarrow\mathcal{U},s_1:\Delta\longrightarrow\mathcal{U},\ldots,s_l:\Delta\longrightarrow\mathcal{U})$$

be a flat family of deformations of $\mathcal{C}$ as in Section 4.2 with $\Delta\subset\mathbb{C}$. Define

$$\Delta^*=\Delta-\{0\},\quad \Delta_{\mathbb{R}}=\Delta\cap\mathbb{R},\quad \Delta_{\mathbb{R}}^*=\Delta^*\cap\mathbb{R},\quad \Delta_{\mathbb{R}}^{\pm}=\Delta\cap\mathbb{R}^{\pm}.$$

Denote by $x_{12}\in\Sigma$ the node of Σ, by $\widetilde{\Sigma}\longrightarrow\Sigma$ its normalization, and by $\Sigma^*\equiv\Sigma-\{x_{12}\}$ its smooth locus.

A neighborhood of x_{12} in $\mathcal{U}$ is isomorphic to

$$\mathcal{U}_0\equiv\big\{(t,z_1,z_2)\in\Delta\times\mathbb{C}^2:\ |z_1|,|z_2|<1,\ z_1z_2=t\big\}.$$

As fibrations over Δ,

$$\mathcal{U}\approx\big(\mathcal{U}_0\sqcup\mathcal{U}'\big)/\sim,\qquad (t,z_1,z_2)\sim\begin{cases}(t,z_1), & \text{if } |z_1|>|z_2|;\\ (t,z_2), & \text{if } |z_1|<|z_2|;\end{cases}\tag{6.10}$$

for some family $\mathcal{U}'$ of deformations of Σ^* over Δ, a choice of coordinates z_i on $\widetilde{\Sigma}$ centered at x_i, and their extensions to $\mathcal{U}$. The local coordinates z_1,z_2 and the family $\mathcal{U}'$ in (6.10) can be chosen so that $\mathcal{U}'$ is preserved by $\widetilde{\mathfrak{c}}$ and the identification in (6.10) intertwines the involution

$$\mathcal{U}_0\longrightarrow\mathcal{U}_0,\qquad (t,z_1,z_2)\longrightarrow\big(\bar{t},\overline{z_2},\overline{z_1}\big)\ \text{ or }\ (t,z_1,z_2)\longrightarrow\big(\bar{t},\overline{z_1},\overline{z_2}\big),\tag{6.11}$$

depending on whether (Σ,x_{12},σ) is of type (E) or (H), with the involution $\widetilde{\mathfrak{c}}$ on $\mathcal{U}$. In particular, $\mathcal{U}$ retracts onto Σ_0 respecting the involution $\widetilde{\mathfrak{c}}$.

Suppose $\pi:\mathcal{U}\longrightarrow\Delta$ and $\widetilde{\mathfrak{c}}$ are as above, $(V,\varphi)\longrightarrow(\mathcal{U},\widetilde{\mathfrak{c}})$ is a real bundle pair, and ∇ and A are a connection and a 0-th order deformation term on (V,φ) as in Section 4.3. The restriction of ∇ and A to $(V,\varphi)|_{(\Sigma_t,\sigma_t)}$ with $t\in\Delta_{\mathbb{R}}$ determines a real CR-operator D_t. The determinant lines of these operators form a line bundle

$$\det D_{(V,\varphi)}\longrightarrow\Delta_{\mathbb{R}};\tag{6.12}$$

see Section 4.3 and Appendix A. We denote by $\det\bar{\partial}_{\mathbb{C}}\longrightarrow\Delta_{\mathbb{R}}$ the determinant line bundle associated with the standard holomorphic structure on $(\mathcal{U}\times\mathbb{C},\widetilde{\mathfrak{c}}\times\mathfrak{c})$.

Corollary 6.7. *Let $(\pi,\widetilde{\mathfrak{c}})$, (V,φ), and (∇,A) be as above. Then a real orientation on (V,φ) as in Definition 5.1 induces an orientation on the line bundle*

$$\widehat{\det}\,D_{(V,\varphi)}\equiv\big(\det D_{(V,\varphi)}\big)\otimes(\det\bar{\partial}_{\mathbb{C}})^{\otimes n}\longrightarrow\Delta_{\mathbb{R}},\tag{6.13}$$

where $n=\mathrm{rk}_{\mathbb{C}}V$. The restriction of this orientation to the fiber over each $t\in\Delta_{\mathbb{R}}^*$ is the orientation on $\widehat{\det}\,D_t$ induced by the restriction of the real orientation to $(V,\varphi)|_{(\Sigma_t,\sigma_t)}$ as in Corollary 5.7.

Proof. By Proposition 6.2, the restriction of the real orientation to $(V,\varphi)|_{(\Sigma_0,\sigma_0)}$ determines a homotopy class of isomorphisms Ψ of real bundle pairs as in (5.5). Since $\mathcal{U}$ retracts onto Σ_0 respecting the involution $\tilde{\mathfrak{c}}$, every isomorphism Ψ_0 over (Σ_0,σ_0) extends to an isomorphism

$$\Psi\colon \big(V\oplus 2L^*, \varphi\oplus 2\widetilde{\phi}^*\big) \approx \big(\mathcal{U}\times\mathbb{C}^{n+2}, \tilde{\mathfrak{c}}\times\mathfrak{c}\big) \tag{6.14}$$

of real bundle pairs over $(\mathcal{U},\tilde{\mathfrak{c}})$. Since an isomorphism Ψ_0 in the homotopy class determined by the restriction of the real orientation to $(V,\varphi)|_{(\Sigma_0,\sigma_0)}$ satisfies the spin structure and $\Lambda_{\mathbb{C}}^{\mathrm{top}}$ conditions at the end of Proposition 5.2, the restriction Ψ_t of (6.14) to $(V,\varphi)|_{(\Sigma_t,\sigma_t)}$ also satisfies these conditions. The restriction of the orientation of the line bundle (6.13) induced by Ψ to the fiber over each $t\in\Delta_{\mathbb{R}}^*$ is the orientation induced by Ψ_t. The latter is the orientation induced by the restriction of the real orientations to $(V,\varphi)|_{(\Sigma_t,\sigma_t)}$. $\square$

Thus, the real line bundle (5.30) extends across the (codimension-one) boundary strata of the moduli spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ and so does its orientation induced by a real orientation on (X,ϕ). The other factor in orienting the line bundle (1.5) over the uncompactified space $\mathfrak{M}_{g,l}(X,B;J)^{\phi}$ is the canonical orientation of the line bundle (5.19). The next lemma makes it possible to extend the orientations induced by the isomorphisms (5.22) used in orienting (5.19) to (but not *across*) the boundary strata.

Let $\widetilde{\Sigma}$ be a smooth Riemann surface and $x\in\widetilde{\Sigma}$. A holomorphic vector field ξ on a neighborhood of x in $\widetilde{\Sigma}$ with $\xi(x)=0$ determines an element

$$\nabla\xi\big|_x \in T_x^*\widetilde{\Sigma}\otimes_{\mathbb{C}} T_x\widetilde{\Sigma} = \mathbb{C}\,.$$

Similarly, a meromorphic one-form η on a neighborhood of x in $\widetilde{\Sigma}$ has a well-defined residue at x, which we denote by $\mathfrak{R}_x\eta$. For a holomorphic line bundle $L\longrightarrow\widetilde{\Sigma}$, we denote by $\Omega(L)$ the sheaf of holomorphic sections of L.

Lemma 6.8. *Let $(\pi\colon\mathcal{U}\longrightarrow\Delta,\tilde{\mathfrak{c}})$ be a flat family of deformations of a one-nodal symmetric Riemann surface (Σ,σ) with $\Delta\subset\mathbb{C}$ and $x_1,x_2\in\widetilde{\Sigma}$ be the preimages of the node $x_{12}\in\Sigma$ in its normalization. There exist holomorphic line bundles $\mathcal{T},\widehat{\mathcal{T}}\longrightarrow\mathcal{U}$ with involutions $\varphi,\widehat{\varphi}$ lifting $\tilde{\mathfrak{c}}$ such that*

$$\begin{gathered}
(\mathcal{T},\varphi)\big|_{\Sigma_t} = \big(T\Sigma_t, \mathrm{d}\tilde{\mathfrak{c}}|_{T\Sigma_t}\big), \quad (\widehat{\mathcal{T}},\widehat{\varphi})\big|_{\Sigma_t} = \big(T^*\Sigma_t, (\mathrm{d}\tilde{\mathfrak{c}}|_{T\Sigma_t})^*\big) \qquad \forall\ t\in\Delta^*,\\
\Omega\big(\mathcal{T}|_{\Sigma_0}\big) = \big\{\xi\in\Omega\big(T\widetilde{\Sigma}(-x_1-x_2)\big)\colon\ \nabla\xi|_{x_1}+\nabla\xi|_{x_2}=0\big\},\\
\Omega\big(\widehat{\mathcal{T}}|_{\Sigma_0}\big) = \big\{\eta\in\Omega\big(T^*\widetilde{\Sigma}(x_1+x_2)\big)\colon\ \mathfrak{R}_{x_1}\eta+\mathfrak{R}_{x_2}\eta=0\big\}.
\end{gathered}$$

Furthermore, $(\widehat{\mathcal{T}},\widehat{\varphi})\approx(\mathcal{T},\varphi)^*$.

Proof. We continue with the notation as in (6.10) and (6.11). Denote by $T^{\mathrm{vrt}}\mathcal{U}'\longrightarrow\mathcal{U}'$ the vertical tangent bundle. Let

$$\mathcal{T} = \big(\mathcal{U}_0\times\mathbb{C}\sqcup T^{\mathrm{vrt}}\mathcal{U}'\big)/\!\sim, \qquad \widehat{\mathcal{T}} = \big(\mathcal{U}_0\times\mathbb{C}\sqcup (T^{\mathrm{vrt}}\mathcal{U}')^*\big)/\!\sim,$$

$$(t,z_1,z_2,c)\sim\begin{cases} c\,z_1\frac{\partial}{\partial z_1}\big|_{(t,z_1)}, & \text{if } |z_1|>|z_2|;\\ -c\,z_2\frac{\partial}{\partial z_2}\big|_{(t,z_2)}, & \text{if } |z_1|<|z_2|;\end{cases} \qquad (t,z_1,z_2,c)\sim\begin{cases} c\,\frac{\mathrm{d}_{(t,z_1)}z_1}{z_1}, & \text{if } |z_1|>|z_2|;\\ -c\,\frac{\mathrm{d}_{(t,z_2)}z_2}{z_2}, & \text{if } |z_1|<|z_2|.\end{cases}$$

Under the identifications (6.10), the vector field and one-form on a neighborhood of the node in $\mathcal{U}$ associated with $(t,z_1,z_2,c)\in\mathcal{U}_0\times\mathbb{C}$ correspond to the vector field and one-form on $\mathcal{U}_0$ given by

$$c\bigg(z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}\bigg)\qquad\text{and}\qquad c\,\frac{\mathrm{d}z_1|_{\Sigma_t}}{z_1}=-c\,\frac{\mathrm{d}z_2|_{\Sigma_t}}{z_2},$$

respectively (the above equality of one-forms holds for $t\neq 0$). Thus, $\mathcal{T}$ and $\widehat{\mathcal{T}}$ have the desired restriction properties. Since the map

$$\mathcal{T}\otimes_{\mathbb{C}}\widehat{\mathcal{T}}\longrightarrow\mathcal{U}\times\mathbb{C},\quad \begin{array}{l}\big[t,z_1,z_2,c_1\big]\otimes\big[t,z_1,z_2,c_2\big]\longrightarrow\big([t,z_1,z_2],c_1c_2\big),\quad (t,z_1,z_2)\in\mathcal{U}_0,\ c_1,c_2\in\mathbb{C},\\ [v]\otimes[\alpha]\longrightarrow\alpha(v)\quad\forall\,v\in T_{z_i}\Sigma_t,\ \alpha\in T^*_{z_i}\Sigma_t,\ (t,z_i)\in\mathcal{U}',\end{array}$$

is a well-defined isomorphism of holomorphic line bundles, $\widehat{\mathcal{T}}\approx\mathcal{T}^*$.

