Notes on J-Holomorphic Maps

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Abstract

These notes present a systematic treatment of local properties of J-holomorphic maps and of Gromov's convergence for sequences of such maps, specifying the assumptions needed for all statements. In particular, only one auxiliary statement depends on the manifold being symplectic. The content of these notes roughly corresponds to Chapters 2 and 4 of McDuff-Salamon's book on the subject.

Contents

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1 Introduction

Gromov's introduction [6] of pseudoholomorphic curves techniques into symplectic topology has revolutionized this field and led to its numerous connections with algebraic geometry. The ideas put forward in [6] have been further elucidated and developed in [14, 17, 11, 15, 16, 10, 3] and in many other works. The most comprehensive introduction to the subject of pseudoholomorphic curves is without a doubt the monumental book [12]. Chapters 2 and 4 of this book concern two of the three fundamental building blocks of this subject, the local structure of J-holomorphic maps and Gromov's convergence for sequences of J-holomorphic maps. The present notes contain an alternative systematic exposition of these two topics with generally sharper specification of the assumptions needed for each statement. Chapter 3 and Sections 6.2 and 6.3 in [12] concern the third fundamental building block of the subject, transversality for J-holomorphic maps. A more streamlined and general treatment of this topic is the concern of [19].

The present notes build on the lecture notes on J-holomorphic maps written for the class the author taught at Stony Brook University in Spring 2014. The lectures themselves were based on the hand-written notes he made while studying [11] back in graduate school and were also influenced by the more thorough exposition of the same topics in [12]. The author would like to thank D. McDuff and D. Salamon for the time and care taken in preparing and updating these books, the students in the Spring 2014 class for their participation that guided the preparation of the original version of the present notes, and X. Chen for thoughtful comments during the revision process.

1.1 Stable maps

A (smooth) Riemann surface (without boundary) is a pair $(\Sigma,\mathfrak{j})$ consisting of a smooth two-dimensional manifold Σ (without boundary) and a complex structure $\mathfrak{j}$ in the fibers of $T\Sigma$. A nodal Riemann surface is a pair $(\Sigma,\mathfrak{j})$ obtained from a Riemann surface $(\widetilde\Sigma,\mathfrak{j})$ by identifying pairs of distinct points in a discrete subset $S_\Sigma\subset\widetilde\Sigma$ (with no point identified with more than one other point); see the left-hand sides of Figures 1 and 2. The pair $(\widetilde\Sigma,\mathfrak{j})$ is called the normalization of $(\Sigma,\mathfrak{j})$; the images of the points of S_Σ in Σ are called the nodes of Σ. We denote their complement in Σ by Σ^*. An irreducible component of $(\Sigma,\mathfrak{j})$ is the image of a topological component of $\widetilde\Sigma$ in Σ. Let

$$\mathfrak{a}(\Sigma)=\frac{2-\chi(\widetilde\Sigma)+|S_\Sigma|}{2},$$

where $\chi(\widetilde\Sigma)$ is the Euler characteristic of $\widetilde\Sigma$, be the (arithmetic) genus of Σ. An equivalence between Riemann surfaces $(\Sigma,\mathfrak{j})$ and $(\Sigma',\mathfrak{j}')$ is a homeomorphism $h:\Sigma\longrightarrow\Sigma'$ induced by a biholomorphic map $\widetilde h$ from $(\widetilde\Sigma,\mathfrak{j})$ to $(\widetilde\Sigma',\mathfrak{j}')$. We denote by $\mathrm{Aut}(\Sigma,\mathfrak{j})$ the group of automorphisms, i.e. self-equivalences, of a Riemann surface $(\Sigma,\mathfrak{j})$.

Let (X,J) be an almost complex manifold. If $(\Sigma,\mathfrak{j})$ is a Riemann surface, a smooth map $u\colon\Sigma\longrightarrow X$ is called J-holomorphic map if

$$\mathrm{d}u\circ\mathfrak{j}=J\circ\mathrm{d}u\colon T\Sigma\longrightarrow u^*TX.$$

