Real Gromov-Witten Theory in All Genera and Real Enumerative Geometry, Appendix

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Real positive-genus GW-invariants of real orientable symplectic manifolds of odd "complex" dimensions are constructed in [3]. For sufficiently positive targets of "complex" dimension 3, such as $\mathbb{P}^3$, these invariants have no contribution from genus 0 curves and thus provide lower bounds for the number of real genus 1 irreducible curves in such manifolds; see [2, Proposition 2.5] for a precise statement. In contrast, the complex genus 1 degree d enumerative and GW-invariants of $\mathbb{P}^3$ are related by the formula

$$\mathrm{E}_{1,d} = \mathrm{GW}_{1,d} + \frac{2d-1}{12}\mathrm{GW}_{0,d}\,. \tag{1}$$

This formula, originally announced as Theorem A in [4], is established as a special case of [7, Theorem 1.1], comparing standard and "reduced" GW-invariants (the latter do not "see" the genus 0 curves in sufficiently positive cases).

The accompanying hand-written notes and a *Mathematica* printout compute the real genus 1 degree 6 GW-invariant of $\mathbb{P}^3$ with 6 pairs of conjugate point constraints. As usual in real enumerative geometry and Gromov-Witten theory, there is a choice of sign involved. The real odd-genus GW-invariants of $\mathbb{P}^3$ are independent of the choice of real orientation on $\mathbb{P}^3$, but there is still an overall sign convention involved. The value of 4 for the real genus 1 degree 6 GW-invariant of $\mathbb{P}^3$ with 6 pairs of conjugate point constraints appearing at the end of the *Mathematica* printout corresponds to the secondary sign change described after [2, Proposition 2.6]. Some version of the primary sign change described at the end of [2, Section 3.2] is necessary for the invariants to be well-defined.

In the hand-written notes and *Mathematica* printout, we apply the equivariant localization theorem of [1] with the standard $(\mathbb{C}^*)^2$-action on $\mathbb{P}^3$ with its standard conjugation τ_4. We denote the weights of this action by $\lambda_1=-\lambda_2$ and $\lambda_3=-\lambda_4$; they correspond to the fixed points P_1,P_2,P_3,P_4 with $P_1=\tau_4(P_2)$ and $P_3=\tau_4(P_4)$. We find that

$$\int_{[\overline{\mathfrak{M}}_{1,6}(\mathbb{P}^3,6)^{\tau_4}]^{\mathrm{vir}}} \prod_{i=1}^{4}\Big(\mathrm{ev}_i*\prod_{k\neq i}(\mathbf{x}-\lambda_k)\Big)\cdot\prod_{i=1}^{2}\Big(\mathrm{ev}_{4+i}^*\prod_{k\neq i}(\mathbf{x}-\lambda_k)\Big)=4, \tag{2}$$

where $\mathbf{x}$ is the equivariant hyperplane class (denoted by H at the top of page 1 of the notes).

In the non-equivariant reduction, $\lambda_k=0$ and (2) becomes integration of pullbacks of the Poincare dual of the point in $\mathbb{P}^3$. We use Pandharipande's trick of twisting by the equivariant weights to reduce the number of contributing torus-fixed loci (the restrictions of the integrand to other loci vanish). This trick works spectacularly in reducing the proof of the Aspinwall-Morrison formula to computing the contribution of the simplest possible fixed locus; see [5, Lemma 27.5.3]. In our case, it leaves only the fixed loci consisting of morphisms passing through all 4 torus-fixed points and severely restricts the possible distributions of the 6 marked points. As the hand-written notes indicate, this still leaves a lot of contributing fixed loci.

As predicted in [6, Section 3.2] and confirmed in [3], the contributions of some of the remaining fixed loci cancel in pairs due to two different types of phenomena (these loci are represented by decorated graphs compatible with at least two different involutions). The normal bundles to the fixed loci are determined in [3] and are as described in [6, Section 3.2].

A careful analysis of our computation might indicate that most of our intermediate signs are wrong. This is indeed the case for the degree 1 edges preserved by the involution. However, nearly all of the contributing graphs with one such edge have precisely two of them, and so a uniform choice of sign for each separate edge does not effect the sign of the product and of the overall contribution of the graph. The only exceptions to this are the two graphs on page 2 of the hand-written notes; they have one involution-invariant degree 1 edge and one involution-invariant degree 3 edge. As the sign for the former is wrong and the sign for the latter is correct, we take the initially computed contribution of these graphs with the negative sign on the last page of the *Mathematica* printout.

The hand-written notes contain all of the graphs corresponding to the relevant fixed loci and determine their contributions. *Mathematica* was used to add up their contributions and to double-check intermediate computations.

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References

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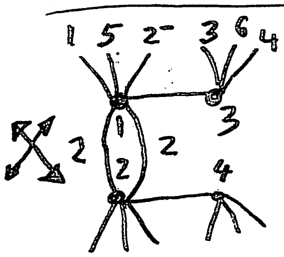
$g=1,\ d=6,\ \mathbb{P}^3$ $\lambda_2=-\lambda_1,\ \lambda_4=-\lambda_3$

$$\int_{\overline{\mathcal{M}}_{1,6}(\mathbb{P}^3,6)^{\mathbb{T}_4}} \prod_{i=1}^{4} ev_i^*\prod_{k\neq i}(H-\lambda_k)\cdot ev_5^*\prod_{k\neq 1}(H-\lambda_k)\cdot ev^*\prod_{k\neq 2}(H-\lambda_k)$$

$\therefore$ #1,5 $\to P_1$ #2 $\to P_2$ #3,6 $\to P_3$ #4 $\to P_4$

top $\to \left(2\lambda_1(\lambda_1^2-\lambda_3^2)\right)^2\cdot\left(2\lambda_3(\lambda_3^2-\lambda_1^2)\right)^2\cdot\left(-2\lambda_1(\lambda_1^2-\lambda_3^2)\right)\left(-2\lambda_3(\lambda_3^2-\lambda_1^2)\right)$

$= -64\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \longrightarrow \dim = 18$



2-pts on top; Aut $=2\times 2$ (left edge aut controls right edge aut)

$\overline{\mathcal{M}}_{0,6}\times\overline{\mathcal{M}}_{0,4}$

$$N\mathcal{Z} = \frac{2\lambda_1\cdot\lambda_1\cdot(-\lambda_1)\cdot(-2\lambda_1)\cdot(\lambda_1^2-\lambda_3^2)^2\cdot(-\lambda_3^2)}{2\lambda_1\cdot(\lambda_1^2-\lambda_3^2)}\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2) = 4\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^2$$

complex orientation of deg 2 edge ← node matching at special vertex

$$\int_{\overline{\mathcal{M}}_{0,4}}\frac{1}{\lambda_3-\lambda_1-\psi_1} = \frac{1}{(\lambda_3-\lambda_1)^2}$$

$$\int_{\overline{\mathcal{M}}_{0,6}}\frac{1}{(\lambda_1-\psi_1)(\lambda_1-\psi_2)(\lambda_1-\lambda_3-\psi_3)} = \frac{1}{\lambda_1^2(\lambda_1-\lambda_3)}\left(\frac{2}{\lambda_1}+\frac{1}{\lambda_1-\lambda_3}\right)^3 = \frac{(3\lambda_1-2\lambda_3)^3}{\lambda_1^5(\lambda_1-\lambda_3)^4}$$

$$C22 = \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{(3\lambda_1-2\lambda_3)^3}{\lambda_1^8\lambda_3^3(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^6}\cdot(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{-4\frac{(\lambda_1+\lambda_3)^4(3\lambda_1-2\lambda_3)^3}{\lambda_1^5(\lambda_1-\lambda_3)^2}}$$

↑ Aut

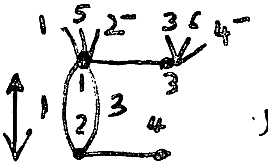
- - - - - - - - - - - - - - - - - - - -

$P_3 \leftrightarrow P_4$: changes λ_3 to $-\lambda_3$ in denominator, adds $-$ for marked pts $\Leftrightarrow$ changing $\lambda_3 \to -\lambda_3$

$$\Longrightarrow \quad (P_3\leftrightarrow P_4) \Leftrightarrow (\lambda_3\to-\lambda_3)$$

$C22 + (\lambda_3\to-\lambda_3) \longrightarrow C22t$

$C22t + (\lambda_1\leftrightarrow\lambda_3) \longrightarrow C22T$



Aut = 3 $\quad \overline{M}_{0,6} \times \overline{M}_{0,4}$

$$NZ = \underbrace{2\lambda_1 \cdot \tfrac{4}{3}\lambda_1 \cdot \tfrac{2}{3}\lambda_1 \cdot (\lambda_1-\lambda_3)(\tfrac{1}{3}\lambda_1-\lambda_3)(\lambda_1+\lambda_3)(\tfrac{1}{3}\lambda_1+\lambda_3)}_{\text{"top half" of deg 3 edge}} \cdot 2\lambda_3(\lambda_3^2-\lambda_1^2)$$

$$= -\frac{32}{9}\lambda_1^3\lambda_3(\lambda_1^2-\lambda_3^2)^2\left(\tfrac{1}{9}\lambda_1^2-\lambda_3^2\right)$$

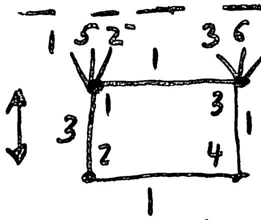
$$\int_{\overline{M}_{0,4}} \frac{1}{\lambda_3-\lambda_1-\psi_1} = \frac{1}{(\lambda_3-\lambda_1)^2}$$

$$\int_{\overline{M}_{0,6}} \frac{1}{(2\lambda_1-\psi_1)(\frac{2\lambda_1}{3}-\psi_2)(\lambda_1-\lambda_3-\psi_3)} = \frac{3}{4\lambda_1^2(\lambda_1-\lambda_3)}\left(\frac{1}{2\lambda_1}+\frac{3}{2\lambda_1}+\frac{1}{\lambda_1-\lambda_3}\right)^3 = \frac{3}{4}\cdot\frac{(3\lambda_1-2\lambda_3)^3}{\lambda_1^5(\lambda_1-\lambda_3)^4}$$

$$C31a = \frac{1}{3}\cdot\frac{-9}{32}\cdot\frac{3}{4}\cdot\frac{(3\lambda_1-2\lambda_3)^3}{\lambda_1^8\lambda_3(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^6(\frac{1}{9}\lambda_1^2-\lambda_3^2)} \times (-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6$$

Aut (↗ $\frac{1}{3}$)

$$= \boxed{\frac{81}{2}\cdot\frac{\lambda_3^2(\lambda_1+\lambda_3)^4(3\lambda_1-2\lambda_3)^3}{\lambda_1^5(\lambda_1-\lambda_3)^2(\lambda_1^2-9\lambda_3^2)}}$$