The identifications in the construction of $\mathcal{T}$ and $\widehat{\mathcal{T}}$ above intertwine the trivial lift of (6.11) to a conjugation on $\mathcal{U}_0\times\mathbb{C}$ with the conjugations on $T^{\mathrm{vrt}}\mathcal{U}'$ and $(T^{\mathrm{vrt}}\mathcal{U}')^*$ induced by $\mathrm{d}\tilde{\mathfrak{c}}$. Thus, they induce conjugations φ and $\hat{\varphi}$ on $\mathcal{T}$ and $\widehat{\mathcal{T}}$. The above trivialization of $\mathcal{T}\otimes_{\mathbb{C}}\widehat{\mathcal{T}}$ intertwines the resulting conjugation on the domain with the conjugation $\tilde{\mathfrak{c}}\times\mathfrak{c}$ on $\mathcal{U}\times\mathbb{C}$. Thus, $(\widehat{\mathcal{T}},\hat{\varphi})$ and $(\mathcal{T},\varphi)^*$ are isomorphic as real bundle pairs over $(\mathcal{U},\tilde{\mathfrak{c}})$. $\square$

Lemma 6.9 (Dolbeault Isomorphism)**.** *Suppose (Σ,σ) and $(\pi:\mathcal{U}\longrightarrow\Delta,\tilde{\mathfrak{c}})$ are as in Lemma 6.8 and $(L,\tilde{\phi})\longrightarrow(\mathcal{U},\tilde{\mathfrak{c}})$ is a holomorphic line bundle so that $\deg L|_\Sigma<0$ and $\deg L|_{\Sigma'}\leqslant 0$ for each irreducible component $\Sigma'\subset\Sigma$. The families of vector spaces $H^1_{\bar\partial}(\Sigma_t;L)$ and $\check{H}^1(\Sigma_t;L)$ then form vector bundles $R^1_{\bar\partial}\pi_*L$ and $\check{R}^1\pi_*L$ over Δ with conjugations lifting $\mathfrak{c}$ which are canonically isomorphic as real bundle pairs over $(\Delta,\mathfrak{c})$.*

Proof. The assumptions on L ensure that $H^0_{\bar\partial}(\Sigma_t;L)=0$ for all $t\in\Delta$. By the Dolbeault Theorem [22, p151], this implies that $\check{H}^0(\Sigma_t;L)=0$ for all $t\in\Delta$. Since $H^0_{\bar\partial}(\Sigma_t;L)=0$ for all $t\in\Delta$, the vector spaces $H^1_{\bar\partial}(\Sigma_t;L)$ naturally form a vector bundle $R^1_{\bar\partial}\pi_*L$ over Δ. By the second statement, the sheaf $R^1\pi_*L$ is locally free over Δ and thus corresponds to a vector bundle $\check{R}^1\pi_*L$ over Δ. The involution $\tilde{\mathfrak{c}}$ and conjugation $\tilde{\phi}$ induce conjugations on the two bundles. The Dolbeault Isomorphism provides an isomorphism between the two resulting real bundle pairs over $(\Delta,\mathfrak{c})$. $\square$

Lemma 6.10 (Serre Duality)**.** *Suppose (Σ,σ), $(\pi:\mathcal{U}\longrightarrow\Delta,\tilde{\mathfrak{c}})$, and $(\widehat{\mathcal{T}},\hat{\varphi})$ are as in Lemma 6.8 and $(L,\tilde{\phi})\longrightarrow(\mathcal{U},\tilde{\mathfrak{c}})$ is a holomorphic line bundle so that $\deg L|_\Sigma>2g_a(\Sigma)-2$ and $\deg L|_{\Sigma'}\geqslant 2g_a(\Sigma')-2$ for each irreducible component $\Sigma'\subset\Sigma$. The family of vector spaces $H^0_{\bar\partial}(\Sigma_t;L)$ then forms a vector bundle $R^0_{\bar\partial}\pi_*L$ over Δ with a conjugation lifting $\mathfrak{c}$ and there is a canonical isomorphism*

$$R^1_{\bar\partial}\pi_*\big(L^*\otimes\widehat{\mathcal{T}}\big)\approx\big(R^0_{\bar\partial}\pi_*L\big)^*\tag{6.15}$$

of real bundle pairs over $(\Delta,\mathfrak{c})$.

Proof. The left-hand side of (6.15) is a vector bundle by Lemma 6.9. The assumptions on L ensure that $H^1_{\bar\partial}(\Sigma_t;L)=0$ for all $t\in\Delta$. Thus, the vector spaces $H^0_{\bar\partial}(\Sigma_t;L)$ with $t\in\Delta$ naturally form a vector bundle $R^0_{\bar\partial}\pi_*L$ over Δ. The involution $\tilde{\mathfrak{c}}$ and conjugation $\tilde{\phi}$ induce a conjugation on the right-hand side of (6.15). The Serre Duality provides an isomorphism between the two bundles in (6.15). Its composition with the multiplication by $\mathfrak{i}$ is an isomorphism between the two bundles in (6.15) as real bundle pairs over $(\Delta,\mathfrak{c})$. $\square$

Remark 6.11. The justification of Dolbeault Isomorphism Theorem in the case of Lemma 6.9 consists of applying the exact sequence of sheaves at the bottom of [22, p150] with $p,q=0$ and $E=L$. As the standard $\bar\partial$-operator on a wedge of two disks is surjective, this sequence is indeed exact over the central fiber $\Sigma_0=\Sigma$ (the exactness is established in [22] over complex manifolds). The Serre Duality for CR-operators over nodal Riemann surfaces appears in [48, Lemma 2.3] and endows the total spaces of the left-hand side in (6.15) and of the bundle $R^1_{\bar\partial}\pi_*L$ in Lemma 6.9 with a topology via the fiberwise SD isomorphisms. The Serre Duality appears on the level of Čech cohomology in the standard algebro-geometric perspective; see [1, p98]. This viewpoint would establish Corollary 6.12 below by applying the Serre Duality first and the Dolbeault Isomorphism second.

Let $(\pi,\widetilde{\mathfrak{c}},s_1,\ldots,s_l)$ be a smoothing of a one-nodal marked symmetric Riemann surface

$$\mathcal{C}\equiv\big(\Sigma,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-)\big), \tag{6.16}$$

$\mathcal{T},\widehat{\mathcal{T}}\longrightarrow\mathcal{U}$ be the holomorphic line bundles with involutions $\varphi,\widehat\varphi$ as in Lemma 6.8, and

$$\mathcal{T}\mathcal{C}=\mathcal{T}\big(-s_1-\widetilde{\mathfrak{c}}\circ s_1-\ldots-s_l-\widetilde{\mathfrak{c}}\circ s_l\big),\qquad \widehat{\mathcal{T}}\mathcal{C}=\widehat{\mathcal{T}}\big(s_1+\widetilde{\mathfrak{c}}\circ s_1+\ldots+s_l+\widetilde{\mathfrak{c}}\circ s_l\big).$$

By the last statement of Lemma 6.8, $\mathcal{T}\mathcal{C}^*=\widehat{\mathcal{T}}\mathcal{C}$.

Corollary 6.12. *If the marked curve (6.16) is stable, the orientation on the restriction of the real line bundle*

$$\Lambda^{\mathrm{top}}_{\mathbb{R}}\big((\check{R}^1\pi_*\mathcal{T}\mathcal{C})^\sigma\big)\otimes\Lambda^{\mathrm{top}}_{\mathbb{R}}\big((R^0_{\bar\partial}\pi_*(\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}}))^\sigma\big)\longrightarrow\Delta_{\mathbb{R}} \tag{6.17}$$

to $\Delta^*_{\mathbb{R}}$ induced by the Dolbeault and SD isomorphisms as in the proof of Proposition 5.9 extends across $t=0$.

Proof. By Lemma 6.9 with $L=\mathcal{T}\mathcal{C}$ and Lemma 6.10 with $L=\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}}$, there are canonical isomorphisms of vector bundles

$$\check{R}^1\pi_*\mathcal{T}\mathcal{C}\approx R^1_{\bar\partial}\pi_*\mathcal{T}\mathcal{C}=R^1_{\bar\partial}\pi_*\big((\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}})^*\otimes\widehat{\mathcal{T}}\big)\approx\big(R^0_{\bar\partial}\pi_*(\widehat{\mathcal{T}}\mathcal{C}\otimes\widehat{\mathcal{T}})\big)^*$$

over Δ which restrict to the Dolbeault and SD isomorphisms over each point. Since they commute with the involutions on the vector bundles, these isomorphisms induce an orientation on the real line bundle (6.17) that restricts to the orientation on each fiber induced by the real parts of the Dolbeault and SD isomorphisms. □

6.3 The orientability of the real Deligne-Mumford space

We now study the extendability of the canonical orientations of the line bundles appearing in the proof of Proposition 5.9 and establish Proposition 6.1. The two main ingredients in this proof are Lemmas 6.14 and 6.17 below. The next lemma summarizes the fundamental difference between the two pairs of cases in Proposition 6.1.