A J-holomorphic map from a nodal Riemann surface $(\Sigma,\mathfrak{j})$ is a tuple $(\Sigma,\mathfrak{j},u)$, where $u\colon\Sigma\longrightarrow X$ is a continuous map induced by a J-holomorphic map $\widetilde u\colon\widetilde\Sigma\longrightarrow X$; see Figures 1 and 2. An equivalence between J-holomorphic maps $(\Sigma,\mathfrak{j},u)$ and $(\Sigma',\mathfrak{j}',u')$ is an equivalence

$$h\colon(\Sigma,\mathfrak{j})\longrightarrow(\Sigma',\mathfrak{j}')$$

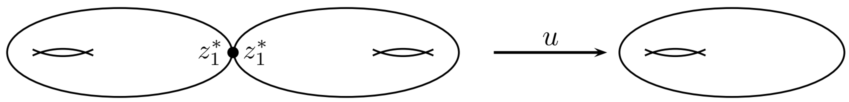


Figure 1: A stable J-holomorphic map

between the underlying Riemann surfaces such that $u=u'\circ h$. We denote by $\mathrm{Aut}(\Sigma,\mathfrak{j},u)$ the group of automorphisms, i.e. self-equivalences, of a J-holomorphic map $(\Sigma,\mathfrak{j},u)$. A J-holomorphic map $(\Sigma,\mathfrak{j},u)$ is called stable if $(\Sigma,\mathfrak{j})$ is compact and $\mathrm{Aut}(\Sigma,\mathfrak{j},u)$ is a finite group.

The Riemann surface $(\Sigma,\mathfrak{j})$ on the left-hand side of Figure 1 is obtained by identifying the marked points of two copies of a smooth elliptic curve $(\Sigma_0,\mathfrak{j}_0,z_1^*)$, i.e. a torus with a complex structure and a marked point. The Riemann surface $(\Sigma_0,\mathfrak{j}_0)$ with the marked point z_1^* is biholomorphic to $\mathbb{C}/\Lambda$ with the marked point 0 for some lattice $\Lambda\subset\mathbb{C}$ and thus has an automorphism of order 2 that preserves z_1^* (it is induced by the map $z\longrightarrow -z$ on $\mathbb{C}$). This is the only non-trivial automorphism of $(\Sigma_0,\mathfrak{j}_0)$ preserving z_1^* if $\mathfrak{j}_0$ is generic; in special cases, the group of such automorphisms is either $\mathbb{Z}_4$ or $\mathbb{Z}_6$. Each automorphism of $(\Sigma_0,\mathfrak{j}_0)$ preserving z_1^* gives rise to an automorphism of $(\Sigma,\mathfrak{j})$ fixing one of the irreducible components. There is also an automorphism of $(\Sigma,\mathfrak{j})$ which interchanges the two irreducible components of Σ. Since it does not commute with the automorphisms preserving one of the components, $\mathrm{Aut}(\Sigma,\mathfrak{j})\approx D_4$ in most cases and contains D_4 in the special cases. If $u\colon\Sigma\longrightarrow\Sigma_0$ is the identity on each irreducible component, $(\Sigma,\mathfrak{j},u)$ is a stable J-holomorphic map; the interchange of the two irreducible components is then the only non-trivial automorphism of $(\Sigma,\mathfrak{j},u)$. The J-holomorphic maps $u:\Sigma\longrightarrow\Sigma_0$ obtained by sending either or both irreducible components of Σ to z_1^* instead are also stable, but have different automorphism groups. If $(\Sigma_0,\mathfrak{j}_0)$ were taken to be the Riemann sphere $\mathbb{P}^1$, the J-holomorphic map $u\colon\Sigma\longrightarrow\Sigma_0$ restricting to the identity on each copy of Σ_0 would still be stable. However, a map $u\colon\Sigma\longrightarrow\Sigma_0$ sending either copy of Σ_0 to z_1^* would not be stable, since the group of automorphisms of $\mathbb{P}^1$ fixing a point is a complex two-dimensional submanifold of PSL_2.

Let $(\Sigma,\mathfrak{j})$ be a compact connected Riemann surface of genus g. If $g\ge 2$, then $\mathrm{Aut}(\Sigma,\mathfrak{j})$ is a finite group. If $g=1$, then $\mathrm{Aut}(\Sigma,\mathfrak{j})$ is an infinite group, but its subgroup fixing any point is finite. If $g=0$, then the subgroup of $\mathrm{Aut}(\Sigma,\mathfrak{j})$ fixing any pair of points is infinite, but the subgroup fixing any triple of points is trivial. If in addition (X,J) is an almost complex manifold and $u\colon\Sigma\longrightarrow X$ is a non-constant J-holomorphic map, then the subgroup of $\mathrm{Aut}(\Sigma,\mathfrak{j})$ consisting of the automorphisms such that $u=u\circ h$ is finite; this is an immediate consequence of Corollary 3.4. If $(\Sigma,\mathfrak{j})$ is a compact nodal Riemann surface, a J-holomorphic map $(\Sigma,\mathfrak{j},u)$ is thus stable if and only if

- every genus 1 topological component of the normalization $\widetilde{\Sigma}$ of Σ such that u restricts to a constant map on its image in Σ contains at least 1 element of S_Σ and
- every genus 0 topological component of $\widetilde{\Sigma}$ such that u restricts to a constant map on its image in Σ contains at least 3 elements of S_Σ.