Aut = 3 $\quad \overline{M}_{0,5} \times \overline{M}_{0,5}$

$$NZ = -\frac{32}{9}\lambda_1^3\lambda_3(\lambda_1^2-\lambda_3^2)^2\left(\tfrac{1}{9}\lambda_1^2-\lambda_3^2\right) \quad \text{as above}$$

left: $$\int_{\overline{M}_{0,5}} \frac{1}{(\frac{2\lambda_1}{3}-\psi_1)(\lambda_1-\lambda_3-\psi_2)} = \frac{3}{2\lambda_1(\lambda_1-\lambda_3)}\left(\frac{3}{2\lambda_1}+\frac{1}{\lambda_1-\lambda_3}\right)^2 = \frac{3}{8}\cdot\frac{(5\lambda_1-3\lambda_3)^2}{\lambda_1^3(\lambda_1-\lambda_3)^3}$$

right: $$\int_{\overline{M}_{0,5}} \frac{1}{(2\lambda_3-\psi_1)(\lambda_3-\lambda_1-\psi_1)} = \frac{1}{2\lambda_3(\lambda_3-\lambda_1)}\left(\frac{1}{2\lambda_3}+\frac{1}{\lambda_3-\lambda_1}\right)^2 = -\frac{1}{8}\cdot\frac{(\lambda_1-3\lambda_3)^2}{\lambda_3^3(\lambda_1-\lambda_3)^3}$$

$$C31b = \frac{1}{3}\cdot\frac{9}{32}\cdot\frac{3}{64}\cdot\frac{(\lambda_1-3\lambda_3)^2(5\lambda_1-3\lambda_3)^2}{\lambda_1^6\lambda_3^4(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^6(\frac{1}{9}\lambda_1^2-\lambda_3^2)}\cdot(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6$$

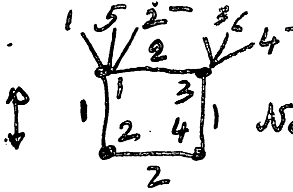
Aut (↗ $\frac{1}{3}$)

$$= \boxed{-\frac{81}{32}\cdot\frac{(\lambda_1+\lambda_3)^4(\lambda_1-3\lambda_3)(5\lambda_1-3\lambda_3)^2}{\lambda_1^3\lambda_3(\lambda_1-\lambda_3)^2(\lambda_1+3\lambda_3)}}$$

C31 = C31a + C31b

C31 + ($\lambda_3 \to -\lambda_3$) ⟶ C31t

C31t + ($\lambda_1 \leftrightarrow \lambda_3$) ⟶ C31T



Aut=2 $\overline{M}_{0,5}\times\overline{M}_{0,5}$

$$NZ = \underbrace{\tfrac{1}{4}(\lambda_1-\lambda_3)^4\cdot 2\lambda_1\cdot 2\lambda_3\cdot(\lambda_1+\lambda_3)^2\cdot\tfrac{1}{4}(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)}_{\text{deg 2 edge}} = \lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^2$$

$$= \tfrac{1}{4}\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^2(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)$$

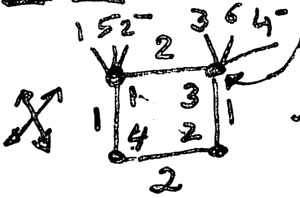
left: $\displaystyle\int_{\overline{M}_{0,5}}\frac{1}{(2\lambda_1-\psi_1)(\frac{\lambda_1-\lambda_3}{2}-\psi_2)} = \frac{1}{\lambda_1(\lambda_1-\lambda_3)}\left(\frac{1}{2\lambda_1}+\frac{2}{\lambda_1-\lambda_3}\right)^2 = \frac{(5\lambda_1-\lambda_3)^2}{4\lambda_1^3(\lambda_1-\lambda_3)^3}$

right: $\displaystyle\int_{\overline{M}_{0,5}}\frac{1}{(2\lambda_3-\psi_1)(\frac{\lambda_3-\lambda_1}{2}-\psi_2)} = -\frac{(\lambda_1-5\lambda_3)^2}{4\lambda_3^3(\lambda_1-\lambda_3)^3}$

$$C12a = \frac{1}{2}\cdot\frac{-1}{4}\cdot\frac{(5\lambda_1-\lambda_3)^2(\lambda_1-5\lambda_3)^2}{\lambda_1^4\lambda_3^4(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^8(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6$$

(Aut → 2)

$$= \boxed{8\,\frac{(\lambda_1+\lambda_3)^4(5\lambda_1-\lambda_3)^2(\lambda_1-5\lambda_3)^2}{\lambda_1\lambda_2(\lambda_1-\lambda_3)^4(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)}}$$



special vertex Aut=2 $\overline{M}_{0,5}\times\overline{M}_{0,5}$

$$NZ = -(\text{above}) = -\tfrac{1}{4}\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^2(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)$$

left: $\displaystyle\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_1+\lambda_3-\psi_1)(\frac{\lambda_1-\lambda_3}{2}-\psi_2)} = \frac{2}{\lambda_1^2-\lambda_3^2}\left(\frac{1}{\lambda_1+\lambda_3}+\frac{2}{\lambda_1-\lambda_3}\right)^2 = 2\,\frac{(3\lambda_1+\lambda_3)^2}{(\lambda_1^2-\lambda_3^2)^3}$

right: $\displaystyle\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_3+\lambda_1-\psi_1)(\frac{\lambda_3-\lambda_1}{2}-\psi_2)} = -2\,\frac{(\lambda_1+3\lambda_3)^2}{(\lambda_1^2-\lambda_3^2)^3}$

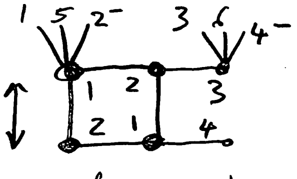
$$C12b = \frac{1}{2}\cdot 16\cdot\frac{(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)}{\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^8(\lambda_1-\lambda_3)^2}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6$$

(Aut → 2)

$$= \boxed{-512\,\frac{\lambda_1^2\lambda_2^2(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)}{(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)^2}}$$

$C12 = C12a + C12b$

$C12 + (\lambda_3\to-\lambda_3) \longrightarrow C31T$



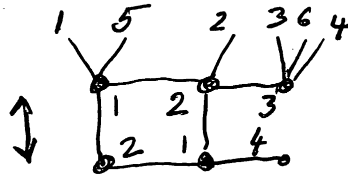
$\overline{M}_{0,5}\times\overline{M}_{0,3}\times\overline{M}_{0,5}$

$$N\mathcal{Z} = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_1)(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2) = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\int_{\overline{M}_{0,5}} \frac{1}{(2\lambda_1-\psi_1)(2\lambda_1-\psi_2)} = \frac{1}{4\lambda_1^2}\left(\frac{1}{2\lambda_1}+\frac{1}{2\lambda_1}\right)^2 = \frac{1}{4\lambda_1^4}$$

$$\int_{\overline{M}_{0,3}} = \frac{1}{(-2\lambda_1)^2(-\lambda_1-\lambda_3)} = -\frac{1}{4\lambda_1^2(\lambda_1+\lambda_3)} \qquad \int_{\overline{M}_{0,4}}\frac{1}{(\lambda_3+\lambda_1-\psi_1)} = \frac{1}{(\lambda_1+\lambda_3)^2}$$

$$C111a = -\frac{1}{8\cdot16}\cdot\frac{1}{\lambda_1^8\lambda_3(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^3}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{2}\,\frac{\lambda_3^2(\lambda_1-\lambda_3)^3}{\lambda_1^5}}$$

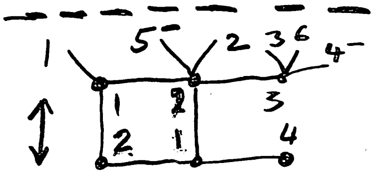


$\times(-3)$ $\overline{M}_{0,4}\times\overline{M}_{0,4}\times\overline{M}_{0,4}$ $\quad N\mathcal{Z} = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$ as above

left: $\int_{\overline{M}_{0,4}} \frac{1}{(2\lambda_1-\psi_1)(2\lambda_1-\psi_2)} = \frac{1}{4\lambda_1^3}$ $\quad$ right: $\int_{\overline{M}_{0,4}} = \frac{1}{(\lambda_1+\lambda_3)^2}$ as above

middle $\int_{\overline{M}_{0,4}} \frac{1}{(-2\lambda_1-\psi_1)(-2\lambda_1-\psi_2)(-\lambda_1-\lambda_3-\psi_3)} = \frac{1}{4\lambda_1^2(\lambda_1+\lambda_3)}\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_1+\lambda_3}\right) = \frac{2\lambda_1+\lambda_3}{4\lambda_1^3(\lambda_1+\lambda_3)^2}$

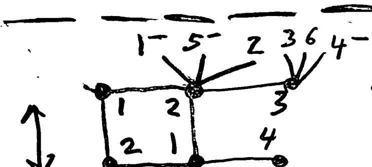
$$C111b = (-3)\cdot\frac{1}{8\cdot16}\cdot\frac{2\lambda_1+\lambda_3}{\lambda_1^8\lambda_3(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^4}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{2}\cdot\frac{\lambda_3^2(\lambda_1-\lambda_3)^3(2\lambda_1+\lambda_2)}{\lambda_1^5(\lambda_1+\lambda_3)}}$$



$\times(3)$ $\overline{M}_{0,3}\times\overline{M}_{0,5}\times\overline{M}_{0,4}$ $\quad N\mathcal{Z} = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$ as above

$\int_{\overline{M}_{0,3}} = \frac{1}{4\lambda_1^2}$ $\quad \int_{\overline{M}_{0,4}} = \frac{1}{(\lambda_1+\lambda_3)^2}$ $\quad \int_{\overline{M}_{0,5}} \frac{1}{(-2\lambda_1-\psi_1)(-2\lambda_1-\psi_2)(-\lambda_1-\lambda_3-\psi_3)} = -\frac{(2\lambda_1+\lambda_3)^2}{4\lambda_1^4(\lambda_1+\lambda_3)^3}$

$$C111c = 3\cdot\frac{-1}{8\cdot16}\cdot\frac{(2\lambda_1+\lambda_3)^2}{\lambda_1^8\lambda_3(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^5}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{2}\cdot\frac{\lambda_3^2(\lambda_1-\lambda_3)^3(2\lambda_1+\lambda_3)^2}{\lambda_1^5(\lambda_1+\lambda_3)^2}}$$



$\times(-1)$ $\overline{M}_{0,2}\times\overline{M}_{0,6}\times\overline{M}_{0,4}$ $\quad N\mathcal{Z} = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

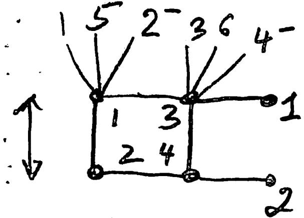
$\frac{1}{4\lambda_1}$ $\quad \frac{1}{(\lambda_1+\lambda_3)^2}$ $\quad \int_{\overline{M}_{0,6}} \frac{1}{(-2\lambda_1-\psi_1)(-2\lambda_1-\psi_2)(-\lambda_1-\lambda_3-\psi_3)} = \frac{(2\lambda_1+\lambda_3)^3}{4\lambda_1^5(\lambda_1+\lambda_3)^4}$

$$C111d = \frac{-1}{8\cdot16}\cdot\frac{(2\lambda_1+\lambda_3)^3}{\lambda_1^8\lambda_3(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^6}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{2}\,\frac{\lambda_3^2(\lambda_1-\lambda_3)^3(2\lambda_1+\lambda_3)^3}{\lambda_1^5(\lambda_1+\lambda_3)^3}}$$

