Real Gromov-Witten Theory in All Genera

Penka Georgieva and Aleksey Zinger

The study of curves in projective varieties has been central to algebraic geometry since the nineteenth century. It was reinvigorated through its introduction into symplectic topology in Gromov's seminal work [4] and now plays prominent roles in symplectic topology and string theory as well. The foundations of (complex) Gromov-Witten invariants, i.e. counts of J-holomorphic curves in symplectic manifolds, were established in the 1990s and have been spectacularly applied ever since. However, there has been much less progress in establishing the foundations of and applying real Gromov-Witten invariants, i.e. counts of J-holomorphic curves in symplectic manifolds preserved by anti-symplectic involutions.

A real symplectic manifold is a triple (X,ω,ϕ) consisting of a symplectic manifold (X,ω) and an anti-symplectic involution ϕ. For such a triple, we denote by $\mathcal{J}_\omega^\phi$ the space of ω-compatible almost complex structures J on X such that $\phi^*J=-J$. The fixed locus X^ϕ of ϕ is then a Lagrangian submanifold of (X,ω) which is totally real with respect to any $J\in\mathcal{J}_\omega^\phi$. The basic example of a real Kahler manifold (X,ω,ϕ,J) is the complex projective space $\mathbb{P}^{n-1}$ with the Fubini-Study symplectic form, the coordinate conjugation

$$\tau_n\colon \mathbb{P}^{n-1}\longrightarrow\mathbb{P}^{n-1}, \qquad \tau_n\big([z_1,\ldots,z_n]\big)=\big[\bar{z}_1,\ldots,\bar{z}_n\big],$$

and the standard complex structure. Another example is a real quintic threefold X_5, i.e. a smooth hypersurface in $\mathbb{P}^4$ cut out by a real equation; it plays a prominent role in the interactions with string theory and algebraic geometry. A symmetric Riemann surface $(\Sigma,\sigma,\mathfrak{j})$ is a connected nodal Riemann surface $(\Sigma,\mathfrak{j})$ with an anti-holomorphic involution σ.

Let (X,ω,ϕ) be a real symplectic manifold, $g,l\in\mathbb{Z}^{\ge0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}_\omega^\phi$. For a symmetric surface (Σ,σ), we denote by

$$\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}\subset\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

the uncompactified moduli space of degree B real J-holomorphic maps from (Σ,σ) to (X,ϕ) with l conjugate pairs of marked points and its stable map compactification. Each codimension-one stratum of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ is either a hypersurface in $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ or a boundary of the spaces $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$ for precisely two topological types of orientation-reversing involutions σ on Σ. Thus, the union of real moduli spaces

$$\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}=\bigcup_\sigma\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi,\sigma}$$

over all topological types of orientation-reversing involutions σ on Σ forms a space without boundary. There is a natural forgetful morphism

$$\mathfrak{f}\colon\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}\longrightarrow\overline{\mathbb{R}\mathcal{M}}_{g,l}$$

to the Deligne-Mumford moduli space of marked real curves. An orientation on $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$ determined by some topological data on (X,ω,ϕ) gives rise to invariants of (X,ω,ϕ) that enumerate real J-holomorphic curves in X, just as happens in the complex Gromov-Witten theory.

The two main obstacles to defining real Gromov-Witten invariants (or any other count of real curves) is the potential non-orientability of the moduli space $\mathfrak{M}_{g,l}(X,B;J)^{\phi,\sigma}$ and the fact that its boundary strata have real codimension one. In contrast, the complex analogues of these spaces are canonically oriented and have boundary of real codimension two. These obstacles were overcome in many genus 0 situations in [5, 6], providing lower bounds for counts of real rational curves in the corresponding settings. In [2], we overcome these obstacles in *all* genera for many real symplectic manifolds.

A real bundle pair (V,φ) over a topological space X with an involution ϕ consists of a complex vector bundle V over X and a conjugation φ on V lifting ϕ. If X is a smooth manifold, then $(TX,\mathrm{d}\phi)$ is a real bundle pair over (X,ϕ). The inspiration for our approach comes in part from the topological classification of real bundle pairs over smooth symmetric surfaces in [1].

Definition ([2, Part I])**.** A real orientation on a real symplectic manifold (X,ω,ϕ) consists of

(RO1) a rank 1 real bundle pair $(L,\tilde{\phi})$ over (X,ϕ) such that
$$w_2(TX^{\phi})=w_1(L^{\tilde{\phi}})^2 \qquad \text{and} \qquad \Lambda^{\mathrm{top}}_{\mathbb{C}}(TX,\mathrm{d}\phi)\approx(L,\tilde{\phi})^{\otimes 2},$$

(RO2) a homotopy class of above isomorphisms of real bundle pairs, and

(RO3) a spin structure on the real vector bundle $TX^{\phi}\oplus 2(L^*)^{\tilde{\phi}^*}$ over X^{ϕ} compatible with the orientation induced by the above homotopy class.

We call a real symplectic manifold (X,ω,ϕ) real-orientable if it admits a real orientation. The examples include $\mathbb{P}^{2n-1}$, X_5, many other projective complete intersections, and simply-connected real symplectic Calabi-Yau and real Kahler Calabi-Yau manifolds with spin fixed locus; see [2, Part III].