1.2 Gromov's topology

Given a Riemann surface $(\Sigma, \mathfrak{j})$, a Riemannian metric g on a smooth manifold X determines the energy $E_g(f)$ for every smooth map $f\colon \Sigma \longrightarrow X$; see (2.5) and (2.6). The fundamental insight in [6] that laid the foundations for the pseudoholomorphic curves techniques in symplectic topology and for the moduli spaces of stable maps and related curve-parametrizing objects in algebraic geometry is that a sequence of stable J-holomorphic maps $(\Sigma_i, \mathfrak{j}_i, u_i)$ into a compact almost complex manifold (X, J) with

$$\liminf_{i\longrightarrow\infty}\Big(\big|\pi_0(\Sigma_i)\big|+\mathfrak{a}(\Sigma_i)+E_g(u_i)\Big) < \infty \tag{1.1}$$

has a subsequence converging in a suitable sense to another stable J-holomorphic map.

The notion of Gromov's convergence of a sequence of stable J-holomorphic maps $(\Sigma_i, \mathfrak{j}_i, u_i)$ to another stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ comes down to

(GC1) $|\pi_0(\Sigma_i)| = |\pi_0(\Sigma_\infty)|$ and $\mathfrak{a}(\Sigma_i) = \mathfrak{a}(\Sigma_\infty)$ for all i large,

(GC2) $(\Sigma_\infty, \mathfrak{j}_\infty)$ is at least as singular as $(\Sigma_i, \mathfrak{j}_i)$ for all i large,

(GC3) the energy is preserved, i.e. $E_g(u_i) \longrightarrow E_g(u_\infty)$ as $i \longrightarrow \infty$, and

(GC4) u_i converges to u_∞ uniformly in the C^∞-topology on compact subsets of Σ_∞^*.

Most applications of the pseudoholomorphic curves techniques in symplectic topology involve J-holomorphic maps from the Riemann sphere $\mathbb{P}^1$. This is a special case of the situation when the complex structures $\mathfrak{j}_i$ on the domains Σ_i of u_i are fixed. The condition (GC4) can then be formally stated in a way clearly indicative of the rescaling procedure of [6].

Definition 1.1 (Gromov's Convergence I). Let (X, J) be an almost complex manifold with Riemannian metric g and $(\Sigma, \mathfrak{j})$ be a compact Riemann surface. A sequence $(\Sigma, \mathfrak{j}, u_i)$ of stable J-holomorphic maps converges to a stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ if

(1) $(\Sigma_\infty, \mathfrak{j}_\infty)$ is obtained from $(\Sigma, \mathfrak{j})$ by identifying a point on each of ℓ trees of Riemann spheres $\mathbb{P}^1$, for some $\ell \in \mathbb{Z}^{\ge 0}$, with distinct points $z_1^*, \ldots, z_\ell^* \in \Sigma$,

(2) $E_g(u_\infty) = \lim\limits_{i\longrightarrow\infty} E_g(u_i)$,

(3) there exist $h_i \in \mathrm{Aut}(\Sigma, \mathfrak{j})$ with $i \in \mathbb{Z}^+$ such that $u_i \circ h_i$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of $\Sigma - \{z_1^*, \ldots, z_\ell^*\}$,

(4) for each $z_1^*, \ldots, z_\ell^* \in \Sigma \subset \Sigma_\infty$ and all $i \in \mathbb{Z}^+$ sufficiently large, there exist a neighborhood $U_j \subset \Sigma$ of z_j^*, an open subset $U_{j;i} \subset \mathbb{C}$, and a biholomorphic map $\psi_{j;i}\colon U_{j;i} \longrightarrow U_j$ such that

   (4a) $U_{j;i} \subset U_{j;i+1}$ and $\mathbb{C} = \bigcup_{i=1}^\infty U_{j;i}$ for every $j = 1, \ldots, \ell$,

   (4b) $u_i \circ h_i \circ \psi_{j;i}$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty, w_{j;1}^*, \ldots, w_{j;k_j}^*$ in the sphere $\mathbb{P}^1_j$ attached at $z_j^* \in \Sigma$,

   (4c) condition (4) applies with Σ, $(z_1^*, \ldots, z_\ell^*)$, and $u_i \circ h_i$ replaced by $\mathbb{P}^1$, $(w_{j;1}^*, \ldots, w_{j;k_j}^*)$, and $u_i \circ h_i \circ \psi_{j;i}$, respectively, for each $j = 1, \ldots, \ell$.