C111ad = C111a + C111b + C111c + C111d

C111ad + $(\lambda_3 \to -\lambda_3)$ $\longrightarrow$ C111adt

C111adt + $(\lambda_1 \leftrightarrow \lambda_3)$ $\longrightarrow$ C111adT



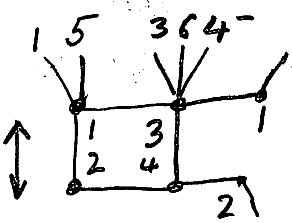
$\overline{M}_{0,5}\times\overline{M}_{0,6}$

$N\mathcal{Z} = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2)\cdot 2\lambda_1(\lambda_1+\lambda_3) = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^2(\lambda_1+\lambda_3)$

$$\int_{\overline{M}_{0,5}} \frac{1}{(2\lambda_1-\psi_1)(\lambda_1-\lambda_3-\psi_2)} = \frac{1}{2\lambda_1(\lambda_1-\lambda_3)}\left(\frac{1}{2\lambda_1}+\frac{1}{\lambda_1-\lambda_3}\right)^2 = \frac{(3\lambda_1-\lambda_3)^2}{8\lambda_1^3(\lambda_1-\lambda_3)^3}$$

$$\int_{\overline{M}_{0,6}} \frac{1}{(2\lambda_3-\psi_1)(\lambda_3-\lambda_1-\psi_2)(\lambda_3-\lambda_1-\psi_3)} = \frac{1}{2\lambda_3(\lambda_1-\lambda_3)^2}\left(\frac{1}{2\lambda_3}+\frac{2}{\lambda_3-\lambda_1}\right)^3 = \frac{(\lambda_1-5\lambda_3)^3}{16\lambda_3^4(\lambda_1-\lambda_3)^5}$$

$$C111e1 = \frac{-1}{64\cdot16}\cdot\frac{(3\lambda_1-\lambda_3)^2(\lambda_1-5\lambda_3)^3}{\lambda_1^5\lambda_3^5(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^7}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{16}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)^2(\lambda_1-5\lambda_3)^3}{\lambda_1^2\lambda_3^2(\lambda_1-\lambda_3)^4}}$$

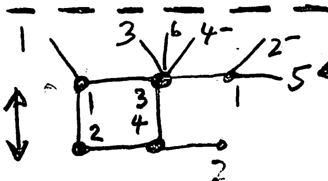


$\times 3$ $\overline{M}_{0,4}\times\overline{M}_{0,6}$ $N\mathcal{Z} = (\text{as above})\cdot(\lambda_1-\lambda_3) = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$$\int_{\overline{M}_{0,4}} \frac{1}{(2\lambda_1-\psi_1)(\lambda_1-\lambda_3-\psi_2)} = \frac{1}{2\lambda_1(\lambda_1-\lambda_3)}\left(\frac{1}{2\lambda_1}+\frac{1}{\lambda_1-\lambda_3}\right) = \frac{3\lambda_1-\lambda_3}{4\lambda_1^2(\lambda_1-\lambda_3)^2}$$

$\int_{\overline{M}_{0,6}}$ as above

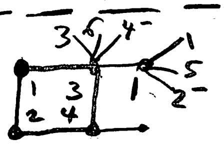
$$C111f1 = 3\cdot\frac{-1}{64\cdot8}\,\frac{(3\lambda_1-\lambda_3)(\lambda_1-5\lambda_3)^3}{\lambda_1^4\lambda_3^5(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^7}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{8}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)(\lambda_1-5\lambda_3)^3}{\lambda_1\lambda_3^2(\lambda_1-\lambda_3)^4}}$$



$\times 3$ $\overline{M}_{0,3}\times\overline{M}_{0,6}\times\overline{M}_{0,3}$ $N\mathcal{Z} = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$\overline{M}_{0,3} \to \frac{1}{2\lambda_1(\lambda_1-\lambda_3)}$, $\overline{M}_{0,6} \to$ as above, $\overline{M}_{0,3} \to \frac{1}{(\lambda_1-\lambda_3)}$

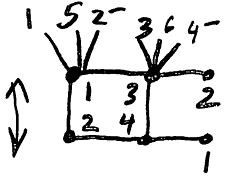
$$C111g1 = 3\cdot\frac{-1}{16\cdot16}\cdot\frac{(\lambda_1-5\lambda_3)^3}{\lambda_1^3\lambda_3^5(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^7}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{4}\cdot\frac{(\lambda_1+\lambda_3)^3(\lambda_1-5\lambda_3)^3}{\lambda_3^2(\lambda_1-\lambda_3)^4}}$$



$\overline{M}_{0,2}\times\overline{M}_{0,6}\times\overline{M}_{0,4}$ $N\mathcal{Z} = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$\overline{M}_{0,2} \to \frac{1}{(3\lambda_1-\lambda_3)}$, $\overline{M}_{0,6} \to$ as above, $\overline{M}_{0,4} \to \frac{1}{(\lambda_1-\lambda_3)^2}$

$$C111h1 = \frac{-1}{8\cdot16}\,\frac{(\lambda_1-5\lambda_3)^3}{\lambda_1^2\lambda_3^5(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^7(3\lambda_1-\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{2}\cdot\frac{\lambda_1(\lambda_1+\lambda_3)^3(\lambda_1-5\lambda_3)^3}{\lambda_3^2(\lambda_1-\lambda_3)^4(3\lambda_1-\lambda_2)}}$$



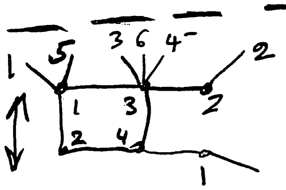
$\overline{\mathcal{M}}_{0,5}\times\overline{\mathcal{M}}_{0,6}$

$$N\mathcal{Z} = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2)\cdot(-2\lambda_1)(-\lambda_1+\lambda_3) = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)$$

$$\int_{\overline{\mathcal{M}}_{0,5}} = \frac{(3\lambda_1-\lambda_3)^2}{8\lambda_1^3(\lambda_1-\lambda_3)^3}$$ as on p5, #1

$$\int_{\overline{\mathcal{M}}_{0,6}} \frac{1}{(2\lambda_3-\psi_1)(\lambda_3-\lambda_1-\psi_2)(\lambda_3+\lambda_1-\psi_3)} = \frac{1}{2\lambda_3(\lambda_3^2-\lambda_1^2)}\left(\frac{1}{2\lambda_3}+\frac{1}{\lambda_3-\lambda_1}+\frac{1}{\lambda_3-\lambda_1}\right)^3 = -\frac{(\lambda_1^2-5\lambda_3^2)^3}{16\lambda_3^4(\lambda_1^2-\lambda_3^2)^4}$$

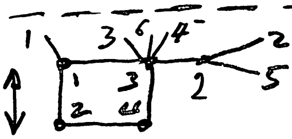
$$C111e2 = \frac{1}{16\cdot 64}\,\frac{(3\lambda_1-\lambda_3)^2(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1^5\lambda_3^5(\lambda_1^2-\lambda_3^2)^6(\lambda_1-\lambda_3)^4}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{-1}{16}\cdot\frac{(3\lambda_1-\lambda_3)^2(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1^2\lambda_3^2(\lambda_1-\lambda_3)^4}}$$



$\times(-3)$ $\overline{\mathcal{M}}_{0,4}\times\overline{\mathcal{M}}_{0,6}$ $N\mathcal{Z}$ = (as above) $\cdot(-\lambda_1-\lambda_3) = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$$\int_{\overline{\mathcal{M}}_{0,4}} = \frac{3\lambda_1-\lambda_2}{4\lambda_1^2(\lambda_1-\lambda_3)^2}$$ as on p5, #2 $\quad \int_{\overline{\mathcal{M}}_{0,6}}$ = as above

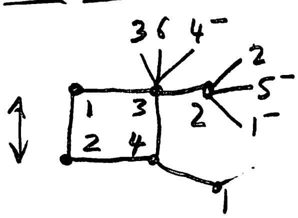
$$C111f2 = \frac{3}{8\cdot 64}\cdot\frac{(3\lambda_1-\lambda_2)(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1^4\lambda_3^5(\lambda_1^2-\lambda_3^2)^7(\lambda_1-\lambda_3)^2}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{-\frac{3}{8}\cdot\frac{(3\lambda_1-\lambda_2)(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_3)^2}}$$



$(\times 3)$ $\overline{\mathcal{M}}_{0,3}\times\overline{\mathcal{M}}_{0,6}\times\overline{\mathcal{M}}_{0,3}$ $N\mathcal{Z} = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$\frac{1}{2\lambda_1(\lambda_1-\lambda_3)}$ ↙ ; as above ↓ ; ↘ $\frac{1}{(-\lambda_1-\lambda_3)} = -\frac{1}{\lambda_1+\lambda_3}$

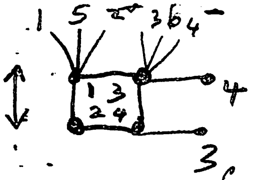
$$C111g2 = \frac{3}{16\cdot 16}\,\frac{(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1^3\lambda_3^5(\lambda_1^2-\lambda_3^2)^8}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{-\frac{3}{4}\,\frac{(\lambda_1^2-5\lambda_3^2)^3}{\lambda_3^2(\lambda_1^2-\lambda_3^2)^2}}$$



(−1) $\overline{\mathcal{M}}_{0,2}\times\overline{\mathcal{M}}_{0,6}\times\overline{\mathcal{M}}_{0,4}$ $N\mathcal{Z} = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$\frac{1}{3\lambda_1-\lambda_3}$; as above ; $\frac{1}{(\lambda_1+\lambda_3)^2}$

$$C111h2 = \frac{1}{16\cdot 8}\cdot\frac{(\lambda_1^2-5\lambda_3^2)^3}{\lambda_1^2\lambda_3^5(\lambda_1^2-\lambda_3^2)^7(\lambda_1+\lambda_3)^2(3\lambda_1-\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{-\frac{1}{2}\cdot\frac{\lambda_1(\lambda_1^2-5\lambda_3^2)^3}{\lambda_3^2(\lambda_1^2-\lambda_3^2)(\lambda_1+\lambda_3)^2(3\lambda_1-\lambda_2)}}$$



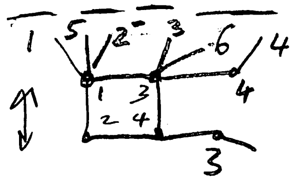
$\overline{M}_{0,5}\times\overline{M}_{0,6}$

$\Omega = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2)\cdot(\lambda_3^2-\lambda_1^2) = 4\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^3$

$\int_{\overline{M}_{0,5}} = \frac{(3\lambda_1-\lambda_3)^2}{8\lambda_1^3(\lambda_1-\lambda_3)^3}$ as on p5 #1

$$\int_{\overline{M}_{0,6}} \frac{1}{(2\lambda_3-\psi_1)(\lambda_3-\lambda_1-\psi_2)(2\lambda_3-\psi_3)} = \frac{1}{4\lambda_3^2(\lambda_3-\lambda_1)}\left(\frac{1}{\lambda_3}+\frac{1}{\lambda_3-\lambda_1}\right)^3 = -\frac{(\lambda_1-2\lambda_3)^3}{4\lambda_3^5(\lambda_1-\lambda_3)^4}$$

$$C111e4 = \frac{-1}{16\cdot 8}\cdot\frac{(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)^3}{\lambda_1^4\lambda_3^6(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^7}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{2}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)^3}{\lambda_1\lambda_3^3(\lambda_1-\lambda_3)^4}}$$