Theorem ([2, Part I])**.** *Let (X,ω,ϕ) be a real-orientable $2n$-manifold, $g,l\in\mathbb{Z}^{\geq 0}$, $B\in H_2(X;\mathbb{Z})$, and $J\in\mathcal{J}^{\phi}_{\omega}$.*

(1) If n is odd, a real orientation on (X,ω,ϕ) orients $\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}$.

(2) If n is even, a real orientation on (X,ω,ϕ) orients the real line bundle
$$\Lambda^{\mathrm{top}}_{\mathbb{R}}(\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi})\otimes\mathfrak{f}^*\Lambda^{\mathrm{top}}_{\mathbb{R}}(\overline{\mathbb{R}\mathcal{M}}_{g,l})\longrightarrow\overline{\mathfrak{M}}_{g,l}(X,B;J)^{\phi}.$$

This theorem is fundamentally about orienting tensor products of determinants of Cauchy-Riemann (or CR) operators on real bundle pairs over symmetric surfaces. The most basic such operator is the standard $\bar{\partial}$-operator on the trivial rank 1 real bundle pair over (Σ,σ), denoted by $\bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)}$. The linearization D_u of the $\bar{\partial}_{J,\mathfrak{j}}$-operator at a real $(J,\mathfrak{j})$-holomorphic map u from (Σ,σ) to (X,ϕ) is a CR-operator on the real bundle pair $u^*(TX,\mathrm{d}\phi)$ over (Σ,σ). A key step in our proof is a classification of automorphisms of real bundle pairs over smooth and one-nodal symmetric surfaces; we extend it to arbitrary nodal symmetric surfaces in [3]. It implies that a real orientation on a real bundle pair (V,φ) over (Σ,σ) *determines* a homotopy class of trivializations of $(V\oplus 2L^*,\varphi\oplus 2\tilde{\phi}^*)$. Thus, a real orientation on (X,ω,ϕ) *orients* the tensor product of $(\det D_u)$ and $(\det \bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)})^{\otimes n}$ for every element $[u]$ of $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ in a continuous fashion. Furthermore, the Kodaira-Spencer map *orients* the tensor product of $T_{(\Sigma,\sigma)}\overline{\mathbb{R}\mathcal{M}}_{g,l}$ and $\det \bar{\partial}_{\mathbb{C}}|_{(\Sigma,\sigma)}$ whenever Σ is smooth. The last orientation extends across $\overline{\mathbb{R}\mathcal{M}}_{g,l}$ after being reversed over smooth symmetric surfaces (Σ,σ) with the parity of the number of the components of the fixed locus Σ^σ equal to the parity of the genus g. The tensor product of the resulting orientations on the above two tensor products orients $\overline{\mathfrak{M}}_{g,l}(X,B;J)^\phi$ if n is odd and the line bundle in (2) if n is even.

Our notion of real orientation on (X,ω,ϕ) can be viewed as the real arbitrary-genus analogue of the now standard notion of relative spin structure in the open genus 0 GW-theory. The latter induces orientations on the moduli spaces of J-holomorphic disks and can in some cases be used to orient the moduli spaces of real J-holomorphic maps from $\mathbb{P}^1$ with the standard involution τ_2. In [2, Part II], we show that in these special cases our orientations on these moduli spaces reduce to the orientations induced by the associated relative spin structure up to a topological sign.

As in the complex case, the curve-counting invariants arising from the above theorem are generally rational numbers. For specific real almost Kahler manifolds (X,ω,ϕ,J), they can be converted into signed counts of genus g degree B real J-holomorphic curves passing through specified conjugate pairs of constraints and thus provide lower bounds in real enumerative geometry. If $n=3$ and (X,ω,ϕ,J) is sufficiently positive, e.g. $X=\mathbb{P}^3$, then the $g=1$ real GW-invariants themselves are such signed counts. This is also the case of the real GW-invariants with real points constraints that arise from the above theorem if $g=1$ and $n=3$.

The equivariant localization data needed to compute the real GW-invariants of $\mathbb{P}^{2n-1}$ is described in [2, Part III]. We use it to show that the real genus g degree d GW-invariants with conjugate pairs of constraints vanish whenever $d-g$ is even. We also find that the absolute value of the signed count of real genus 1 degree d curves through d pairs of conjugate points in $\mathbb{P}^3$ is 0 for $d=2$, 1 for $d=4$, and 4 for $d=6$; the details of the last computation appear in [2, Appendix]. These values are consistent with the corresponding complex counts: 0, 1, and 2860.

References

[1] I. Biswas, J. Huisman, and J. Hurtubise, *The moduli space of stable vector bundles over a real algebraic curve*, Math. Ann. 347 (2010), no. 1, 201–233

[2] P. Georgieva and A. Zinger, *Real Gromov-Witten theory in all genera and real enumerative geometry*, I (math/1504.06617), II (preprint), III (in preparation), Appendix (available from the authors' websites)

[3] P. Georgieva and A. Zinger, *On the topology of real bundle pairs over nodal symmetric surfaces*, in preparation

[4] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no 2, 307–347

[5] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. 162 (2005), no. 1, 195–234

[6] J.-Y. Welschinger, *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants*, Duke Math. J. 127 (2005), no. 1, 89-121