Lemma 6.13. *Let (Σ,x_{12},σ), $(\pi,\widetilde{\mathfrak{c}})$, and $(\mathcal{T},\varphi)$ be as in Lemma 6.8. The restriction of the real line bundle $\mathcal{T}^\varphi\longrightarrow\Sigma^\sigma$ to the singular topological component $\Sigma_1^\sigma\subset\Sigma^\sigma$ is orientable if the one-nodal symmetric surface (Σ,x_{12},σ) is of type (E) or (H1) and is not orientable if (Σ,x_{12},σ) is of type (H2) or (H3).*

Proof. If (Σ, x_{12}, σ) is of type (E), Σ_1^σ consists of the node x_{12} and there is nothing to prove. Otherwise, a local section of $\mathcal{T}^\varphi$ near x_{12} is given by $x\frac{\partial}{\partial x}$ along the x-axis and $-y\frac{\partial}{\partial y}$ along the y-axis. It points away from the origin along the x-axis and towards along the y-axis. The claims in the (H1) and (H2)/(H3) are thus immediate from the middle diagrams in Figures 2 and 1, respectively. $\square$

Suppose (Σ, x_{12}, σ) is of type (E) or (H1). By the first part of the proof of Corollary 5.6, the restriction of the real bundle pair

$$\big(\widehat{\mathcal{T}}^{\otimes 2}\oplus 2\mathcal{T}, \widehat{\varphi}^{\otimes 2}\oplus 2\varphi\big) \longrightarrow (\mathcal{U}, \widetilde{\mathfrak{c}}) \tag{6.18}$$

to the central fiber (Σ, σ) thus has a canonical real orientation. It extends to a real orientation on (6.18) which restricts to the canonical real orientation over each fiber (Σ_t, σ_t) with $t\in\Delta_{\mathbb{R}}^*$.

Suppose (Σ, x_{12}, σ) is of type (H2) or (H3). The singular component Σ_1^σ of Σ^σ consists of two copies of S^1 with a point x_1 on the first copy identified with a point x_2 on the second copy. By Corollary 5.6, there are then *four* natural real orientations on the restriction of (6.18) to (Σ, σ). They correspond to the two orientations of each of the two irreducible components of Σ_1^σ. Each of the four real orientations extends to a real orientation on the real bundle pair (6.18) over $(\mathcal{U}, \widetilde{\mathfrak{c}})$.

Lemma 6.14. *Let $\mathcal{C}$, $(\pi, \widetilde{\mathfrak{c}})$, and $\mathcal{T}, \widehat{\mathcal{T}} \longrightarrow \mathcal{U}$ be as in Lemma 6.8 with (Σ, x_{12}, σ) of type (H2) or (H3). For each of the four natural real orientations on the restriction of (6.18) to (Σ, σ), there exists $\varepsilon\in\{\pm 1\}$ such that the restriction over (Σ_t, σ_t) of the extension of this real orientation over $(\mathcal{U}, \widetilde{\mathfrak{c}})$ is the canonical real orientation if $\varepsilon t\in\Delta_{\mathbb{R}}^+$ and differs from the canonical real orientation by the spin structure over precisely one component of $\Sigma_t^{\sigma_t}$ if $\varepsilon t\in\Delta_{\mathbb{R}}^-$.*

Proof. For $t\in\Delta_{\mathbb{R}}^*$, the topological component $\Sigma_{t;1}^{\sigma_t}$ of $\Sigma_t^{\sigma_t}$ corresponding to Σ_1^σ is obtained as follows. Cut the first copy of S^1 at x_1 into a closed interval S_1^1 with endpoints 1^- and 1^+; cut the second copy of S^1 at x_2 into a closed interval S_2^1 with endpoints 2^- and 2^+. For $t\in\Delta_{\mathbb{R}}^+$, $\Sigma_{t;1}^{\sigma_t}\approx S^1$ is formed from S_1^1 and S_2^1 by identifying either 1^- with 2^+ and 1^+ with 2^- or 1^- with 2^- and 1^+ with 2^+. For $t\in\Delta_{\mathbb{R}}^-$, $\Sigma_{t;1}^{\sigma_t}$ is formed by the other identification. Thus, the transition from $\Sigma_{t;1}^{\sigma_t}$ with $t\in\Delta_{\mathbb{R}}^-$ to $\Sigma_{t;1}^{\sigma_t}$ with $t\in\Delta_{\mathbb{R}}^+$ is equivalent to flipping the second copy of S^1 around x_2 and another point. This flips the orientation on S_2^1. By the second part of the proof of Corollary 5.6, this is equivalent to flipping the spin structure on the restriction of the real part of (6.18) to half of $\Sigma_t^{\sigma_t}\approx S^1$ with $t\in\Delta_{\mathbb{R}}^*$. Thus, precisely one of the two spin structures (either before or after the flip) on the restriction of the real part of (6.18) to $\Sigma_t^{\sigma_t}$ is the canonical one. $\square$

Remark 6.15. Suppose both copies of S^1 in the proof of Lemma 6.14 are oriented from the $-$ to $+$ end. These orientations determine spin structures on the restrictions of the real part of (6.18) to the two irreducible components of Σ_1^σ. The spin structure over $\Sigma_t^{\sigma_t}$ is then the canonical one if $\Sigma_t^{\sigma_t}$ is obtained by gluing 1^- with 2^- and 1^+ with 2^+. This gluing untwists back a half-spin of $\mathbb{R}$ in $\mathbb{R}^2$ over the first circle, instead of completing it to a full twist.

Corollary 6.16. *Let (Σ, σ), $(\widetilde{\Sigma}, \widetilde{\sigma})$, $(\pi, \widetilde{\mathfrak{c}})$, and $\mathcal{T}, \widehat{\mathcal{T}} \longrightarrow \mathcal{U}$ be as in Lemma 6.8. The orientation on the restriction of the real line bundle*

$$\big(\det\bar\partial_{(\widehat{\mathcal{T}},\widehat{\varphi})^{\otimes 2}}\big)\otimes\big(\det\bar\partial_{\mathbb{C}}\big) \longrightarrow \Delta_{\mathbb{R}} \tag{6.19}$$

to $\Delta_{\mathbb{R}}^*$ determined by the canonical isomorphisms of Corollary 5.6 extends across $t=0$ if (Σ, x_{12}, σ) is of type (E) or (H1) and flips if (Σ, x_{12}, σ) is of type (H2) or (H3).

Proof. Since $\mathcal{U}$ retracts onto Σ respecting the involution $\widetilde{\mathfrak{c}}$, a real orientation on the restriction of the real bundle pair (6.18) to the central fiber (Σ,σ) extends to a real orientation on (6.18). By Corollary 6.7, the former induces an orientation on the real line bundle (6.19) over $\Delta_{\mathbb{R}}$. The restriction of this orientation to the fiber over each $t\in\Delta_{\mathbb{R}}^*$ is the orientation induced by the restriction of the extended real orientation to the fiber of (6.18) as in Corollary 5.7.

Suppose (Σ,x_{12},σ) is of type (E) or (H1). The canonical real orientation on (6.18) over (Σ,σ) then induces the canonical real orientation on the restriction of (6.18) over (Σ_t,σ_t) with $t\in\Delta_{\mathbb{R}}^*$. Thus, the orientation on (6.19) induced by the canonical real orientation on (6.18) over (Σ,σ) restricts to the canonical orientation over $t\in\Delta_{\mathbb{R}}^*$. This establishes the claim for types (E) and (H1).

Suppose (Σ,x_{12},σ) is of type (H2) or (H3). Fix one of the four natural real orientations on (6.18) over (Σ,σ) and let $\varepsilon\in\{\pm1\}$ be as in Lemma 6.14. Since this real orientation induces the canonical real orientation on (6.18) over (Σ,σ) if $\varepsilon t\in\Delta_{\mathbb{R}}^+$, the orientation on (6.19) induced by the former restricts to the canonical orientation if $\varepsilon t\in\Delta_{\mathbb{R}}^+$. Since the chosen real orientation on (6.18) over (Σ,σ) induces an orientation on (6.18) differing from the canonical one by the spin structure over precisely one component of $\Sigma_t^{\sigma_t}$ if $\varepsilon t\in\Delta_{\mathbb{R}}^-$, the orientation on (6.19) induced by the former restricts to the opposite of the canonical orientation if $\varepsilon t\in\Delta_{\mathbb{R}}^-$; see Corollary 5.7. This establishes the claim for types (H2) and (H3). $\square$

Lemma 6.17. *Suppose $g,l\in\mathbb{Z}^{\geqslant0}$ with $g+l\geqslant2$ and (Σ,x_{12},σ), $\mathcal{C}$, and $(\pi,\widetilde{\mathfrak{c}},s_1,\ldots,s_l)$ are as in (6.16) with $\mathcal{U}|_{\Delta_{\mathbb{R}}}\longrightarrow\Delta_{\mathbb{R}}$ embedded inside of the universal curve fibration over $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The orientation on the restriction of the real line bundle*

$$\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}(T\mathbb{R}\overline{\mathcal{M}}_{g,l})\big)^*\otimes\Lambda_{\mathbb{R}}^{\mathrm{top}}\big((\check{R}^1\pi_*\mathcal{T}\mathcal{C})^\sigma\big)\longrightarrow\Delta_{\mathbb{R}} \tag{6.20}$$

to $\Delta_{\mathbb{R}}^*$ induced by the KS isomorphism as in (5.22) flips across $t=0$.

Proof. Let $x_1,x_2\in\widetilde{\Sigma}$ be the preimages of the node $x_{12}\in\Sigma$ as before and

$$T\widetilde{\mathcal{C}}=T\widetilde{\Sigma}\big(-z_1^+-z_1^--\ldots-z_l^+-z_l^--x_1-x_2\big).$$

Denote by $\mathcal{N}_{g,l}\subset\overline{\mathcal{M}}_{g,l}$ and $\mathbb{R}\mathcal{N}_{g,l}\subset\mathbb{R}\overline{\mathcal{M}}_{g,l}$ the one-node strata, by $L^{\mathbb{R}}\longrightarrow\mathbb{R}\mathcal{N}_{g,l}$ the normal bundle of $\mathbb{R}\mathcal{N}_{g,l}$ in $\mathbb{R}\overline{\mathcal{M}}_{g,l}$, and by $\mathcal{T}\widetilde{\mathcal{C}}\longrightarrow\widetilde{\mathcal{U}}_{g-2,l+2}$ the twisted down vertical tangent bundle over the universal curve $\pi\colon\widetilde{\mathcal{U}}_{g-2,l+2}\longrightarrow\mathcal{N}_{g,l}$. Let $\mathbb{C}_{x_{12}}\longrightarrow\Sigma$ be the skyscraper sheaf over x_{12}.