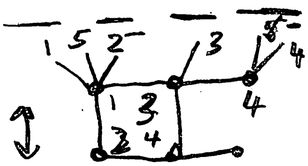


$\times(-3)$ $\overline{M}_{0,5}\times\overline{M}_{0,5}\times\overline{M}_{0,2}$ $\Omega = (\text{as above})\times(-2\lambda_3) = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

as above

$$\int_{\overline{M}_{0,5}} \frac{1}{(2\lambda_3-\psi_1)(\lambda_3-\lambda_1-\psi_2)(2\lambda_3-\psi_3)} = -\frac{(\lambda_1-2\lambda_3)^2}{4\lambda_3^4(\lambda_1-\lambda_3)^3}$$

$$C111f4 = \frac{-3}{4\cdot 64}\cdot\frac{(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)^2}{\lambda_1^4\lambda_3^6(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^6}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{4}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)^2}{\lambda_1\lambda_3^3(\lambda_1-\lambda_3)^3}}$$

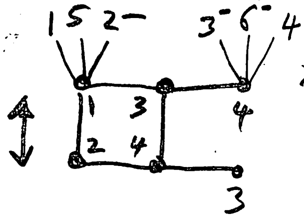


$(\times 3)$ $\overline{M}_{0,5}\times\overline{M}_{0,4}\times\overline{M}_{0,3}\longrightarrow -\frac{1}{2\lambda_3}$ $\Omega = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

as above

$$\frac{1}{4\lambda_3^2(\lambda_3-\lambda_1)}\left(\frac{1}{2\lambda_3}+\frac{1}{2\lambda_3}+\frac{1}{\lambda_3-\lambda_1}\right) = -\frac{\lambda_1-2\lambda_3}{4\lambda_3^3(\lambda_1-\lambda_3)^2}$$

$$C111g4 = \frac{-3}{8\cdot 64}\cdot\frac{(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)}{\lambda_1^4\lambda_3^6(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^5}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{3}{8}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)^2(\lambda_1-2\lambda_3)}{\lambda_1\lambda_3^3(\lambda_1-\lambda_3)^2}}$$



$\times(-1)$ $\overline{M}_{0,5}\times\overline{M}_{0,3}\times\overline{M}_{0,4}$ $\Omega = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

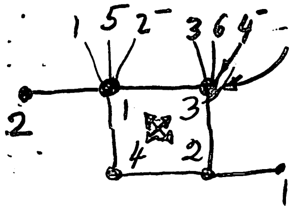
as above; $\overline{M}_{0,3}\to -\frac{1}{4\lambda_3^2(\lambda_1-\lambda_3)}$; $\overline{M}_{0,4}\to\frac{1}{4\lambda_3^2}$

$$C111h4 = \frac{-1}{16\cdot 64}\cdot\frac{(3\lambda_1-\lambda_3)^2}{\lambda_1^4\lambda_3^6(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)^4}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{\frac{1}{16}\cdot\frac{(\lambda_1+\lambda_3)^3(3\lambda_1-\lambda_3)^2}{\lambda_1\lambda_3^3(\lambda_1-\lambda_3)}}$$

C111eh = sum of 12 terms on pp5-7

C111eh + $(\lambda_3\to-\lambda_3)$ $\longrightarrow$ C111eht

C111eht + $(\lambda_1\leftrightarrow\lambda_3)$ $\longrightarrow$ C111ehT



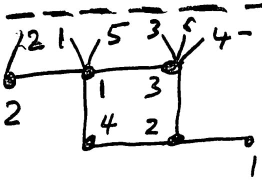
$\overline{M}_{0,6}\times\overline{M}_{0,5}$ — at bottom left vertex; matching at top right

$$N2 = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_3)(\lambda_3^2-\lambda_1^2)\cdot(\lambda_1^2-\lambda_3^2) = 4\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\int_{\overline{M}_{0,6}}\frac{1}{(\lambda_1-\lambda_3-\psi_1)(\lambda_1+\lambda_3-\psi_2)(2\lambda_1-\psi_3)} = \frac{1}{2\lambda_1(\lambda_1^2-\lambda_3^2)}\left(\frac{1}{2\lambda_1}+\frac{1}{\lambda_1-\lambda_3}+\frac{1}{\lambda_1+\lambda_3}\right)^3 = \frac{(5\lambda_1^2-\lambda_3^2)^3}{16\lambda_1^4(\lambda_1^2-\lambda_3^2)^4}$$

$$\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_3-\lambda_1-\psi_1)(\lambda_3+\lambda_1-\psi_2)} = \frac{1}{\lambda_3^2-\lambda_1^2}\left(\frac{1}{\lambda_3-\lambda_1}+\frac{1}{\lambda_3+\lambda_1}\right)^2 = -\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$$

$$C1111i2 = -\frac{1}{16}\frac{\lambda_3(5\lambda_1^2-\lambda_3^2)^3}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^{10}}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{4\frac{\lambda_3^4(5\lambda_1^2-\lambda_3^2)^3}{\lambda_1^2(\lambda_1^2-\lambda_3^2)^4}}$$



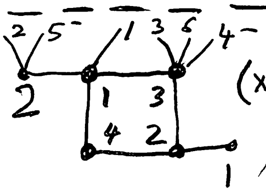
$\times(-3)$ $\overline{M}_{0,2}\times\overline{M}_{0,5}\times\overline{M}_{0,5}$

$$N2 = (\text{as above})\times(-2\lambda_1) = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$\to -\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$ as above

$$\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_1-\lambda_3-\psi_1)(\lambda_1+\lambda_3-\psi_2)(2\lambda_1-\psi_2)} = \frac{(5\lambda_1^2-\lambda_3^2)^2}{8\lambda_1^3(\lambda_1^2-\lambda_3^2)^3}$$

$$C111j2 = -\frac{3}{16}\cdot\frac{\lambda_3(5\lambda_1^2-\lambda_3^2)^2}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^9}\times(-64)\lambda_1^2\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{12\frac{\lambda_3^4(5\lambda_1^2-\lambda_3^2)^2}{\lambda_1^2(\lambda_1^2-\lambda_3^2)^3}}$$



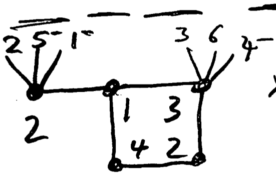
$(\times 3)$ $\overline{M}_{0,3}\times\overline{M}_{0,4}\times\overline{M}_{0,5}$

$$N2 = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$\overline{M}_{0,3}\to -\frac{1}{2\lambda_1}$; $\overline{M}_{0,5}\to$ as above

$$\int_{\overline{M}_{0,4}}\frac{1}{(\lambda_1-\lambda_3-\psi_1)(\lambda_1+\lambda_3-\psi_2)(2\lambda_1-\psi_3)} = \frac{(5\lambda_1^2-\lambda_3^2)}{4\lambda_1^2(\lambda_1^2-\lambda_3^2)^2}$$

$$C111k2 = \frac{-3}{16}\cdot\frac{\lambda_3(5\lambda_1^2-\lambda_3^2)}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^8}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{12\frac{\lambda_3^4(5\lambda_1^2-\lambda_3^2)}{\lambda_1^2(\lambda_1^2-\lambda_3^2)^2}}$$



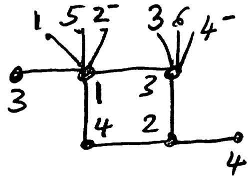
$\times(-1)$ $\overline{M}_{0,4}\times\overline{M}_{0,3}\times\overline{M}_{0,5}$

$$N2 = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$\overline{M}_{0,5}\to -\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$ as above

$\overline{M}_{0,4}\to \frac{1}{4\lambda_1^2}$, $\overline{M}_{0,3}\to \frac{1}{2\lambda_1(\lambda_1^2-\lambda_3^2)}$

$$C111l2 = -\frac{1}{16}\cdot\frac{\lambda_3}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^7}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{4\frac{\lambda_3^4}{\lambda_1^2(\lambda_1^2-\lambda_3^2)}}$$



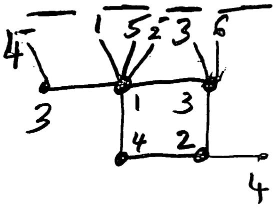
$\overline{M}_{0,6}\times\overline{M}_{0,5}$

$$NZ = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_3)(\lambda_3^2-\lambda_1^2)\cdot 2\lambda_3(\lambda_3+\lambda_1) = 8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^2(\lambda_1+\lambda_3)$$

$$\int_{\overline{M}_{0,6}}\frac{1}{(\lambda_1-\lambda_3-\psi_1)(\lambda_1+\lambda_3-\psi_2)(\lambda_1-\lambda_3-\psi_3)} = \frac{1}{(\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_3)}\left(\frac{1}{\lambda_1+\lambda_3}+\frac{2}{\lambda_1-\lambda_3}\right)^3 = \frac{(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1-\lambda_3)}$$

$$\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_3-\lambda_1-\psi_1)(\lambda_3+\lambda_1-\psi_2)} \Longrightarrow -\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$$ as p8, #1

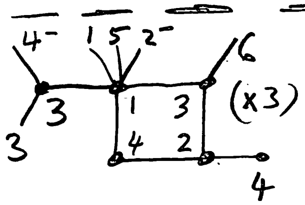
$$C111i3 = \frac{-1}{2}\cdot\frac{(3\lambda_1+\lambda_3)^3}{\lambda_1(\lambda_1^2-\lambda_3^2)^{10}}\cdot(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{32\,\frac{\lambda_1^2\lambda_3^3(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4}}$$



×3 $\overline{M}_{0,2}\times\overline{M}_{0,6}\times\overline{M}_{0,4}$ $NZ = (\text{as above})\cdot(\lambda_3-\lambda_1) = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\frac{(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1-\lambda_3)} \qquad \int_{\overline{M}_{0,4}}\frac{1}{(\lambda_3-\lambda_1-\psi_1)(\lambda_3+\lambda_1-\psi_2)} = \frac{2\lambda_3}{(\lambda_1^2-\lambda_3^2)^2}$$

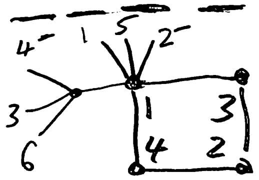
$$C111j3 = \frac{-3}{4}\cdot\frac{(3\lambda_1+\lambda_3)^3}{\lambda_1\lambda_2(\lambda_1^2-\lambda_3^2)^9(\lambda_1-\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{48\,\frac{\lambda_1^2\lambda_2^2(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^3(\lambda_1-\lambda_3)}}$$



(×3) $\overline{M}_{0,3}\times\overline{M}_{0,6}\times\overline{M}_{0,3}$ $NZ = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\frac{1}{\lambda_3-\lambda_1} \qquad \text{as above} \qquad \frac{1}{\lambda_3^2-\lambda_1^2}$$