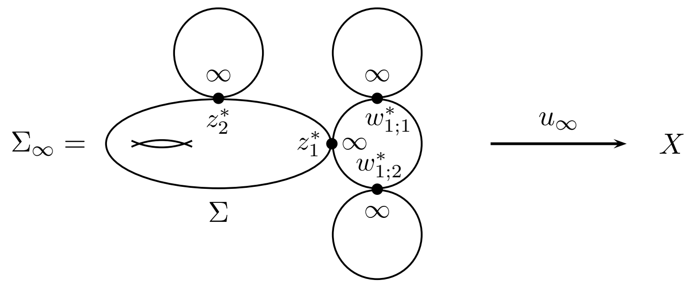


Figure 2: Gromov's limit of a sequence of J-holomorphic maps $u_i\colon \Sigma \longrightarrow X$

An example of a possible limiting map with $\ell=2$ trees of spheres is shown in Figure 2. The recursive condition (4) in Definition 1.1 is equivalent to the *Rescaling* axiom in [12, Definition 5.2.1] on sequences of automorphisms ϕ_α^i of $\mathbb{P}^1$; they correspond to compositions of the maps $\psi_{j;i}$ associated with different irreducible components of Σ_∞. The single energy condition (2) in Definition 1.1 is replaced in [12, Definition 5.2.1] by multiple conditions of the *Energy* axiom. These multiple conditions are equivalent to (2) if the other three axioms in [12, Definition 5.2.1] are satisfied.

Theorem 1.2 (Gromov's Compactness I)**.** *Let (X,J) be a compact almost complex manifold with Riemannian metric g, $(\Sigma,\mathfrak{j})$ be a compact Riemann surface, and $u_i:\Sigma\longrightarrow X$ be a sequence of non-constant J-holomorphic maps. If $\liminf E_g(u_i)<\infty$, then the sequence $(\Sigma,\mathfrak{j},u_i)$ contains a subsequence converging to some stable J-holomorphic map $(\Sigma_\infty,\mathfrak{j}_\infty,u_\infty)$ in the sense of Definition 1.1.*

This theorem is established in Section 5.3 by assembling together a number of geometric statements obtained earlier in these notes. In Section 5.4, we relate the convergence notion of Definition 1.1 in the case of holomorphic maps from $\mathbb{CP}^1$ to $\mathbb{CP}^n$, which can always be represented by $(n+1)$-tuples of homogeneous polynomials in two variables, to the behavior of the linear factors of the associated polynomials.

The convergence notion of Definition 1.1 can be equivalently reformulated in terms of deformations of the limiting domain $(\Sigma_\infty,\mathfrak{j}_\infty)$ so that it readily extends to sequences of stable J-holomorphic maps with varying complex structures $\mathfrak{j}_i$ on the domains Σ_i. This was formally done in the algebraic geometry category by [4], several years after this perspective had been introduced into the field informally, and adapted to the almost complex category by [10]. We summarize this perspective below.

Let $(\Sigma,\mathfrak{j})$ be a nodal Riemann surface. A flat family of deformations of $(\Sigma,\mathfrak{j})$ is a holomorphic map $\pi\colon\mathcal{U}\longrightarrow\Delta$, where $\mathcal{U}$ is a complex manifold and $\Delta\subset\mathbb{C}^N$ is a neighborhood of 0, such that