The short exact sequence of sheaves

$$0\longrightarrow\mathcal{O}(\mathcal{T}\mathcal{C}|_\Sigma)\longrightarrow\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\longrightarrow\mathbb{C}_{x_{12}}\longrightarrow0 \tag{6.21}$$

induces an exact sequence

$$0\longrightarrow\mathbb{C}\longrightarrow\check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{T}\mathcal{C}|_\Sigma)\big)\longrightarrow\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)\longrightarrow0$$

of complex vector spaces. Its real part is a short exact sequence

$$0\longrightarrow\mathbb{R}\longrightarrow\check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{T}\mathcal{C}|_\Sigma)\big)^\sigma\longrightarrow\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^\sigma\longrightarrow0 \tag{6.22}$$

of real vector spaces. By the definition of $L^{\mathbb{R}}$, there is also a natural short exact sequence

$$0\longrightarrow T_{\mathcal{C}}\mathbb{R}\mathcal{N}_{g,l}\longrightarrow T_{\mathcal{C}}\mathbb{R}\overline{\mathcal{M}}_{g,l}\longrightarrow L^{\mathbb{R}}|_{\mathcal{C}}\longrightarrow0 \tag{6.23}$$

of real vector spaces.

By (6.22) and (6.23), there is a canonical isomorphism

$$\begin{aligned}\Lambda_{\mathbb{R}}^{\mathrm{top}}(T_{\mathcal{C}}\mathbb{R}\mathcal{N}_{g,l})\otimes\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(\check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^{\sigma}\big)&\\ \approx\Big(\Lambda_{\mathbb{R}}^{\mathrm{top}}(T_{\mathcal{C}}\mathbb{R}\overline{\mathcal{M}}_{g,l})\otimes\Lambda_{\mathbb{R}}^{\mathrm{top}}\big(\check{H}^1\big(\Sigma;\mathcal{O}(\mathcal{T}\mathcal{C}|_{\Sigma})\big)^{\sigma}\big)\Big)\otimes L^{\mathbb{R}}\otimes\mathbb{R}\,.&\end{aligned}\tag{6.24}$$

The complex vector bundles

$$T\mathcal{N}_{g,l},\check{R}^1\pi_*\big(\mathcal{T}\widetilde{\mathcal{C}}\big)\longrightarrow\mathcal{N}_{g,l}$$

extend over a neighborhood of $\mathcal{N}_{g,l}$ in $\overline{\mathcal{M}}_{g,l}$ as a subbundle of $T\overline{\mathcal{M}}_{g,l}$ and a quotient bundle of $\check{R}^1\pi_*\mathcal{T}\mathcal{C}$. The KS map induces an isomorphism between these two extensions. Over a neighborhood of $\mathcal{C}$, these extensions can be chosen to be σ-invariant. We then obtain a diagram

$$\begin{array}{ccccc}T_{\mathcal{C}}\mathbb{R}\mathcal{N}_{g,l}&\longrightarrow&T_{\mathcal{C}_t}\mathbb{R}\overline{\mathcal{M}}_{g,l}&\longrightarrow&L^{\mathbb{R}}|_{\mathcal{C}}\\ {\scriptstyle\mathrm{KS}}\big\downarrow{\scriptstyle\approx}&&{\scriptstyle\mathrm{KS}}\big\downarrow{\scriptstyle\approx}&&{\scriptstyle\mathrm{KS}}\big\downarrow{\scriptstyle\approx}\\ \check{H}^1\big(\widetilde{\Sigma};\mathcal{O}\big(T\widetilde{\mathcal{C}}\big)\big)^{\sigma}&\longleftarrow&\check{H}^1\big(\Sigma_t;\mathcal{O}(\mathcal{T}\mathcal{C}|_{\Sigma_t})\big)^{\sigma_t}&\longleftarrow&\mathbb{R}\end{array}$$

of vector space homomorphisms which commutes up to homotopy of the isomorphisms given by the vertical arrows. The KS map for $(\widetilde{\Sigma},x_1,x_2)$ induces a continuous orientation on the first tensor product on the right-side side in (6.24) and its extension over $\Delta_{\mathbb{R}}$. Thus, it is sufficient to show that for small values of $t\in\Delta_{\mathbb{R}}^*$ the KS map for (Σ_t,σ_t) associates the radial vector

$$\frac{\partial}{\partial|t|}\in T_{\Sigma_t}\mathbb{R}\mathcal{M}_{g,l}\tag{6.25}$$

with the same direction of the factor $\mathbb{R}$ in (6.24), regardless of whether $t\in\Delta_{\mathbb{R}}^+$ or $t\in\Delta_{\mathbb{R}}^-$ (in these two cases, the radial vector field determines opposite orientations on $L^{\mathbb{R}}|_{\Sigma}$).

We use the explicit description of the KS map at the bottom of page 11 in [28] and continue with the notation in the proof of Lemma 6.8. We cover a neighborhood of Σ_t in $\mathcal{U}$ by the open sets

$$\mathcal{U}_1=\big\{(t,z_1,z_2)\in\mathcal{U}_0\colon\,2|z_2|<1\big\}\qquad\text{and}\qquad\mathcal{U}_2=\big\{(t,z_1,z_2)\in\mathcal{U}_0\colon\,2|z_1|<1\big\},$$

along with coordinate charts each of which intersects at most one of $\mathcal{U}_1$ and $\mathcal{U}_2$. Since $z_1z_2=t$ on $\mathcal{U}_0$, the overlaps between the coordinates z_1 on $\mathcal{U}_1$ and z_2 on $\mathcal{U}_2$ are given by

$$z_1\equiv f_{12}(t,z_2)=tz_2^{-1}\qquad\text{and}\qquad z_2\equiv f_{21}(t,z_1)=tz_1^{-1};$$

all other overlap maps do not depend on t. Thus, the KS map takes the tangent vector (6.25) to the Čech 1-cocycle on Σ_t given by

$$\theta_{t;12}\equiv\frac{\partial f_{12}}{\partial|t|}\frac{\partial}{\partial z_1}=|t|^{-1}z_1\frac{\partial}{\partial z_1},\qquad\theta_{t;21}\equiv\frac{\partial f_{21}}{\partial|t|}\frac{\partial}{\partial z_2}=|t|^{-1}z_2\frac{\partial}{\partial z_2},$$

and vanishing on all remaining overlaps. The positive factor of $|t|^{-1}$ does not effect the orientation on the fiber of (6.20) over $t\in\Delta_{\mathbb{R}}^*$ induced by the KS map and can be dropped above. The resulting Čech 1-cocycle $\widehat{\theta}_t$ is then an extension of the Čech 1-cocycle $\widehat{\theta}_0$ on Σ given by

$$\widehat{\theta}_{0;12}\equiv z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2},\qquad\widehat{\theta}_{0;21}\equiv-z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2},\tag{6.26}$$

and vanishing on all remaining overlaps. For $t \in \Delta_{\mathbb{R}^*}$, the positive direction of the last tensor product on the right-hand side of (6.24) is thus given by

$$\frac{\partial}{\partial |t|} \otimes \widehat{\theta}_t;$$

this orientation does not extend across $t=0$. $\square$

Proof of Proposition 6.1. Suppose (Σ, x_{12}, σ), $\mathcal{C}$, and $(\pi, \widetilde{\mathfrak{c}}, s_1, \ldots, s_l)$ are as in (6.16) with $\mathcal{U}|_{\Delta_{\mathbb{R}}} \longrightarrow \Delta_{\mathbb{R}}$ embedded inside of the universal curve fibration over $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The orientation on the restriction of the real line bundle (3.1) to $\Delta_{\mathbb{R}}^*$ provided by Proposition 5.9 is the tensor product of

(1) the orientation on the restriction of the real line bundle (6.20) to $\Delta_{\mathbb{R}}^*$ induced by the KS isomorphism,

(2) the orientation on the restriction of the real line bundle (6.17) to $\Delta_{\mathbb{R}}^*$ induced by the Dolbeault and SD isomorphisms,

(3) the orientation on the restriction of the real line bundle

$$\big(\det \bar{\partial}_{(\hat{\mathcal{T}}\mathcal{C}, \hat{\varphi}) \otimes (\hat{\mathcal{T}}, \hat{\varphi})}\big) \otimes \big(\det \bar{\partial}_{(\hat{\mathcal{T}}, \hat{\varphi})^{\otimes 2}}\big) \longrightarrow \Delta_{\mathbb{R}}$$

to $\Delta_{\mathbb{R}}^*$ induced by the short exact sequences (5.24) and the specified orientations of (5.21),

(4) the orientation on the restriction of the real line bundle (6.19) to $\Delta_{\mathbb{R}}^*$ determined by the canonical isomorphisms of Corollary 5.6.

Since the family of the short exact sequences (5.24) and the specified orientations of (5.21) extend across $t=0$, so does the orientation in (3). By Corollary 6.12, the orientation in (2) also extends across $t=0$. By Lemma 6.17, the orientation in (1) flips across $t=0$. By Corollary 6.16, the orientation in (4) extends across $t=0$ if (Σ, x_{12}, σ) is of type (E) or (H1) and flips if (Σ, x_{12}, σ) is of type (H2) or (H3). Combining these four statements, we obtain the claim. $\square$

6.4 Proofs of the main statements

We now establish the main statements of this paper, Theorems 1.3 and 1.5.

Proof of Theorem 1.3. By Corollary 5.10, a real orientation on (X, ω, ϕ) determines an orientation on the restriction of the real line bundle (1.5) to the uncompactified moduli space

$$\mathfrak{M}_{g,l}(X, B; J)^{\phi, \sigma} \subset \overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi}$$

for every topological type σ of genus g orientation-reversing involutions. We show that these orientations multiplied by $(-1)^{g+|\sigma|_0+1}$ extend across the codimension-one strata of $\overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi}$.

Suppose $[u, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j}]$ is a stable real morphism from a one-nodal symmetric surface (Σ, σ). Since the fibers of the forgetful morphism

$$\overline{\mathfrak{M}}_{g,l+1}(X, B; J)^{\phi} \longrightarrow \overline{\mathfrak{M}}_{g,l}(X, B; J)^{\phi}$$

are canonically oriented, we can assume that

$$\mathcal{C} \equiv \big(\Sigma, (z_1^+, z_1^-), \ldots, (z_l^+, z_l^-), \mathfrak{j}\big)$$

is a stable symmetric surface and thus defines an element of $\mathbb{R}\overline{\mathcal{M}}_{g,l}$. The canonical isomorphism (5.29) then extends across $[u]$. By Corollary 6.7, the canonical orientation on the restriction of the real line bundle (5.30) to $\mathfrak{M}_{g,l}(X,B;J)^{\phi}$ also extends across $[u]$. Since $(-1)^{g+|\sigma|_0+1}$ flips across the codimension-one boundary strata of types (E) and (H1) and extends across the codimension-one boundary strata of types (H2) and (H3), the claim now follows from Proposition 6.1. $\square$

Proof of Theorem 1.5. For $J \in \mathcal{J}_\omega^\phi$ and a (real) perturbation ν of the real $\bar{\partial}_J$-equation, we denote by $\overline{\mathfrak{M}}_{1,l;k}(X,B;J,\nu)^{\phi}$ the moduli space of real genus 1 degree B (J,ν)-maps with l conjugate pairs of marked points and k real marked points. For $k=0$, we omit the corresponding subscript. If (X,ω,ϕ) is semi-positive in the sense of [50, Definition 1.2], then ν can be taken to be a real Ruan-Tian perturbation as defined in [50, Section 3.1]. In general, ν is a perturbation in the sense of Kuranishi structures.