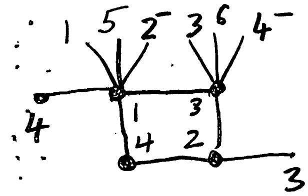
$$C111k3 = \frac{-3}{8}\cdot\frac{(3\lambda_1+\lambda_3)^3}{\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^8(\lambda_1-\lambda_2)^2}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{24\,\frac{\lambda_1^2\lambda_2(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_2)^2}}$$



$\overline{M}_{0,4}\times\overline{M}_{0,6}\times\overline{M}_{0,2}$ $NZ = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\frac{1}{(\lambda_1-\lambda_3)^2} \qquad \frac{(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1-\lambda_3)} \qquad \frac{1}{2\lambda_3}$$

$$C111l3 = \frac{-1}{16}\cdot\frac{(3\lambda_1+\lambda_3)^3}{\lambda_1\lambda_3^3(\lambda_1^2-\lambda_3^2)^7(\lambda_1-\lambda_3)^3}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{4\,\frac{\lambda_1^2(3\lambda_1+\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_3)^3}}$$



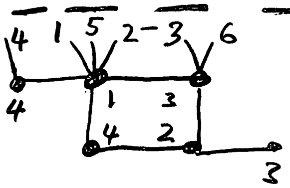
$\overline{M}_{0,6} \times \overline{M}_{0,5}$

$$NZ = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_3)(\lambda_3^2-\lambda_1^2)\cdot(-2\lambda_3)(-\lambda_3+\lambda_1) = -8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^2(\lambda_1-\lambda_3)$$

$$\int_{\overline{M}_{0,6}} \frac{1}{(\lambda_1-\lambda_3-\psi_1)(\lambda_1+\lambda_3-\psi_2)(\lambda_1+\lambda_3-\psi_3)} = \frac{(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1+\lambda_3)}$$ similar to p19, #1

$$\int_{\overline{M}_{0,5}} = -\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$$ as on p8, #1

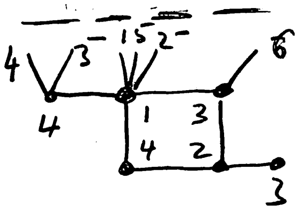
$$C111i4 = \frac{1}{2}\cdot\frac{(3\lambda_1-\lambda_3)^3}{\lambda_1(\lambda_1^2-\lambda_3^2)^{10}} \times (-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \Rightarrow \boxed{-32\,\frac{\lambda_1^2\lambda_3^3(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4}}$$



$\times(-3)$ $\quad \overline{M}_{0,2}\times\overline{M}_{0,6}\times\overline{M}_{0,4}$ $\quad NZ = (\text{as above})\cdot(-\lambda_3-\lambda_1) = 8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\overline{M}_{0,6} \to \frac{(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1+\lambda_3)} \qquad \overline{M}_{0,4} \to \frac{2\lambda_3}{(\lambda_1^2-\lambda_3^2)^2}$$ as p19, #2

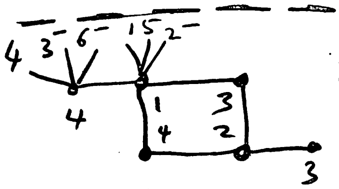
$$C111j4 = \frac{-3}{4}\cdot\frac{(3\lambda_1-\lambda_3)^3}{\lambda_1\lambda_3(\lambda_1^2-\lambda_3^2)^9(\lambda_1+\lambda_3)} \times (-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \Rightarrow \boxed{48\,\frac{\lambda_1^2\lambda_3^2(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)}}$$



$\times(3)$ $\quad \overline{M}_{0,3}\times\overline{M}_{0,6}\times\overline{M}_{0,3}$ $\quad NZ = 8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\overline{M}_{0,3} \to -\frac{1}{\lambda_1+\lambda_3} \qquad \overline{M}_{0,6} \to \frac{(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1+\lambda_3)} \qquad \overline{M}_{0,3} \to -\frac{1}{\lambda_1^2-\lambda_3^2}$$

$$C111k4 = \frac{3}{8}\cdot\frac{(3\lambda_1-\lambda_3)^3}{\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^8(\lambda_1+\lambda_3)^2} \times (-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \Rightarrow \boxed{-24\,\frac{\lambda_1^2\lambda_2(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^2(\lambda_1+\lambda_3)^2}}$$



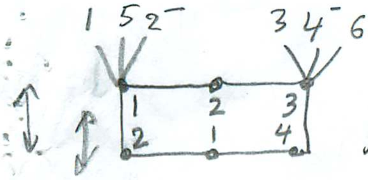
$\times(-1)$ $\quad \overline{M}_{0,4}\times\overline{M}_{0,6}\times\overline{M}_{0,2}$ $\quad NZ = 8\lambda_1\lambda_3^2(\lambda_1^2-\lambda_3^2)^3$

$$\overline{M}_{0,4} \to \frac{1}{(\lambda_1+\lambda_3)^2} \qquad \overline{M}_{0,6} \to \frac{(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)^4(\lambda_1+\lambda_3)} \qquad \overline{M}_{0,2} \to \frac{1}{2\lambda_3}$$

$$C111l4 = \frac{-1}{16}\cdot\frac{(3\lambda_1-\lambda_3)^3}{\lambda_1\lambda_3^3(\lambda_1^2-\lambda_3^2)^7(\lambda_1+\lambda_3)^3} \times (-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 = \boxed{4\,\frac{\lambda_1^2(3\lambda_1-\lambda_3)^3}{(\lambda_1^2-\lambda_3^2)(\lambda_1+\lambda_3)^3}}$$

C111il = sum of 12 terms on pp 8-10

C111il + $(\lambda_1 \leftrightarrow \lambda_3)$ $\longrightarrow$ C111ilT



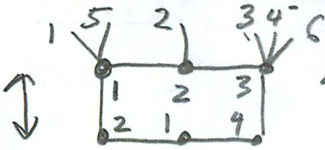
$$\overline{M}_{0,5}\times\overline{M}_{0,2}\times\overline{M}_{0,5} \longrightarrow \frac{1}{-2\lambda_1-\lambda_1-\lambda_3} = -\frac{1}{3\lambda_1+\lambda_3}$$

$$NZ = 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_1)(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2) = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\int_{\overline{M}_{0,5}} \frac{1}{(2\lambda_1-\psi_1)(2\lambda_1-\psi_2)} = \frac{1}{4\lambda_1^2}\left(\frac{1}{2\lambda_1}+\frac{1}{2\lambda_1}\right)^2 = \frac{1}{4\lambda_1^4}$$

$$\int_{\overline{M}_{0,5}} \frac{1}{(\lambda_3+\lambda_1-\psi_1)(2\lambda_3-\psi_2)} = \frac{(\lambda_1+3\lambda_3)^2}{8\lambda_3^3(\lambda_1+\lambda_3)^3}$$

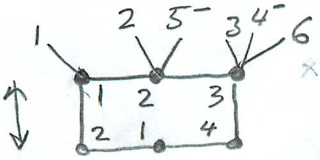
$$C111m0 = \frac{-1}{4\cdot 64}\cdot\frac{(\lambda_1+3\lambda_3)^2}{\lambda_1^6\lambda_3^4(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^3(3\lambda_1+\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \longrightarrow \boxed{\frac{1}{4}\cdot\frac{(\lambda_1-\lambda_3)^3(\lambda_1+3\lambda_3)^2}{\lambda_1^3\lambda_3(3\lambda_1+\lambda_3)}}$$



$$\times(-3)\quad \overline{M}_{0,4}\times\overline{M}_{0,3}\times\overline{M}_{0,5} \qquad NZ = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\frac{1}{4\lambda_1^3} \qquad \frac{1}{(-2\lambda_1)(-\lambda_1-\lambda_3)} = \frac{1}{2\lambda_1(\lambda_1+\lambda_3)} \qquad \frac{(\lambda_1+3\lambda_3)^2}{8\lambda_3^3(\lambda_1+\lambda_3)^3}$$

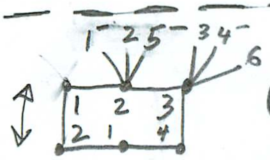
$$C111m1 = \frac{-3}{8\cdot 64}\cdot\frac{(\lambda_1+3\lambda_3)^2}{\lambda_1^6\lambda_3^4(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^4}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \longrightarrow \boxed{\frac{3}{8}\,\frac{(\lambda_1-\lambda_3)^3(\lambda_1+3\lambda_3)^2}{\lambda_1^3\lambda_3(\lambda_1+\lambda_3)}}$$



$$\times 3\quad \overline{M}_{0,3}\times\overline{M}_{0,4}\times\overline{M}_{0,5} \qquad NZ = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\frac{1}{4\lambda_1^2} \qquad -\frac{3\lambda_1+\lambda_3}{4\lambda_1^2(\lambda_1+\lambda_3)^2} \qquad \frac{(\lambda_1+3\lambda_3)^2}{8\lambda_3^3(\lambda_1+\lambda_3)^3}$$

$$C111m2 = \frac{-3}{16\cdot 64}\cdot\frac{(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)^2}{\lambda_1^6\lambda_3^4(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^5}\times(-64)\lambda_1^3\lambda_2^3(\lambda_1^2-\lambda_3^2)^6 \longrightarrow \boxed{\frac{3}{16}\cdot\frac{(\lambda_1-\lambda_3)^3(3\lambda_1+\lambda_3)(\lambda_1+3\lambda_3)^2}{\lambda_1^3\lambda_3(\lambda_1+\lambda_3)^2}}$$



$$(-1)\quad \overline{M}_{0,2}\times\overline{M}_{0,5}\times\overline{M}_{0,5} \qquad NZ = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\frac{1}{4\lambda_1} \qquad \frac{(\lambda_1+3\lambda_3)^2}{8\lambda_3^3(\lambda_1+\lambda_3)^3}$$

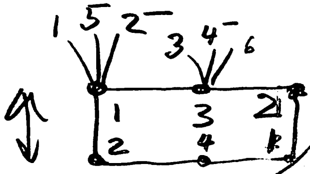
$$\int_{\overline{M}_{0,5}} \frac{1}{(-2\lambda_1-\psi_1)(-\lambda_1-\lambda_3-\psi_2)} = \frac{(3\lambda_1+\lambda_3)^2}{8\lambda_1^3(\lambda_1+\lambda_3)^3}$$

$$C111m3 = \frac{-1}{32\cdot 64}\cdot\frac{(3\lambda_1+\lambda_3)^2(\lambda_1+3\lambda_3)^2}{\lambda_1^6\lambda_3^4(\lambda_1^2-\lambda_3^2)^3(\lambda_1+\lambda_3)^6}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6 \longrightarrow \boxed{\frac{1}{32}\,\frac{(\lambda_1-\lambda_3)^3(3\lambda_1+\lambda_3)^2(\lambda_1+3\lambda_3)^2}{\lambda_1^3\lambda_3(\lambda_1+\lambda_3)^3}}$$