- $\pi^{-1}(\lambda)$ is a nodal Riemann surface for each $\lambda\in\Delta$ and $\pi^{-1}(0)=(\Sigma,\mathfrak{j})$,
- π is a submersion outside of the nodes of the fibers of π,
- for every $\lambda^*\equiv(\lambda_1^*,\ldots,\lambda_N^*)\in\Delta$ and every node $z^*\in\pi^{-1}(\lambda^*)$, there exist $i\in\{1,\ldots,N\}$ with $\lambda_i^*=0$, neighborhoods Δ_{λ^*} of λ^* in Δ and $\mathcal{U}_{z^*}$ of z^* in $\mathcal{U}$, and a holomorphic map
$$\Psi\colon\mathcal{U}_{z^*}\longrightarrow\big\{\big((\lambda_1,\ldots,\lambda_N),x,y\big)\in\Delta_{\lambda^*}\times\mathbb{C}^2\colon xy=\lambda_i\big\}$$
such that Ψ is a homeomorphism onto a neighborhood of $(\lambda^*,0,0)$ and the composition of Ψ with the projection to Δ_{λ^*} equals $\pi|_{\mathcal{U}_{z^*}}$.

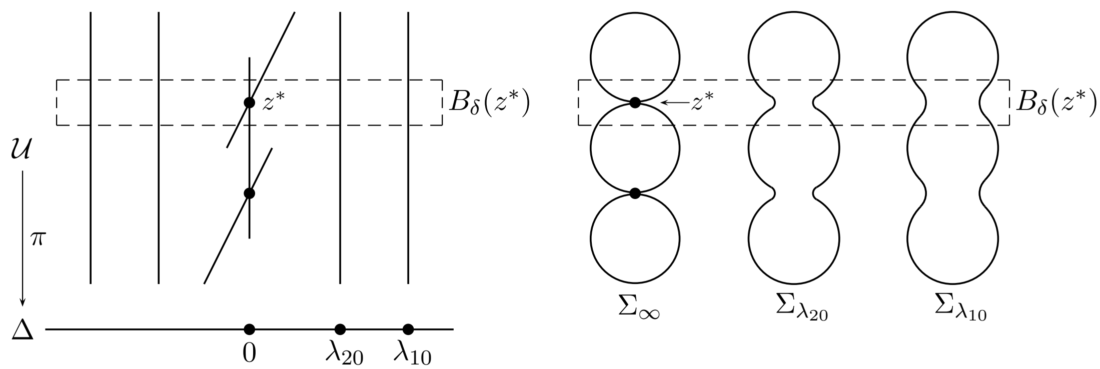


Figure 3: A complex-geometric presentation of a flat family of deformations of $(\Sigma_\infty, \mathfrak{j}_\infty) = \pi^{-1}(0)$ and a differential-geometric presentation of the domains of the maps u_i in Definition 1.3.

If $\pi: \mathcal{U} \longrightarrow \Delta$ is a flat family of deformations of $(\Sigma, \mathfrak{j})$ and Σ is compact, there exists a neighborhood $\mathcal{U}^* \subset \mathcal{U}$ of $\Sigma^* \subset \pi^{-1}(0)$ such that

$$\pi|_{\mathcal{U}^*}: \mathcal{U}^* \longrightarrow \Delta_0 \equiv \pi(\mathcal{U}^*) \subset \Delta$$

is a trivializable Σ^*-fiber bundle in the smooth category. For each $\lambda \in \Delta_0$, let

$$\psi_\lambda: \Sigma^* \longrightarrow \pi^{-1}(\lambda) \cap \mathcal{U}^*$$

be the corresponding smooth identification. If $\lambda_i \in \Delta$ is a sequence converging to $0 \in \Delta$ and $u_i: \pi^{-1}(\lambda_i) \longrightarrow X$ is a sequence of continuous maps that are smooth on the complements of the nodes of $\pi^{-1}(\lambda_i)$, we say that the sequence u_i converges to a smooth map $u: \Sigma^* \longrightarrow X$ u.c.s. (uniformly on compact subsets) if the sequence of maps

$$u_i \circ \psi_{\lambda_i}: \Sigma^* \longrightarrow X$$

converges to u uniformly in the C^∞-topology on compact subsets of Σ^*. This notion is independent of the choices of $\mathcal{U}^*$ and trivialization of $\pi|_{\mathcal{U}^*}$.

Definition 1.3 (Gromov's Convergence II)**.** Let (X, J) be an almost complex manifold with a Riemannian metric g. A sequence $(\Sigma_i, \mathfrak{j}_i, u_i)$ of stable J-holomorphic maps converges to a stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ if $E_g(u_i) \longrightarrow E_g(u_\infty)$ as $i \longrightarrow \infty$ and there exist

(a) a flat family of deformations $\pi: \mathcal{U} \longrightarrow \Delta$ of $(\Sigma_\infty, \mathfrak{j}_\infty)$,

(b) a sequence $\lambda_i \in \Delta$ converging to $0 \in \Delta$, and

(c) equivalences $h_i: \pi^{-1}(\lambda_i) \longrightarrow (\Sigma_i, \mathfrak{j}_i)$

such that $u_i \circ h_i$ converges to $u_\infty|_{\Sigma^*_\infty}$ u.c.s.