By Theorem 1.3, the compactified moduli space $\overline{\mathfrak{M}}_{1,l}(X,B;J,\nu)^{\phi}$ is orientable. Thus, the orientability of $\overline{\mathfrak{M}}_{1,l;k}(X,B;J,\nu)^{\phi}$ is determined by the orientability of the vertical tangent bundle of the forgetful morphism

$$\overline{\mathfrak{M}}_{1,l;k}(X,B;J,\nu)^{\phi} \longrightarrow \overline{\mathfrak{M}}_{1,l}(X,B;J,\nu)^{\phi} \tag{6.27}$$

dropping the real marked points. The fibers of (6.27) over the main strata

$$\mathfrak{M}_{1,l}(X,B;J,\nu)^{\phi,\sigma} \subset \overline{\mathfrak{M}}_{1,l}(X,B;J,\nu)^{\phi}$$

are open subsets of $(S^1)^k$.

Since there are diffeomorphisms $h \in \mathcal{D}_\sigma$ which reverse an orientation on the fixed locus, the vertical tangent bundle of (6.27) is not orientable over $\mathfrak{M}_{1,l}(X,B;J,\nu)^{\phi,\sigma}$ if k is odd. If k is even, the fibers of (6.27) are canonically oriented as follows. If $|\sigma|_0 = 1$, an orientation on the fixed locus determines an orientation on each fiber of (6.27) which is independent of the choice of the first orientation. If $|\sigma|_0 = 2$, the fixed locus Σ^σ splits Σ into two annuli; let Σ^b be either of these annuli. Endow one of the boundary circles of Σ^b with the induced boundary orientation and the other with the opposite of the induced boundary orientation. These choices determine an orientation on each fiber of (6.27). Since k is even, this orientation is independent of which circle is oriented as a boundary and thus of the choice of the half Σ^b. We determine the orientability of the vertical tangent bundle over $\overline{\mathfrak{M}}_{1,l}(X,B;J,\nu)^{\phi,\sigma}$ by studying how these canonical orientations change across the codimension-one boundary strata.

If $g=1$, the codimension-one boundary strata can be of types (E), (H1), and (H3) only. If $k>0$, the domains of all morphisms of type (E) are one-nodal symmetric surfaces (Σ, x_{12}, σ) with the fixed locus consisting of the node x_{12} and a fixed circle Σ_1^σ containing all of the real marked points. The canonical orientations on the fibers of (6.27) extend across such strata.

In the (H1) case, the nodal symmetric surface (Σ,σ) is $(\mathbb{P}^1,\tau)$ with two real points identified. In particular, the fixed locus Σ^σ splits Σ into two copies of a disk with two boundary points identified;

denote by Σ^b either of these copies and by $x_{12}\in\Sigma^\sigma$ the node. Let $(\mathcal{T},\varphi)$ be the real bundle pair over a one-parameter family of smoothings of (Σ,σ) as in Lemma 6.8. An orientation on $\mathcal{T}^\varphi|_{\Sigma^\sigma}\longrightarrow\Sigma^\sigma$ induces an orientation on $T\Sigma_t^{\sigma_t}$ for every smoothing of (Σ_t,σ_t). By the matching condition on $\Omega(\mathcal{T}|_{\Sigma_0})$ in Lemma 6.8, the orientation on

$$\mathcal{T}^\varphi|_{\Sigma^\sigma-x_{12}} = T\big(\Sigma^\sigma-x_{12}\big)$$

as the boundary of Σ^b does not extend over x_{12}. This implies that the orientation on $\Sigma_t^{\sigma_t}$ with $|\sigma_t|_0=2$ induced by an orientation on $\mathcal{T}^\varphi|_{\Sigma^\sigma}$ is not the boundary orientation from either of the annuli obtained by cutting Σ_t along $\Sigma_t^{\sigma_t}$. Thus, the canonical orientations on the fibers of (6.27) extend across the (H1) boundary strata as well.

In the (H3) case, the nodal symmetric surface (Σ,σ) consists of a genus 1 surface with a sphere bubble attached. A choice of an orientation on Σ^σ is compatible with the orientation of the fixed locus on only one side of the boundary. If the number of the real marked points on either the torus or the sphere is even, then the orientation of the fibers of (6.27) still extends across this stratum. We will call the codimension-one boundary strata of type (H3) with odd numbers of real marked points on the torus and the sphere to be of type (H3$^-$). Following the approach of [4, 37], we show that in a generic one-parameter family the cut-down moduli space does not cross such strata and thus the counting invariant (1.5) is well-defined.

Let

$$\begin{gathered}\mathrm{ev}\colon \mathfrak{M}^*_{1,l;k}(X,B;J,\nu)^\phi \longrightarrow X^l\times(X^\phi)^k,\\ \big[u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),x_1,\ldots,x_k,\mathfrak{j}\big]\longrightarrow \big(u(z_1^+),\ldots,u(z_l^+),u(x_1),\ldots,u(x_k)\big),\end{gathered}$$

be the total evaluation map from the moduli space of simple (J,ν)-maps. Choose pseudocycle representatives

$$h_1\colon Y_1\longrightarrow X,\quad \ldots,\quad h_l\colon Y_l\longrightarrow X$$

for the Poincare duals of $\mu_1,\ldots,\mu_l$; this is possible to do by [43, Theorem 1.1]. We can assume that

$$\sum_{i=1}^{l}(\deg\mu_i-2)+2k=\big\langle c_1(X),B\big\rangle \tag{6.28}$$

and so k is even under our assumptions. Choose k real points $p_1,\ldots,p_k\in X^\phi$. If (X,ω,ϕ) is semi-positive, (J,ν) is generic, and $h_1,\ldots,h_l,p_1,\ldots,p_k$ are chosen generically, then ev is transverse to the pseudocycle

$$\prod_{i=1}^{l}Y_i\longrightarrow X^l\times(X^\phi)^k,\quad (y_1,\ldots,y_l)\longrightarrow\big(h_1(y_1),\ldots,h_l(y_l),p_1,\ldots,p_k\big)\,.$$

The intersection of ev with this pseudocycle, i.e.

$$\begin{aligned}&\mathfrak{M}^*_{1,l;k}(X,B;J,\nu)^\phi_{h_1,\ldots,h_l;p_1,\ldots,p_k}\\ &\quad\equiv\Big\{\big([u,(z_1^+,z_1^-),\ldots,(z_l^+,z_l^-),x_1,\ldots,x_k,\mathfrak{j}],y_1,\ldots,y_l\big)\in\mathfrak{M}^*_{1,l;k}(X,B;J,\nu)^\phi\times\prod_{i=1}^{l}Y_i\colon\\ &\qquad\qquad u(z_i^+)=h_i(y_i)\ \forall\,i=1,\ldots,\ell,\ u(x_i)=p_i\ \forall\,i=1,\ldots,k\Big\},\end{aligned}$$

is then a zero-dimensional manifold. A real orientation on (X,ω,ϕ) and the canonical orientation on the vertical tangent bundle of (6.27) determine an orientation of this manifold. We set

$$\big\langle\mu_1,\ldots,\mu_l;\mathrm{pt}^k;J,\nu\big\rangle_{1,B}^{\phi} = {}^{\pm}\big|\mathfrak{M}^*_{1,l;k}(X,B;J,\nu)^{\phi}_{h_1,\ldots,h_l;p_1,\ldots,p_k}\big|$$

to be the signed cardinality of this set.

Let (J_1,ν_1) and (J_2,ν_2) be two regular ϕ-invariant pairs and $\{J_t,\nu_t\}$ be a generic path between them. If (X,ω,ϕ) is semi-positive, the image of the ϕ-multiply covered maps is of codimension at least 2; a generic path of cut-down moduli spaces thus avoids them. Along the path $\{J_t,\nu_t\}$, the cut-down moduli space forms a one-dimensional bordism and contains finitely many points in the codimension-one boundary strata of type (H3$^-$). We orient this bordism outside of the (H3$^-$) elements as the preimage of the submanifold

$$\big\{(q_1,\ldots,q_l,p_1,\ldots,p_k,q_1,\ldots,q_l)\colon\, q_1,\ldots,q_l\in X\big\}\subset X^l\times(X^{\phi})^k\times X^l$$

under the transverse morphism

$$\mathrm{ev}\times h_1\times\ldots\times h_l\colon \bigcup_{t\in[0,1]}\{t\}\times\mathfrak{M}^*_{1,l;k}(X,B;J_t,\nu_t)\times\prod_{i=1}^{l}Y_i\longrightarrow X^l\times(X^{\phi})^k\times X^l.$$

The signed cardinalities of the boundaries of this bordism over $t=0$ and $t=1$ are

$$-\big\langle\mu_1,\ldots,\mu_l;\mathrm{pt}^k;J_0,\nu_0\big\rangle_{1,B}^{\phi}\qquad\text{and}\qquad\big\langle\mu_1,\ldots,\mu_l;\mathrm{pt}^k;J_1,\nu_1\big\rangle_{1,B}^{\phi},\tag{6.29}$$

respectively.

Suppose that in a one-parameter family the cut-down moduli space crosses a codimension-one boundary stratum of type (H3) with the map degree splitting into classes $B_1,B_2\in H_2(X;\mathbb{Z})$ between the genus 1 surface and the sphere bubble, respectively. Let $l_1,l_2\in\mathbb{Z}^{\geqslant0}$ be the numbers of conjugate pairs of marked points carried by the two components and $k_1,k_2\in\mathbb{Z}^{\geqslant0}$ be the numbers of real marked points carried by them. Thus,

$$B_1+B_2=B,\qquad l_1+l_2=l,\qquad k_1+k_2=k.$$

By a dimension count, this can happen only if

$$\sum_{i=1}^{l_1}(\deg\mu_{j_i}-2)+2k_1\leqslant\big\langle c_1(X),B_1\big\rangle+1\quad\text{and}\quad\sum_{i=1}^{l_2}(\deg\mu_{j_i}-2)+2k_2\leqslant\big\langle c_1(X),B_2\big\rangle+1.$$

Using (6.28), we obtain

$$\sum_{i=1}^{l_2}(\deg\mu_{j_i}-2)+2k_2-1\leqslant\big\langle c_1(X),B_2\big\rangle\leqslant\sum_{i=1}^{l_2}(\deg\mu_{j_i}-2)+2k_2+1\,.$$

Since $\deg\mu_{j_i}-2$ and $\langle c_1(X),B_2\rangle$ are divisible by 4, this implies that k_2 is even and that the codimension-one boundary strata of type (H3$^-$) are never crossed. Thus, the canonical orientations extend over the whole cobordism and the two counts in (6.29) are equal.