C111m03 = C111m0 + C111m1 + C111m2 + C111m3

C111m03 + ($\lambda_3 \to -\lambda_3$) $\longrightarrow$ C111m03t

C111m03t + ($\lambda_1 \leftrightarrow \lambda_3$) $\longrightarrow$ C111m03T



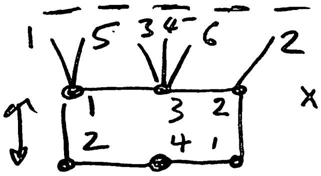
$$\overline{M}_{0,5}\times\overline{M}_{0,5}\times\overline{M}_{0,2}\longrightarrow\frac{-1}{3\lambda_1+\lambda_3}$$

$$NZ=2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_3(\lambda_3^2-\lambda_1^2)\cdot(-2\lambda_1)(\lambda_1^2-\lambda_3^2)=8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\int_{\overline{M}_{0,5}}\frac{1}{(\lambda_3-\lambda_1-\psi_1)(\lambda_3+\lambda_1-\psi_1)}=-\frac{4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3}$$ p8, #1

$$\int_{\overline{M}_{0,5}}\frac{1}{(2\lambda_1-\psi_1)(\lambda_1-\lambda_3-\psi_1)}\longrightarrow\frac{(3\lambda_1-\lambda_3)^2}{8\lambda_1^3(\lambda_1-\lambda_3)^3}$$ similar to p17, #1

$$C111n0=\frac{1}{16}\cdot\frac{\lambda_3(3\lambda_1-\lambda_3)^2}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^6(\lambda_1-\lambda_3)^3(3\lambda_1+\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6\longrightarrow\boxed{-4\,\frac{\lambda_3^4(3\lambda_1-\lambda_3)^2}{\lambda_1^2(\lambda_1-\lambda_3)^3(3\lambda_1+\lambda_3)}}$$



$\times(-3)$

$$\overline{M}_{0,4}\times\overline{M}_{0,5}\times\overline{M}_{0,3}\qquad NZ=8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

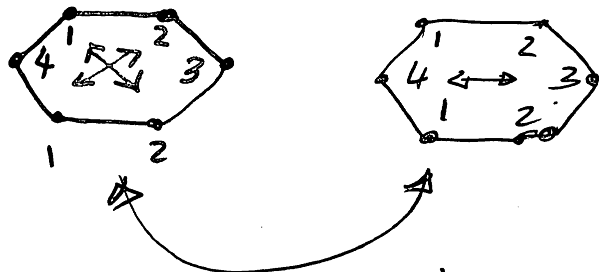
$$\overline{M}_{0,4}\to\frac{3\lambda_1-\lambda_3}{4\lambda_1^2(\lambda_1-\lambda_3)^2},\qquad \overline{M}_{0,5}\to\frac{-4\lambda_3^2}{(\lambda_1^2-\lambda_3^2)^3},\qquad \overline{M}_{0,3}\to\frac{1}{2\lambda_1(\lambda_1+\lambda_3)}$$

$$C111n1=\frac{3}{16}\cdot\frac{\lambda_3(3\lambda_1-\lambda_3)}{\lambda_1^5(\lambda_1^2-\lambda_3^2)^7(\lambda_1-\lambda_3)}\times(-64)\lambda_1^3\lambda_3^3(\lambda_1^2-\lambda_3^2)^6\longrightarrow\boxed{-12\,\frac{\lambda_3^4(3\lambda_1-\lambda_3)}{\lambda_1^2(\lambda_1^2-\lambda_3^2)(\lambda_1-\lambda_3)}}$$

C111n01 = C111n0 + C111n1

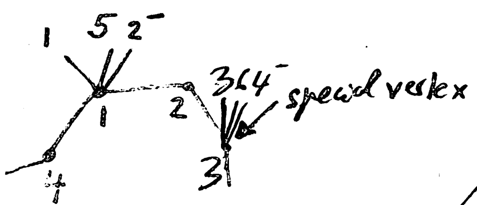
C111n01 + $(\lambda_3\longrightarrow-\lambda_3)\longrightarrow$ C111n01t

C111n01t + $(\lambda_1\leftrightarrow\lambda_3)\longrightarrow$ C111n01T



Opposite sign

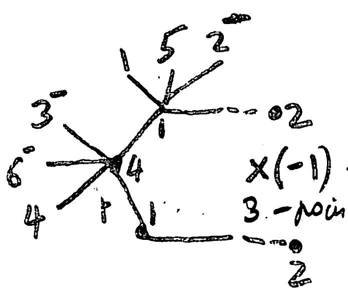
Example of Cancellation



$$N\chi = (-2\lambda_3)(\lambda_3^2-\lambda_1^2)\cdot 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot(-2\lambda_1)(\lambda_1^2-\lambda_3^2) = -8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\overline{M}_{0,5}\times\overline{M}_{0,2}\times\overline{M}_{0,5}$$

$$\int_{\overline{M}_{0,5}}\frac{1}{(2\lambda_1-\psi_1)(\lambda_1+\lambda_3-\psi_2)} \qquad -\frac{1}{3\lambda_1+\lambda_3} \qquad \int_{\overline{M}_{0,5}}\frac{1}{(\lambda_3+\lambda_1-\psi_1)(\lambda_3+\lambda_1-\psi_2)}$$



×(−1) for 3-points

$$N\chi = (-2\lambda_3)(\lambda_3^2-\lambda_1^2)\cdot 2\lambda_1(\lambda_1^2-\lambda_3^2)\cdot 2\lambda_1(\lambda_1^2-\lambda_3^2) = 8\lambda_1^2\lambda_3(\lambda_1^2-\lambda_3^2)^3$$

$$\overline{M}_{0,5}\times\overline{M}_{0,2}\times\overline{M}_{0,5}$$

$$\int_{\overline{M}_{0,5}}\frac{1}{(2\lambda_1-\psi_1)(\lambda_1+\lambda_3-\psi_2)} \qquad \frac{1}{3\lambda_1+\lambda_3} \qquad \int_{\overline{M}_{0,5}}\frac{1}{(-\lambda_3-\lambda_1-\psi_1)(-\lambda_3-\lambda_1-\psi_2)}$$

$$\frac{1}{(\lambda_3+\lambda_1+\psi_1)(\lambda_3+\lambda_1+\psi_2)} \xrightarrow{\deg 2} \text{same sign as above}$$

∴ Cancel

```
In[1]:= num = -64 x^3 y^3 (x^2 - y^2)^6;

D22 = 4 x^3 y^3 (x^2 - y^2)^2;
Normal[Series[1 / ((x - t * z1) (x - t * z2) (x - y - t * z3)), {t, 0, 3}]];
I22 = 1 / (x - y)^2 *
  Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
     z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C22 = Simplify[(1 / 4) * num * I22 / D22]
C22t = Factor[C22 + ((C22 /. {y → -z}) /. {z → y})]
C22T = Factor[C22t + ((C22t /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[4]= $$\frac{(3x-2y)^3}{x^5 (x-y)^6}$$

Out[5]= $$-\frac{4 (3x-2y)^3 (x+y)^4}{x^5 (x-y)^2}$$

Out[6]= $$-\frac{8 \left(3x^2-y^2\right) \left(9x^6+42x^4y^2-47x^2y^4+12y^6\right)}{x^4 (x-y)^2 (x+y)^2}$$

Out[7]= $$\left(8 \left(12x^{12}-83x^{10}y^2+156x^8y^4-234x^6y^6+156x^4y^8-83x^2y^{10}+12y^{12}\right)\right) / \left(x^4 y^4 (-x+y)^2 (x+y)^2\right)$$

```
In[8]:= D31 = -32 / 9 x^3 y (x^2 - y^2) ^2 (x^2 / 9 - y^2);
Normal[Series[1 / ((2 x - t * z1) (2 x / 3 - t * z2) (x - y - t * z3)), {t, 0, 3}]];
I31a = 1 / (x - y) ^2 *
  Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C31a = Simplify[(1 / 3) * num * I31a / D31]

Normal[Series[1 / ((2 x / 3 - t * z1) (x - y - t * z2)), {t, 0, 2}]];
I31b1 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((2 y - t * z1) (y - x - t * z2)), {t, 0, 2}]];
I31b2 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C31b = Simplify[(1 / 3) * num * I31b1 * I31b2 / D31]

C31 = Factor[C31a + C31b]
C31t = Factor[C31 + ((C31 /. {y → -z}) /. {z → y})]
C31T = Factor[C31t + ((C31t /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[10]= $$\frac{3\,(3x-2y)^3}{4x^5\,(x-y)^6}$$

Out[11]= $$\frac{81\,(3x-2y)^3\,y^2\,(x+y)^4}{2x^5\,(x-y)^2\,\left(x^2-9y^2\right)}$$

Out[13]= $$\frac{3\,(5x-3y)^2}{8x^3\,(x-y)^3}$$

Out[15]= $$\frac{(x-3y)^2}{8y^3\,(-x+y)^3}$$

Out[16]= $$-\frac{81\,(x-3y)\,(5x-3y)^2\,(x+y)^4}{32x^3\,(x-y)^2\,y\,(x+3y)}$$

Out[17]= $$-\left(\left(81\,(x+y)^4\,\left(25x^5-155x^4y+259x^3y^2-497x^2y^3+448xy^4-128y^5\right)\right)\Big/\left(32x^5\,(x-3y)\,(x-y)\,y\,(x+3y)\right)\right)$$

Out[18]= $$\frac{81\,\left(15x^8+251x^6y^2+509x^4y^4-487x^2y^6+96y^8\right)}{8x^4\,(x-3y)\,(x-y)\,(x+y)\,(x+3y)}$$

Out[19]= $$\left(81\,\left(96x^{12}-1255x^{10}y^2+3772x^8y^4+1686x^6y^6+3772x^4y^8-1255x^2y^{10}+96y^{12}\right)\right)\Big/\left(8x^4y^4\,(-3x+y)\,(3x+y)\,(-x+3y)\,(x+3y)\right)$$

In[20]:=
```
D12a = 1 / 4 x y (x^2 - y^2) ^2 (x - y) ^2 (3 x + y) (x + 3 y);
Normal[Series[1 / ((2 x - t * z1) ((x - y) / 2 - t * z2)), {t, 0, 2}]];
I12a1 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((2 y - t * z1) ((y - x) / 2 - t * z2)), {t, 0, 2}]];
I12a2 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C12a = Simplify[(1 / 2) * num * I12a1 * I12a2 / D12a]

D12b = -D12a;
Normal[Series[1 / ((x + y - t * z1) ((x - y) / 2 - t * z2)), {t, 0, 2}]];
I12b1 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((y + x - t * z1) ((y - x) / 2 - t * z2)), {t, 0, 2}]];
I12b2 = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C12b = Simplify[(1 / 2) * num * I12b1 * I12b2 / D12b]

C12 = Factor[C12a + C12b]
C12T = Factor[C12 + ((C12 /. {y → -z}) /. {z → y})]
```

Out[22]= $$\frac{(-5x+y)^2}{4x^3(x-y)^3}$$

Out[24]= $$\frac{(x-5y)^2}{4y^3(-x+y)^3}$$

Out[25]= $$\frac{8(x-5y)^2(-5x+y)^2(x+y)^4}{x(x-y)^4 y(3x+y)(x+3y)}$$

Out[28]= $$\frac{2(3x+y)^2}{(x-y)^3(x+y)^3}$$

Out[30]= $$-\frac{2(x+3y)^2}{(x-y)^3(x+y)^3}$$

Out[31]= $$-\frac{512x^2y^2(3x+y)(x+3y)}{(x-y)^4(x+y)^2}$$

Out[32]= $$\left(8\left(25x^8-60x^7y-604x^6y^2-1028x^5y^3-762x^4y^4-1028x^3y^5-604x^2y^6-60xy^7+25y^8\right)\right)/\left(x(x-y)^2y(x+y)^2(3x+y)(x+3y)\right)$$

Out[33]= $$-\frac{32\left(215x^8-1388x^6y^2-726x^4y^4-1388x^2y^6+215y^8\right)}{(x-3y)(x-y)^2(3x-y)(x+y)^2(3x+y)(x+3y)}$$

In[47]:=
```
D111a = 8 x^2 y (x^2 - y^2) ^3;
I111a = -1 / (16 x^6 (x + y) ^3);
C111a = Simplify[num * I111a / D111a]

I111b = (2 x + y) / (16 x^6 (x + y) ^4);
C111b = Simplify[(-3) num * I111b / D111a]

Normal[Series[1 / ((-2 x - t * z1) (-2 x - t * z2) (-x - y - t * z3)), {t, 0, 2}]];
I111c = 1 / (4 x^2 (x + y) ^2) * Simplify[Expand[Coefficient[%, t, 2]] /.
    {z1^2 → 1, z2^2 → 1, z3^2 → 1, z1 z2 → 2, z1 z3 → 2, z2 z3 → 2}]
C111c = Simplify[3 num * I111c / D111a]

Normal[Series[1 / ((-2 x - t * z1) (-2 x - t * z2) (-x - y - t * z3)), {t, 0, 3}]];
I111d = 1 / (4 x (x + y) ^2) *
  Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
     z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111d = Simplify[-num * I111d / D111a]

C111ad = Factor[C111a + C111b + C111c + C111d]
C111adt = Factor[C111ad + ((C111ad /. {y → -z}) /. {z → y})]
C111adT = Factor[C111adt + ((C111adt /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[49]= $\frac{(x-y)^3 y^2}{2 x^5}$

Out[51]= $\frac{3 (x-y)^3 y^2 (2 x+y)}{2 x^5 (x+y)}$

Out[53]= $-\frac{(2 x+y)^2}{16 x^6 (x+y)^5}$

Out[54]= $\frac{3 (x-y)^3 y^2 (2 x+y)^2}{2 x^5 (x+y)^2}$

Out[56]= $\frac{(2 x+y)^3}{16 x^6 (x+y)^6}$

Out[57]= $\frac{(x-y)^3 y^2 (2 x+y)^3}{2 x^5 (x+y)^3}$

Out[58]= $\frac{(x-y)^3 y^2 (3 x+2 y)^3}{2 x^5 (x+y)^3}$

Out[59]= $\frac{y^2 \left(3 x^2-y^2\right) \left(9 x^6+42 x^4 y^2-47 x^2 y^4+12 y^6\right)}{x^4 (x-y)^3 (x+y)^3}$

Out[60]= $\left(12 x^{12}-71 x^{10} y^2+112 x^8 y^4+22 x^6 y^6+112 x^4 y^8-71 x^2 y^{10}+12 y^{12}\right) / \left(x^4 (x-y)^2 y^4 (x+y)^2\right)$

In[61]:=
```
D111e1 = -4 x y (x^2 - y^2)^2 * 2 x (x + y);
Normal[Series[1 / ((2 x - t * z1) (x - y - t * z2)), {t, 0, 2}]];
I111e = Simplify[Expand[Coefficient[%, t, 2]] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((2 y - t * z1) (y - x - t * z2) (y - x - t * z3)), {t, 0, 3}]];
I111e1 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111e1 = Simplify[num * I111e * I111e1 / D111e1]

D111f1 = -4 x y (x^2 - y^2)^2 * 2 x (x^2 - y^2);
Normal[Series[1 / ((2 x - t * z1) (x - y - t * z2)), {t, 0, 2}]];
I111f = Simplify[Expand[Coefficient[%, t, 1]] /. {z1 → 1, z2 → 1}]
C111f1 = Simplify[3 num * I111f * I111e1 / D111f1]

I111g1 = 1 / (2 x (x - y)^2);
C111g1 = Simplify[3 num * I111g1 * I111e1 / D111f1]

I111h1 = 1 / ((3 x - y) (x - y)^2);
C111h1 = Simplify[num * I111h1 * I111e1 / D111f1]
```

Out[63]= $\frac{(-3\,x+y)^2}{8\,x^3\,(x-y)^3}$

Out[65]= $\frac{(x-5\,y)^3}{16\,(x-y)^5\,y^4}$

Out[66]= $\frac{(x-5\,y)^3\,(-3\,x+y)^2\,(x+y)^3}{16\,x^2\,(x-y)^4\,y^2}$

Out[69]= $-\frac{-3\,x+y}{4\,x^2\,(x-y)^2}$

Out[70]= $\frac{3\,(x-5\,y)^3\,(3\,x-y)\,(x+y)^3}{8\,x\,(x-y)^4\,y^2}$

Out[72]= $\frac{3\,(x-5\,y)^3\,(x+y)^3}{4\,(x-y)^4\,y^2}$

Out[74]= $\frac{x\,(x-5\,y)^3\,(x+y)^3}{2\,(x-y)^4\,(3\,x-y)\,y^2}$

In[75]:=
```
D111e2 = -4 x y (x^2 - y^2)^2 * (-2 x) (-x + y);
Normal[Series[1 / ((2 y - t * z1) (y - x - t * z2) (y + x - t * z3)), {t, 0, 3}]];
I111e2 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111e2 = Simplify[num * I111e * I111e2 / D111e2]

D111f2 = -4 x y (x^2 - y^2)^2 * (-2 x) (x^2 - y^2);
C111f2 = Simplify[-3 num * I111f * I111e2 / D111f2]

I111g2 = -1 / (2 x (x^2 - y^2));
C111g2 = Simplify[3 num * I111g2 * I111e2 / D111f2]

I111h2 = 1 / ((3 x - y) (x + y)^2);
C111h2 = Simplify[-num * I111h2 * I111e2 / D111f2]
```

Out[77]= $-\frac{\left(x^2-5 y^2\right)^3}{16 (x-y)^4 y^4 (x+y)^4}$

Out[78]= $-\frac{(-3 x+y)^2 \left(x^2-5 y^2\right)^3}{16 x^2 (x-y)^4 y^2}$

Out[80]= $-\frac{3 (3 x-y) \left(x^2-5 y^2\right)^3}{8 x (x-y)^3 y^2 (x+y)}$

Out[82]= $-\frac{3 \left(x^2-5 y^2\right)^3}{4 (x-y)^2 y^2 (x+y)^2}$

Out[84]= $-\frac{x \left(x^2-5 y^2\right)^3}{2 y^2 (x+y)^3 \left(3 x^2-4 x y+y^2\right)}$

In[85]:=
```
D111e4 = -4 x y (x^2 - y^2)^2 * (y^2 - x^2);
Normal[Series[1 / ((2 y - t * z1) (y - x - t * z2) (y + y - t * z3)), {t, 0, 3}]];
I111e4 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111e4 = Simplify[num * I111e * I111e4 / D111e4]

D111f4 = -4 x y (x^2 - y^2)^2 * (-2 y) (y^2 - x^2);
Normal[Series[1 / ((2 y - t * z1) (y - x - t * z2) (y + y - t * z3)), {t, 0, 3}]];
I111f4 = Simplify[Expand[Coefficient[%, t, 2]] /.
   {z1^2 → 1, z2^2 → 1, z3^2 → 1, z1 z2 → 2, z1 z3 → 2, z2 z3 → 2}]
C111f4 = Simplify[-3 num * I111e * I111f4 / D111f4]

I111g4 = (x - 2 y) / (8 y^4 (x - y)^2);
C111g4 = Simplify[3 num * I111e * I111g4 / D111f4]

I111h4 = -1 / (16 y^4 (x - y));
C111h4 = Simplify[-num * I111e * I111h4 / D111f4]

C111eh = Simplify[Expand[C111e1 + C111f1 + C111g1 + C111h1 +
    C111e2 + C111f2 + C111g2 + C111h2 + C111e4 + C111f4 + C111g4 + C111h4]]
C111eht = Factor[C111eh + ((C111eh /. {y → -z}) /. {z → y})]
C111ehT = Factor[C111eht + ((C111eht /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[87]= $-\frac{(x-2y)^3}{4(x-y)^4 y^5}$

Out[88]= $\frac{(x-2y)^3(-3x+y)^2(x+y)^3}{2x(x-y)^4 y^3}$

Out[91]= $-\frac{(x-2y)^2}{4(x-y)^3 y^4}$

Out[92]= $\frac{3(x-2y)^2(-3x+y)^2(x+y)^3}{4x(x-y)^3 y^3}$

Out[94]= $\frac{3(x-2y)(-3x+y)^2(x+y)^3}{8x(x-y)^2 y^3}$

Out[96]= $\frac{(-3x+y)^2(x+y)^3}{16x(x-y)y^3}$

Out[97]= $\left(729x^{12} - 6546x^{10}y^2 + 696x^9y^3 + 29147x^8y^4 + 1248x^7y^5 - 94060x^6y^6 - 129520x^5y^7 - 75489x^4y^8 - 3488x^3y^9 + 14174x^2y^{10} + 2040xy^{11} - 1075y^{12}\right) / \left(16x(x-y)^4(3x-y)y^3(x+y)^3\right)$

Out[98]= $\left(729x^{12} - 6024x^{10}y^2 + 30257x^8y^4 - 190888x^6y^6 - 110485x^4y^8 + 14832x^2y^{10} - 565y^{12}\right) / \left(2(x-y)^4(3x-y)y^2(x+y)^4(3x+y)\right)$

Out[99]= $$\left(729\, x^{16} - 7500\, x^{14}\, y^2 - 49\,580\, x^{12}\, y^4 + 545\,996\, x^{10}\, y^6 + 3\,215\,014\, x^8\, y^8 + 545\,996\, x^6\, y^{10} - 49\,580\, x^4\, y^{12} - 7500\, x^2\, y^{14} + 729\, y^{16}\right) \Big/ \left(2\, x^2\, (x - 3\, y)\, (x - y)^4\, (3\, x - y)\, y^2\, (x + y)^4\, (3\, x + y)\, (x + 3\, y)\right)$$

In[100]:=
```
D111i2 = 4 x y (x^2 - y^2)^2 * (x^2 - y^2);
Normal[Series[1 / ((y - x - t * z1) (y + x - t * z2)), {t, 0, 2}]];
I111i = Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((x - y - t * z1) (x + y - t * z2) (2 x - t * z3)), {t, 0, 3}]];
I111i2 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111i2 = Simplify[num * I111i * I111i2 / D111i2]

D111j2 = 4 x y (x^2 - y^2)^2 * (x^2 - y^2) * (-2 x);
Normal[Series[1 / ((x - y - t * z1) (x + y - t * z2) (2 x - t * z3)), {t, 0, 2}]];
I111j2 = Simplify[Expand[Coefficient[%, t, 2]] /.
   {z1^2 → 1, z2^2 → 1, z3^2 → 1, z1 z2 → 2, z1 z3 → 2, z2 z3 → 2}]
C111j2 = Simplify[(-3) num * I111i * I111j2 / D111j2]

Normal[Series[1 / ((x - y - t * z1) (x + y - t * z2) (2 x - t * z3)), {t, 0, 2}]];
I111k2 = -1 / (2 x) * Simplify[Expand[Coefficient[%, t, 1]] /. {z1 → 1, z2 → 1, z3 → 1}]
C111k2 = Simplify[3 num * I111i * I111k2 / D111j2]

I111l2 = 1 / (8 x^3 (x^2 - y^2));
C111l2 = Simplify[-num * I111i * I111l2 / D111j2]
```