A similar cobordism argument holds for a semi-positive deformation of ω and for a change of the pseudocycle representatives. The general case is treated using Kuranishi structures similarly to [37, Section 7]. $\square$

A Topologizing determinant line bundles

The existence of topologies on the total space of (4.9) with good properties is readily implied by the main algebraic conclusion of [25] in combination with some of the analytic results obtained in [27, Section 3]. The latter ensure that the kernels of surjective CR-operators $D_{(V,\varphi);\mathbf{t}}$ in (4.8) and their extensions $D_{\Theta;\mathbf{t}}$ in (A.12) form vector bundles over Δ and thus so do the determinants of these operators (the determinants are then the top exterior powers of the kernels); see Proposition A.2. In the general case, the bundle isomorphisms (A.20) would topologize the total space of (4.9) from the determinant line bundles of surjective operators if the resulting overlap maps between the latter are continuous. For the property (D2) on page 23 to hold, *all* isomorphisms (A.20) and thus the aforementioned overlap maps *need to* be continuous.

The isomorphisms (A.20) are special cases of the isomorphisms $\widehat{\mathcal{I}}_{\Theta;D}$ in (A.4); the latter are induced by the isomorphisms (4.6) associated with the exact triples (A.1) of Fredholm operators. The isomorphisms (4.6) associated with the exact triples (A.2) induce the isomorphisms $\mathcal{I}_{\Theta;D}$ in (A.4) going in the opposite direction. The property (D2) and thus the continuity of the isomorphisms in (A.4) would be implied by the two purely algebraic Compositions properties of [49, Section 2] for homomorphisms between *finite*-dimensional vector spaces. By the main algebraic conclusion of [25], it is possible to *choose* the isomorphisms (4.6) for exact triples of operators between *finite*-dimensional vector spaces so that they satisfy these two properties. Furthermore, the resulting topologies on the determinant line bundles (4.9) satisfy all properties in [49, Section 2] with Fredholm spaces replaced by real bundle pairs. In fact, the choice of a good collection of the isomorphisms (4.6) for non-surjective operators between finite-dimensional vector spaces is not unique. However, any two choices induce topologies that differ by homeomorphisms intertwining all isomorphisms between determinant line bundles listed in [49, Section 2]; see [49, Theorem 2].

As shown in [24, Appendix D.2] and [49, Section 3.2], the topologies on the determinant line bundles over families of Fredholm operators between *fixed* Banach spaces arise from *exactly* the same algebraic considerations. The only difference is that the analogue of Proposition A.2 for continuous families of surjective Fredholm operators between *fixed* Banach spaces is straightforward.

A.1 Linear algebra

We begin by recalling the relevant algebraic facts from [49]. We denote by

$$\Omega_N \equiv e_1 \wedge \ldots \wedge e_N$$

the standard volume tensor on $\mathbb{R}^N$ and by $\Omega_N^* \in (\Lambda_{\mathbb{R}}^N \mathbb{R}^N)^*$ its dual. For a Banach space X and $N, N_1, N_2 \in \mathbb{Z}^{\geqslant 0}$, define

$$\begin{aligned} \iota_{X;N} \colon X &\longrightarrow X \oplus \mathbb{R}^N, & \iota_{X;N}(x) &= (x,0), \\ R_{X;N_1,N_2} \colon X \oplus \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2} &\longrightarrow X \oplus \mathbb{R}^{N_2} \oplus \mathbb{R}^{N_1}, & R_{X;N_1,N_2}(x,v_1,v_2) &= (x,v_2,v_1). \end{aligned}$$

For vector space homomorphisms $\Theta: \mathbb{R}^N \longrightarrow Y$ and $R: \mathbb{R}^{N'} \longrightarrow \mathbb{R}^N$ and a Fredholm operator $D \colon X \longrightarrow Y$, define

$$\begin{aligned} D_\Theta \colon X \oplus \mathbb{R}^N &\longrightarrow Y, & D_\Theta(x,v) &= Dx + \Theta(v), \\ R_{\Theta;D} \colon \ker D_{\Theta\circ R} &\longrightarrow \ker D_\Theta, & R_{\Theta;D}(x,v') &= (x,Rv'). \end{aligned}$$

In particular, the triple $\mathfrak{t}_{\Theta;D}$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "D"] & X\oplus\mathbb{R}^N \arrow[r, "\pi_2"] \arrow[d, "D_\Theta"] & \mathbb{R}^N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r] & Y \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{A.1}$$

of Fredholm operators is exact.

A homomorphism between Fredholm operators $D\colon X\longrightarrow Y$ and $D'\colon X'\longrightarrow Y'$ is a pair of homomorphisms $\phi\colon X\longrightarrow X'$ and $\psi\colon Y\longrightarrow Y'$ so that $D'\circ\phi=\psi\circ D$; an isomorphism between Fredholm operators D and D' is a homomorphism $(\phi,\psi)\colon D\longrightarrow D'$ so that ϕ and ψ are isomorphisms. Such an isomorphism induces isomorphisms

$$\det\phi\colon \Lambda_{\mathbb{R}}^{\mathrm{top}}(\ker D)\xrightarrow{\approx}\Lambda_{\mathbb{R}}^{\mathrm{top}}(\ker D'),\qquad \det\psi^{-1}\colon \Lambda_{\mathbb{R}}^{\mathrm{top}}(\operatorname{cok} D')\xrightarrow{\approx}\Lambda_{\mathbb{R}}^{\mathrm{top}}(\operatorname{cok} D),$$
$$\mathcal{I}_{\phi,\psi;D}\colon \det D\xrightarrow{\approx}\det D',\quad x\wedge\alpha\longrightarrow\big(\{\det\phi\}(x)\big)\wedge\big(\alpha\circ\{\det\psi^{-1}\}\big).$$

For homomorphisms $\Theta_1\colon\mathbb{R}^{N_1}\longrightarrow Y$ and $\Theta_2\colon\mathbb{R}^{N_2}\longrightarrow Y$, let

$$\mathcal{R}_{\Theta_1,\Theta_2;D}=\mathcal{I}_{R_{X;N_1,N_2},\mathrm{id}_Y;D_{\Theta_1\oplus\Theta_2}}\colon \det D_{\Theta_1\oplus\Theta_2}\xrightarrow{\approx}\det D_{\Theta_2\oplus\Theta_1}\,.$$

A pair of Fredholm operators $D_1\colon X_1\longrightarrow X_2$ and $D_2\colon X_2\longrightarrow X_3$ determines an exact triple

$$\begin{tikzcd} 0 \arrow[r] & X_1 \arrow[r, "\mathfrak{i}_X"] \arrow[d, "D_1"] & X_1\oplus X_2 \arrow[r, "\mathfrak{j}_X"] \arrow[d, "D_2\circ D_1\oplus\mathrm{id}_{X_2}"] & X_2 \arrow[r] \arrow[d, "D_2"] & 0 \\ 0 \arrow[r] & X_2 \arrow[r, "\mathfrak{i}_Y"] & X_3\oplus X_2 \arrow[r, "\mathfrak{j}_Y"] & X_3 \arrow[r] & 0 \end{tikzcd} \qquad \begin{aligned} \mathfrak{i}_X(x_1)&=(x_1,D_1x_1)\\ \mathfrak{j}_X(x_1,x_2)&=D_1x_1-x_2\\ \mathfrak{i}_Y(x_2)&=(D_2x_2,x_2)\\ \mathfrak{j}_Y(x_3,x_2)&=x_3-D_2x_2\,, \end{aligned} \tag{A.2}$$

of Fredholm operators. The isomorphism (4.6) for this triple becomes

$$\widetilde{\mathcal{C}}_{D_1,D_2}\colon\big(\det D_1\big)\otimes\big(\det D_2\big)\xrightarrow{\approx}\det(D_2\circ D_1)\,. \tag{A.3}$$

If D, Θ, and π_2 are as above, $D=D_\Theta\circ\iota_{X;N}$ and the projection

$$\pi_2\colon\operatorname{cok}\iota_{X;N}\longrightarrow\mathbb{R}^N$$

is an isomorphism. We thus obtain two isomorphisms induced by (4.6),

$$\begin{aligned} \widehat{\mathcal{I}}_{\Theta;D}\colon\det D&\xrightarrow{\approx}\det D_\Theta, & \widehat{\mathcal{I}}_{\Theta;D}(\varpi)&=\Psi_{\mathfrak{t}_{\Theta;D}}\big(\varpi\otimes\Omega_N\otimes1^*\big),\\ \mathcal{I}_{\Theta;D}\colon\det D_\Theta&\xrightarrow{\approx}\det D, & \mathcal{I}_{\Theta;D}(\varpi)&=\mathcal{C}_{\iota_{X;N},D_\Theta}\big(1\otimes(\Omega_N^*\circ\{\det\pi_2\})\otimes\varpi\big), \end{aligned} \tag{A.4}$$

via (A.1) and (A.3), respectively.

Every short exact sequence

$$0\longrightarrow V'\longrightarrow V\longrightarrow V''\longrightarrow 0 \tag{A.5}$$

of finite-dimensional vector spaces determines an isomorphism

$$\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}V'\big)\otimes\big(\Lambda_{\mathbb{R}}^{\mathrm{top}}V''\big)\xrightarrow{\approx}\Lambda_{\mathbb{R}}^{\mathrm{top}}V \tag{A.6}$$

between the top exterior powers of the vector spaces involved; see [49, Lemma 4.1]. By the Snake Lemma, an exact triple (4.5) induces an exact sequence

$$0 \longrightarrow \ker D' \xrightarrow{\mathfrak{i}_X} \ker D \xrightarrow{\mathfrak{j}_X} \ker D'' \xrightarrow{\delta} \operatorname{cok} D' \xrightarrow{\mathfrak{i}_Y} \operatorname{cok} D \xrightarrow{\mathfrak{j}_Y} \operatorname{cok} D'' \longrightarrow 0 \tag{A.7}$$

of finite-dimensional vector spaces. It is equivalent to four *short* exact sequences, such as

$$0 \longrightarrow \ker D' \xrightarrow{\mathfrak{i}_X} \ker D \xrightarrow{\mathfrak{j}_X} \operatorname{Im} \mathfrak{j}_X \longrightarrow 0.$$

The isomorphisms (4.6) should clearly be induced by the isomorphisms (A.6) corresponding to these four short exact sequences. However, there are at least choices of signs involved in putting the four resulting isomorphisms together, depending on the dimensions of the vector spaces appearing in the four sequences. Choosing these signs in some compatible fashion is necessary to ensure that the isomorphisms (A.20) used to topologize determinant line bundles overlap continuously.

If the operators in (4.5) are surjective, the exact sequence (A.7) reduces to the exact sequence

$$0 \longrightarrow \ker D' \xrightarrow{\mathfrak{i}_X} \ker D \xrightarrow{\mathfrak{j}_X} \ker D'' \longrightarrow 0.$$

It is then standard to require that the corresponding isomorphism (4.6) be given by the isomorphism (A.6) associated with this exact sequence of kernels; this property is Normalization II in [49, Section 2]. An explicit formula for the isomorphism (4.6) in the general case with this property is given by [49, (4.10)]. The induced isomorphisms (A.3) satisfy the two algebraic Compositions properties in [49, Section 2] and thus the remaining algebraic properties listed there (Naturality II and III and Exact Squares); see the paragraph after Theorem 1 in [49, Section 2]. The associated isomorphisms (A.4) satisfy

$$\mathcal{I}_{\Theta;D}\circ\widehat{\mathcal{I}}_{\Theta;D} = (-1)^{(\operatorname{ind} D)N}\mathrm{id}\colon \det D \xrightarrow{\approx} \det D, \tag{A.8}$$

$$\mathcal{I}_{\Theta_2;D}^{-1}\circ\mathcal{I}_{\Theta_1;D} = (-1)^{N_1N_2}\mathcal{I}_{\Theta_1;D_{\Theta_2}}\circ\mathcal{R}_{\Theta_1,\Theta_2;D}\circ\mathcal{I}_{\Theta_2;D_{\Theta_1}}^{-1}\colon \det D_{\Theta_1} \xrightarrow{\approx} \det D_{\Theta_2} \tag{A.9}$$

for all Fredholm operators $D\colon X\longrightarrow Y$ and homomorphisms

$$\Theta\colon \mathbb{R}^N \longrightarrow Y, \qquad \Theta_1\colon \mathbb{R}^{N_1} \longrightarrow Y, \qquad \Theta_2\colon \mathbb{R}^{N_2} \longrightarrow Y\,;$$

see [49, Lemma 4.11] and the end of the proof of [49, Proposition 5.3].

A.2 Analysis and topology

Let $(\Sigma_0,\sigma_0,\mathfrak{j}_0)$, $(\pi,\widetilde{\mathfrak{c}})$, and (V,φ) be as above (4.9). Fix $p>2$, $\widetilde{\mathfrak{c}}$-invariant Riemannian metric on $\mathcal{U}$, and a φ-invariant metric on V. For each $\mathbf{t}\in\Delta_{\mathbb{R}}$, we denote by

$$\mathcal{E}_{\mathbf{t}}(V)^{\varphi} \supset \Gamma\big(\Sigma_{\mathbf{t}};V|_{\Sigma_{\mathbf{t}}}\big)^{\varphi} \qquad \text{and} \qquad \mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi} \supset \Gamma^{0,1}_{\mathfrak{j}_{\mathbf{t}}}\big(\Sigma_{\mathbf{t}};V|_{\Sigma_{\mathbf{t}}}\big)^{\varphi}$$

the completions of the spaces of smooth $(\varphi,\sigma_{\mathbf{t}})$-invariant bundle sections in the modified L^p_1- and L^p-norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$, respectively, introduced in [27, Section 3]. The norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$ dominate the usual L^p_1- and L^p-norms, but are equivalent to them away from the nodes of $\Sigma_{\mathbf{t}}$. Some of the key properties of these norms are summarized by the next statement. Let

$$D_{\mathbf{t}}\colon \mathcal{E}_{\mathbf{t}}(V)^{\varphi} \longrightarrow \mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi} \tag{A.10}$$

be the operator induced by $D_{(V,\varphi);\mathbf{t}}$.

Lemma A.1. *For every $\mathbf{t}^*\in\Delta_{\mathbb{R}}$, there exist a neighborhood $\Delta_{\mathbf{t}^*}$ of $\mathbf{t}^*$ in $\Delta_{\mathbb{R}}$ and $C_{\mathbf{t}^*}\in\mathbb{R}^+$ such that*

$$\|\xi\|_{C^0}\leqslant C_{\mathbf{t}^*}\|\xi\|_{p,1},\quad \big\|D_{\mathbf{t}}\xi\big\|_p\leqslant C_{\mathbf{t}^*}\|\xi\|_{p,1},\quad \|\xi\|_{p,1}\leqslant C_{\mathbf{t}^*}\big(\big\|D_{\mathbf{t}}\xi\big\|_p+\|\xi\|_p\big)$$

for all $\xi\in\mathcal{E}_{\mathbf{t}}(V)^{\varphi}$ and $\mathbf{t}\in\Delta_{\mathbf{t}^*}$.

The second inequality above is immediate from the definition of the norms $\|\cdot\|_{p,1}$ and $\|\cdot\|_p$. The first inequality holds even with the standard L^p_1-norm on the right-hand side; see [27, Lemma 3.2] and [47, Proposition 4.10]. The last inequality is the crucial uniform elliptic estimate of [27, Lemma 3.9]; see the proof of [41, Proposition 5.11] and [47, Section 4.3] for more details. By Lemma A.1, (A.10) is a Fredholm operator; its index, which we denote by ind $D_{(V,\varphi)}$, does not depend on $\mathbf{t}$.

The normed topologies on the fibers of the projections

$$\mathcal{E}(V)^{\varphi}\equiv\bigsqcup_{\mathbf{t}\in\Delta_{\mathbb{R}}}\big(\{\mathbf{t}\}\times\mathcal{E}_{\mathbf{t}}(V)^{\varphi}\big)\longrightarrow\Delta_{\mathbb{R}}\quad\text{and}\quad\mathcal{E}^{0,1}(V)^{\varphi}\equiv\bigsqcup_{\mathbf{t}\in\Delta_{\mathbb{R}}}\big(\{\mathbf{t}\}\times\mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi}\big)\longrightarrow\Delta_{\mathbb{R}}$$

are extended to topologies on $\mathcal{E}(V)^{\varphi}$ and $\mathcal{E}^{0,1}(V)^{\varphi}$ in [27, Section 3]. These topologies are described as follows. Let $\mathbf{t}\in\Delta_{\mathbb{R}}$ and

$$\psi_{\mathbf{t}'}:\Sigma^*_{\mathbf{t}}\longrightarrow q^{-1}(\Sigma^*_{\mathbf{t}})\cap\Sigma_{\mathbf{t}'}$$

be analogues of the diffeomorphisms (4.3) defined for $\mathbf{t}'\in\Delta$ in a neighborhood of $\mathbf{t}$. For each $\delta\in\mathbb{R}^+$, denote by $B'_{\mathbf{t};\delta}\subset\mathcal{U}$ the δ-neighborhood of the nodes of $\Sigma_{\mathbf{t}}$. Suppose $\mathbf{t}_r\in\Delta_{\mathbb{R}}$ is a sequence converging to $\mathbf{t}$. A sequence $\xi_r\in\mathcal{E}_{\mathbf{t}_r}(V)^{\varphi}$ converges to $\xi\in\mathcal{E}_{\mathbf{t}}(V)^{\varphi}$ if

(a) the sequence $\xi_r\circ\psi_{\mathbf{t}_r}$ converges to ξ in the L^p_1-norm on compact subsets of $\Sigma^*_{\mathbf{t}}$ and

(b) $\lim\limits_{\delta\longrightarrow0}\lim\limits_{r\longrightarrow\infty}\big\|\xi_r|_{B'_{\mathbf{t};\delta}\cap\Sigma_{\mathbf{t}_r}}\big\|_{p,1}=0$.

The topology on $\mathcal{E}^{0,1}(V)^{\varphi}$ introduced in [27] is described analogously, with the L^p_1-norms replaced by L^p-norms.

For any bundle homomorphism

$$\Theta\colon\Delta_{\mathbb{R}}\times\mathbb{R}^N\longrightarrow\mathcal{E}^{0,1}(V)^{\varphi},\qquad\Theta(\mathbf{t},v)=\big(\mathbf{t},\zeta_{\mathbf{t},v}\big),\tag{A.11}$$

and $\mathbf{t}\in\Delta_{\mathbb{R}}$, let

$$\Theta_{\mathbf{t}}:\mathbb{R}^N\longrightarrow\mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi},\qquad\Theta_{\mathbf{t}}(v)=\zeta_{\mathbf{t},v}\,,$$

be the restriction of Θ to the fiber over $\mathbf{t}$. Define

$$D_{\Theta;\mathbf{t}}=(D_{\mathbf{t}})_{\Theta_{\mathbf{t}}}:\mathcal{E}_{\mathbf{t}}(V)^{\varphi}\oplus\mathbb{R}^N\longrightarrow\mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi},\quad U_\Theta=\big\{\mathbf{t}\in\Delta_{\mathbb{R}}:\operatorname{cok}D_{\Theta;\mathbf{t}}=\{0\}\big\},\tag{A.12}$$

$$\ker D_\Theta=\big\{(\mathbf{t},\xi,v)\in\mathcal{E}(V)^{\varphi}\times\mathbb{R}^N\colon\mathbf{t}\in U_\Theta,\ D_{\Theta;\mathbf{t}}(\xi,v)=0\big\}\longrightarrow U_\Theta\,,\tag{A.13}$$

$$\widehat{\mathcal{I}}_{\Theta;\mathbf{t}}=\widehat{\mathcal{I}}_{\Theta_{\mathbf{t}};D_{\mathbf{t}}}:\det D_{\mathbf{t}}\xrightarrow{\approx}\det D_{\Theta;\mathbf{t}},\quad\mathcal{I}_{\Theta;\mathbf{t}}=\mathcal{I}_{\Theta_{\mathbf{t}};D_{\mathbf{t}}}:\det D_{\Theta;\mathbf{t}}\xrightarrow{\approx}\det D_{\mathbf{t}}\,.$$

If in addition

$$R:\Delta_{\mathbb{R}}\times\mathbb{R}^{N'}\longrightarrow\Delta_{\mathbb{R}}\times\mathbb{R}^N,\tag{A.14}$$

$$\Theta_1:\Delta_{\mathbb{R}}\times\mathbb{R}^{N_1}\longrightarrow\mathcal{E}^{0,1}(V)^{\varphi},\quad\Theta_2:\Delta_{\mathbb{R}}\times\mathbb{R}^{N_2}\longrightarrow\mathcal{E}^{0,1}(V)^{\varphi}\tag{A.15}$$

are bundle homomorphisms, let

$$R_{\Theta;\mathbf{t}}=R_{\Theta_{\mathbf{t}};D_{\mathbf{t}}}\colon \ker D_{\Theta\circ R;\mathbf{t}} \longrightarrow \ker D_{\Theta;\mathbf{t}},$$
$$\mathcal{R}_{\Theta_1,\Theta_2;\mathbf{t}}=\mathcal{R}_{(\Theta_1)_{\mathbf{t}},(\Theta_2)_{\mathbf{t}};D_{\mathbf{t}}}\colon \det D_{\Theta_1\oplus\Theta_2;\mathbf{t}} \xrightarrow{\approx} \det D_{\Theta_2\oplus\Theta_1;\mathbf{t}}\,.$$

By (A.8) and (A.9),

$$\mathcal{I}_{(\Theta_2)_{\mathbf{t}};D_{\Theta_1;\mathbf{t}}}\circ\widehat{\mathcal{I}}_{(\Theta_2)_{\mathbf{t}};D_{\Theta_1;\mathbf{t}}} = (-1)^{(\operatorname{ind} D_{(V,\varphi)}+N_1)N_2}\mathrm{id}\colon \det D_{\Theta_1;\mathbf{t}} \xrightarrow{\approx} \det D_{\Theta_1;\mathbf{t}}, \tag{A.16}$$
$$\mathcal{I}_{\Theta_2;\mathbf{t}}^{-1}\circ\mathcal{I}_{\Theta_1;\mathbf{t}} = (-1)^{N_1N_2}\mathcal{I}_{(\Theta_1)_{\mathbf{t}};D_{\Theta_2;\mathbf{t}}}\circ\mathcal{R}_{\Theta_1,\Theta_2;\mathbf{t}}\circ\mathcal{I}^{-1}_{(\Theta_2)_{\mathbf{t}};D_{\Theta_1;\mathbf{t}}}\colon \det D_{\Theta_1;\mathbf{t}} \xrightarrow{\approx} \det D_{\Theta_2;\mathbf{t}} \tag{A.17}$$

for all $\mathbf{t}\in\Delta_{\mathbb{R}}$.