Out[102]= $\frac{4\,y^2}{(-x+y)^3\,(x+y)^3}$

Out[104]= $\frac{\left(5\,x^2-y^2\right)^3}{16\,x^4\,(x-y)^4\,(x+y)^4}$

Out[105]= $-\frac{4\,y^4\,\left(-5\,x^2+y^2\right)^3}{x^2\,(x-y)^4\,(x+y)^4}$

Out[108]= $\frac{\left(-5\,x^2+y^2\right)^2}{8\,x^3\,(x-y)^3\,(x+y)^3}$

Out[109]= $\frac{12\,y^4\,\left(-5\,x^2+y^2\right)^2}{x^2\,(x-y)^3\,(x+y)^3}$

Out[111]= $\frac{-5\,x^2+y^2}{8\,x^3\,(x-y)^2\,(x+y)^2}$

Out[112]= $-\frac{12\,y^4\,\left(-5\,x^2+y^2\right)}{x^2\,(x-y)^2\,(x+y)^2}$

Out[114]= $\frac{4\,y^4}{x^4-x^2\,y^2}$

In[115]:=
```
D111i3 = 4 x y (x^2 - y^2)^2 * 2 y (y + x);
Normal[Series[1 / ((x - y - t * z1) (x + y - t * z2) (x - y - t * z3)), {t, 0, 3}]];
I111i3 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111i3 = Simplify[num * I111i * I111i3 / D111i3]

D111j3 = 4 x y (x^2 - y^2)^2 * 2 y (y + x) (y - x);
I111j3 = 2 y / (x^2 - y^2)^2;
C111j3 = Simplify[3 num * I111j3 * I111i3 / D111j3]

I111k3 = 1 / ((x^2 - y^2) (x - y));
C111k3 = Simplify[3 num * I111k3 * I111i3 / D111j3]

I111l3 = 1 / (2 y (x - y)^2);
C111l3 = Simplify[num * I111l3 * I111i3 / D111j3]
```

Out[117]= $\frac{(3x+y)^3}{(x-y)^5 (x+y)^4}$

Out[118]= $\frac{32 x^2 y^3 (3x+y)^3}{(x-y)^4 (x+y)^4}$

Out[121]= $\frac{48 x^2 y^2 (3x+y)^3}{(x-y)^4 (x+y)^3}$

Out[123]= $\frac{24 x^2 y (3x+y)^3}{(x-y)^4 (x+y)^2}$

Out[125]= $\frac{4 x^2 (3x+y)^3}{(x-y)^4 (x+y)}$

In[126]:=
```
D111i4 = 4 x y (x^2 - y^2)^2 * (-2 y) (-y + x);
Normal[Series[1 / ((x - y - t * z1) (x + y - t * z2) (x + y - t * z3)), {t, 0, 3}]];
I111i4 =
 Simplify[Expand[Coefficient[%, t, 3]] /. {z1^3 → 1, z2^3 → 1, z3^3 → 1, z1^2 z2 → 3,
    z1^2 z3 → 3, z2^2 z1 → 3, z2^2 z3 → 3, z3^2 z1 → 3, z3^2 z2 → 3, z1 z2 z3 → 6}]
C111i4 = Simplify[num * I111i * I111i4 / D111i4]

D111j4 = 4 x y (x^2 - y^2)^2 * (-2 y) (-y + x) (-y - x);
I111j4 = 2 y / (x^2 - y^2)^2;
C111j4 = Simplify[-3 num * I111i4 * I111j4 / D111j4]

I111k4 = 1 / ((x^2 - y^2) (x + y));
C111k4 = Simplify[3 num * I111i4 * I111k4 / D111j4]

I111l4 = 1 / (2 y (x + y)^2);
C111l4 = Simplify[-num * I111i4 * I111l4 / D111j4]

C111il = Simplify[Expand[C111i2 + C111j2 + C111k2 + C111l2 +
    C111i3 + C111j3 + C111k3 + C111l3 + C111i4 + C111j4 + C111k4 + C111l4]]
C111ilT = Factor[C111il + ((C111il /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[128]= $\frac{(3x-y)^3}{(x-y)^4 (x+y)^5}$

Out[129]= $-\frac{32 x^2 (3x-y)^3 y^3}{(x-y)^4 (x+y)^4}$

Out[132]= $\frac{48 x^2 (3x-y)^3 y^2}{(x-y)^3 (x+y)^4}$

Out[134]= $-\frac{24 x^2 (3x-y)^3 y}{(x-y)^2 (x+y)^4}$

Out[136]= $\frac{4 x^2 (3x-y)^3}{(x-y) (x+y)^4}$

Out[137]= $\frac{8 \left(27 x^{10} + 981 x^8 y^2 + 1089 x^6 y^4 - 81 x^4 y^6 + 36 x^2 y^8 - 4 y^{10}\right)}{x^2 (x-y)^4 (x+y)^4}$

Out[138]= $-\left(\left(8 \left(4 x^{12} - 63 x^{10} y^2 - 900 x^8 y^4 - 2178 x^6 y^6 - 900 x^4 y^8 - 63 x^2 y^{10} + 4 y^{12}\right)\right) / \left(x^2 (x-y)^4 y^2 (x+y)^4\right)\right)$

In[139]:=
```
D111m = 8 x^2 y (x^2 - y^2) ^3;
Normal[Series[1 / ((y + x - t * z1) (2 y - t * z2)), {t, 0, 2}]];
I111m = Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((2 x - t * z1) (2 x - t * z2)), {t, 0, 2}]];
I111m0 =
 -1 / (3 x + y) * Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C111m0 = Simplify[num * I111m * I111m0 / D111m]

Normal[Series[1 / ((2 x - t * z1) (2 x - t * z2)), {t, 0, 2}]];
I111m1 = 1 / (2 x (x + y)) * Simplify[Coefficient[%, t, 1] /. {z1 → 1, z2 → 1}]
C111m1 = Simplify[- 3 num * I111m * I111m1 / D111m]

Normal[Series[1 / ((-2 x - t * z1) (-x - y - t * z2)), {t, 0, 2}]];
I111m2 = 1 / (4 x^2) * Simplify[Coefficient[%, t, 1] /. {z1 → 1, z2 → 1}]
C111m2 = Simplify[3 num * I111m * I111m2 / D111m]

Normal[Series[1 / ((-2 x - t * z1) (-x - y - t * z2)), {t, 0, 2}]];
I111m3 = 1 / (4 x) * Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C111m3 = Simplify[- num * I111m * I111m3 / D111m]

C111m03 = Simplify[Expand[C111m0 + C111m1 + C111m2 + C111m3]]
C111m03t = Factor[C111m03 + ((C111m03 /. {y → -z}) /. {z → y})]
C111m03T = Factor[C111m03t + ((C111m03t /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[141]= $\frac{(x+3y)^2}{8y^3(x+y)^3}$

Out[143]= $-\frac{1}{4x^4(3x+y)}$

Out[144]= $\frac{(x-y)^3(x+3y)^2}{4x^3y(3x+y)}$

Out[146]= $\frac{1}{8x^4(x+y)}$

Out[147]= $\frac{3(x-y)^3(x+3y)^2}{8x^3y(x+y)}$

Out[149]= $-\frac{3x+y}{16x^4(x+y)^2}$

Out[150]= $\frac{3(x-y)^3(3x+y)(x+3y)^2}{16x^3y(x+y)^2}$

Out[152]= $\frac{(3x+y)^2}{32x^4(x+y)^3}$

Out[153]= $\frac{(x-y)^3(3x+y)^2(x+3y)^2}{32x^3y(x+y)^3}$

Out[154]= $$\frac{(x+3y)^2\left(5x^2-2xy-3y^2\right)^3}{32x^3y(x+y)^3(3x+y)}$$

Out[155]= $$\left(275x^{10}-327x^8y^2-7986x^6y^4+8834x^4y^6-3249x^2y^8+405y^{10}\right)\Big/\left(8x^2(x-y)^3(3x-y)(x+y)^3(3x+y)\right)$$

Out[156]= $$\left(3645x^{12}-25\,726x^{10}y^2+54\,227x^8y^4-31\,524x^6y^6+54\,227x^4y^8-25\,726x^2y^{10}+3645y^{12}\right)\Big/\left(8x^2y^2(-3x+y)(-x+y)^2(x+y)^2(3x+y)(-x+3y)(x+3y)\right)$$

In[157]:=
```
D111n = 8 x^2 y (x^2 - y^2) ^3;
Normal[Series[1 / ((y - x - t * z1) (y + x - t * z2)), {t, 0, 2}]];
I111n = Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
Normal[Series[1 / ((2 x - t * z1) (x - y - t * z2)), {t, 0, 2}]];
I111n0 =
 -1 / (3 x + y) * Simplify[Coefficient[%, t, 2] /. {z1^2 → 1, z2^2 → 1, z1 z2 → 2}]
C111n0 = Simplify[num * I111n * I111n0 / D111n]

Normal[Series[1 / ((2 x - t * z1) (x - y - t * z2)), {t, 0, 2}]];
I111n1 = 1 / (2 x (x + y)) * Simplify[Coefficient[%, t, 1] /. {z1 → 1, z2 → 1}]
C111n1 = Simplify[- 3 num * I111n * I111n1 / D111n]

C111n01 = Simplify[Expand[C111n0 + C111n1]]
C111n01t = Factor[C111n01 + ((C111n01 /. {y → -z}) /. {z → y})]
C111n01T = Factor[C111n01t + ((C111n01t /. {y → z, x → w}) /. {z → x, w → y})]
```

Out[159]= $$\frac{4y^2}{(-x+y)^3(x+y)^3}$$

Out[161]= $$-\frac{(-3x+y)^2}{8x^3(x-y)^3(3x+y)}$$

Out[162]= $$-\frac{4y^4(-3x+y)^2}{x^2(x-y)^3(3x+y)}$$

Out[164]= $$-\frac{-3x+y}{8x^3(x-y)^2(x+y)}$$

Out[165]= $$-\frac{12(3x-y)y^4}{x^2(x-y)^2(x+y)}$$

Out[166]= $$-\frac{16y^4\left(9x^3-6x^2y-2xy^2+y^3\right)}{x^2(x-y)^3(x+y)(3x+y)}$$

Out[167]= $$-\frac{32y^4\left(3x^2-y^2\right)^3}{x^2(x-y)^3(3x-y)(x+y)^3(3x+y)}$$

Out[168]= $$\frac{32\left(9x^{12}-73x^{10}y^2+179x^8y^4-118x^6y^6+179x^4y^8-73x^2y^{10}+9y^{12}\right)}{x^2(x-3y)(x-y)^2(3x-y)y^2(x+y)^2(3x+y)(x+3y)}$$

In[169]:=
```
Tot =
 Simplify[Expand[C22T - C31T + C12T + C111adT + C111ehT + C111ilT + C111m03T + C111n01T]]
```

Out[169]= 4