We call a bundle homomorphism as in (A.11) smoothly supported if $\zeta_{\mathbf{t},v}\in\mathcal{E}^{0,1}_{\mathbf{t}}(V)^{\varphi}$ is smooth and $\mathrm{supp}(\zeta_{\mathbf{t},v})\subset\Sigma^*_{\mathbf{t}}$ for all $\mathbf{t}\in\Delta_{\mathbb{R}}$ and $v\in\mathbb{R}^N$.

Proposition A.2. *For every continuous smoothly supported bundle homomorphism Θ as in (A.11), $U_\Theta\subset\Delta_{\mathbb{R}}$ is an open subset and (A.13) is a vector bundle. If in addition R is a continuous bundle homomorphism as in (A.14), then $U_{\Theta\circ R}\subset U_\Theta$ and*

$$R_\Theta\colon \ker D_{\Theta\circ R} \longrightarrow \ker D_\Theta\big|_{U_{\Theta\circ R}}, \qquad R_\Theta(\mathbf{t},\xi,v) = \big(\mathbf{t}, R_{\Theta;\mathbf{t}}(\xi,v)\big),$$

is a continuous bundle map.

This proposition follows from Lemma A.1, as demonstrated by the gluing construction of [27, Section 3] for (J,ν)-holomorphic maps instead of bundle sections. The greatly simplified, linear version of this construction (without the quadratic term of the first equation in the proof of [27, Proposition 3.4]) provides local trivializations for the projection

$$\mathcal{E}(V)^{\varphi}\times\mathbb{R}^N \supset \ker D_\Theta \longrightarrow U_\Theta$$

around every point $\mathbf{t}\in\Delta_{\mathbb{R}}$ and thus that $U_\Theta\subset\Delta_{\mathbb{R}}$ is open. This construction in the $N=0$ case and without restricting to the invariant sections is carried out in [42, Section 3.2]. By the smooth support assumption on Θ (which is in line with the setup in [27]), the reasoning in [42] applies in the general case, including for invariant sections, and implies the first statement of the proposition. The claim $U_{\Theta\circ R}\subset U_\Theta$ is immediate from the definitions. Factoring R through its graph reduces the remaining claim of the proposition to the case that R has constant rank. This case in turn reduces to showing that $\ker D_{\Theta\circ R}$ is a subbundle of $\ker D_\Theta|_{U_{\Theta\circ R}}$ if R is induced by the inclusion of a coordinate subspace of $\mathbb{R}^N$. This follows readily from the setup in [42, Section 3.2].

By the first statement of Proposition A.2, the total space of the projection

$$\det D_\Theta\big|_{U_\Theta}=\Lambda_{\mathbb{R}}^{\operatorname{ind} D_{(V,\varphi)}+N}(\ker D_\Theta) \longrightarrow U_\Theta$$

is a real line bundle with a natural topology. The isomorphisms (A.6) associated with short exact sequences of vector spaces as in (A.5) induce continuous isomorphisms of the same kind for short exact sequences of vector bundles. Along with the second statement of Proposition A.2, this implies that the bundle isomorphisms

$$\widehat{\mathcal{I}}_{\Theta_2;D_{\Theta_1}}\colon \det D_{\Theta_1} \longrightarrow \det D_{\Theta_1\oplus\Theta_2}\big|_{U_{\Theta_1}}, \qquad \widehat{\mathcal{I}}_{\Theta_2;D_{\Theta_1}}(\mathbf{t},\varpi) = \big(\mathbf{t},\widehat{\mathcal{I}}_{(\Theta_2)_{\mathbf{t}};D_{\Theta_1;\mathbf{t}}}(\varpi)\big),$$
$$\mathcal{R}_{\Theta_1,\Theta_2}\colon \det D_{\Theta_1\oplus\Theta_2}\big|_{U_{\Theta_1\oplus\Theta_2}} \longrightarrow \det D_{\Theta_2\oplus\Theta_1}\big|_{U_{\Theta_1\oplus\Theta_2}}, \quad \mathcal{R}_{\Theta_1,\Theta_2}(\mathbf{t},\varpi) = \big(\mathbf{t},\mathcal{R}_{\Theta_1,\Theta_2;\mathbf{t}}(\varpi)\big),$$

are continuous with respect to the natural topologies on the domains and targets for all continuous smoothly supported bundle homomorphisms Θ_1 and Θ_2 as in (A.15). Combining this with (A.16) and (A.17), we obtain the following.

Corollary A.3. *For all continuous smoothly supported bundle homomorphisms Θ_1 and Θ_2 as in (A.15), the bundle map*

$$\widehat{\mathcal{I}}_{\Theta_1\Theta_2}\colon \det D_{\Theta_2}\big|_{U_{\Theta_1}\cap U_{\Theta_2}} \longrightarrow \det D_{\Theta_1}\big|_{U_{\Theta_1}\cap U_{\Theta_2}}, \quad \widehat{\mathcal{I}}_{\Theta_1\Theta_2}(\mathbf{t},\varpi) = \big(\mathbf{t},\widehat{\mathcal{I}}_{\Theta_1;\mathbf{t}}\big(\widehat{\mathcal{I}}_{\Theta_2;\mathbf{t}}^{-1}(\varpi)\big)\big),$$

is continuous with respect to the natural topologies on its domain and target.

We now topologize the total space of the projection (4.9). Let $\mathbf{t}^*\in\Delta_{\mathbb{R}}$. By the elliptic regularity of $D_{\mathbf{t}^*}$, there exists a homomorphism

$$\Theta_{\mathbf{t}^*}\colon \mathbb{R}^N \longrightarrow \mathcal{E}^{0,1}_{\mathbf{t}^*}(V)^{\varphi}, \qquad \Theta_{\mathbf{t}^*}(v) = \zeta_{\mathbf{t}^*,v}, \tag{A.18}$$

such that every $(0,1)$-form $\zeta_{\mathbf{t}^*,v}$ is smooth and supported in $\Sigma^*_{\mathbf{t}^*}$ and the operator

$$\big(D_{\mathbf{t}^*}\big)_{\Theta_{\mathbf{t}^*}}\colon \mathcal{E}_{\mathbf{t}^*}(V)^{\varphi}\oplus\mathbb{R}^N \longrightarrow \mathcal{E}^{0,1}_{\mathbf{t}^*}(V)^{\varphi}, \qquad \big(D_{\mathbf{t}^*}\big)_{\Theta_{\mathbf{t}^*}}(\xi,v) = D_{\mathbf{t}^*}\xi+\zeta_{\mathbf{t}^*,v}, \tag{A.19}$$

is surjective. Choose a continuous smoothly supported homomorphism Θ as in (A.11) which restricts to (A.18) over $\mathbf{t}^*$. By (A.19), $\mathbf{t}^*\in U_\Theta$. We topologize $\det D_{(V,\varphi)}|_{U_\Theta}$ by requiring that the bundle map

$$\widehat{\mathcal{I}}_{\Theta}\colon \det D_{(V,\varphi)}\big|_{U_\Theta} \longrightarrow \det D_{\Theta}\big|_{U_\Theta}, \qquad \widehat{\mathcal{I}}_{\Theta}(\mathbf{t},\varpi) = \big(\mathbf{t},\widehat{\mathcal{I}}_{\Theta;\mathbf{t}}(\varpi)\big), \tag{A.20}$$

be a homeomorphism with respect to the natural topology on its target. By Corollary A.3, the overlaps between these maps are continuous. Thus, these maps define a topology on the total space of the projection (4.9).

It is immediate from the construction that the resulting topologies on the determinant line bundle corresponding to different real bundle pairs (V,φ) satisfy (D1) on page 23. By the proof of [49, Corollary 5.4], these topologies also satisfy (D2).

Remark A.4. Other topologies on the total space of the projection (4.9) with good properties can be obtained by modifying the isomorphisms (4.6) associated with exact sequences (A.7) as described above Theorem 2 in [49, Section 3.2]. This would modify the topologizing maps (A.20) and would thus generally change the topology on the total space of (4.9). The two topologies would differ by a homeomorphism which restricts to the identity over the points $\mathbf{t}\in\Delta_{\mathbb{R}}$ such that $D_{\mathbf{t}}$ is surjective.

Remark A.5. A connection ∇ as above (4.9) induces a splitting

$$TV \approx \pi^*V\oplus\pi^*T\mathcal{U}\,.$$

The complex structure $\mathfrak{i}$ in the fibers of $\pi: V\longrightarrow\mathcal{U}$, the complex structure $\mathfrak{j}_{\mathcal{U}}$ on $\mathcal{U}$, and the zeroth-order deformation term (4.7) induce a complex structure J_V on the total space of V by

$$J_V\big|_v(\dot v,\dot x) = \big(\mathfrak{i}\dot v+\{Av\}(\dot x),\mathfrak{j}_{\mathcal{U}}\dot x\big).$$

For each $\mathbf{t}\in\Delta_{\mathbb{R}}$, $D_{\mathbf{t}}$ is then the $\bar\partial_{J_V}$-operator on the space of real maps from $(\Sigma_{\mathbf{t}},\mathfrak{j}_{\mathbf{t}})$ to the total space of V. In particular, $\ker D_{\mathbf{t}}$ consists of real $(J_V,\mathfrak{j}_{\mathbf{t}})$-holomorphic maps. By the smooth support assumption on Θ, the subspace topology on $\ker D_\Theta$ can thus be described in terms of convergence of sequences similarly to Definition 4.2.

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