By the compactness of Σ_∞, the notion of convergence of Definition 1.3 is independent of the choice of metric g on X. It is illustrated in Figure 3. If the Riemann surfaces $(\Sigma_i, \mathfrak{j}_i)$ are smooth, the limiting Riemann surface $(\Sigma_\infty, \mathfrak{j}_\infty)$ is obtained by pinching some disjoint embedded circles in the smooth two-dimensional manifold Σ underlying these Riemann surfaces.

If $(\Sigma_i, \mathfrak{j}_i) = (\Sigma, \mathfrak{j})$ for all i as in Definition 1.1, only contractible circles are pinched to produce Σ_∞; it then consists of Σ with trees of spheres attached. The family $\pi: \mathcal{U} \longrightarrow \Delta$ is obtained by starting with the family

$$\pi_0 : \mathcal{U}_0 \equiv \mathbb{C} \times \Sigma \longrightarrow \mathbb{C},$$

then blowing up $\mathcal{U}_0$ at a point of $\{0\} \times \Sigma$ to obtain a family $\pi_1 : \mathcal{U}_1 \longrightarrow \mathbb{C}$ with the central fiber $\Sigma_1 \equiv \pi_1^{-1}(0)$ consisting of Σ with $\mathbb{P}^1$ attached, then blowing up a smooth point of Σ_1, and so on. The number of blowups involved is precisely the number of nodes of Σ_∞, i.e. four in the case of Figure 2 and two in the case of Figure 3. The pinched annuli on the right-hand side of Figure 3 correspond to $\phi_\alpha(B_\delta(z_{\alpha\beta})) \cup \phi_\beta(B_\delta(z_{\beta\alpha}))$ in the notation of [12, Chapters 4,5].

With the setup of Definition 1.3, let $B_\delta(z^*) \subset \mathcal{U}$ denote the ball of radius $\delta \in \mathbb{R}^+$ around a point $z^* \in \mathcal{U}$ with respect to some metric on $\mathcal{U}$. Then,

$$\lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} \mathrm{diam}_g\big(u_i\big(h_i(\pi^{-1}(\lambda_i) \cap B_\delta(z^*))\big)\big) = 0 \qquad \forall\, z^* \in \Sigma_\infty\,. \tag{1.2}$$

This is immediate from the last condition in Definition 1.3 if $z^* \in \Sigma_\infty^*$. If $z^* \in \Sigma_\infty - \Sigma_\infty^*$ is a node of Σ_∞, (1.2) is a consequence of both convergence conditions of Definition 1.3 and the maps u_i being J-holomorphic. It is a reflection of the fact that bubbling or any other kind of erratic C^0-behavior of a sequence of J-holomorphic maps requires a nonzero amount of energy in the limit, but the two convergence conditions of Definition 1.3 ensure that all limiting energy is absorbed by $u|_{\Sigma_\infty^*}$ and thus none is left for bubbling around the nodes of Σ_∞. An immediate implication of (1.2) is that $u_i(h_i(\pi^{-1}(\lambda_i) \cap B_\delta(z^*)))$ is contained in a geodesic ball around $u_\infty(z^*)$ in X. Thus,

$$u_{i*}[\Sigma_i] = u_{\infty *}[\Sigma_\infty] \in H_2(X; \mathbb{Z})$$

for all $i \in \mathbb{Z}^+$ sufficiently large. If Σ_∞ is a tree of spheres (and thus so is each Σ_i), then u_i with i sufficiently large lies in the equivalence class in $\pi_2(X)$ determined by u_∞ for the same reason.

Theorem 1.4 (Gromov's Compactness II)**.** *Let (X, J) be a compact almost complex manifold with a Riemannian metric g and $(\Sigma_i, \mathfrak{j}_i, u_i)$ be a sequence of stable J-holomorphic maps. If it satisfies (1.1), then it contains a subsequence converging to some stable J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ in the sense of Definition 1.3.*

This theorem is obtained by combining the compactness of the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{1,1}$ of stable (possibly) nodal elliptic curves and $\overline{\mathcal{M}}_g$ of stable nodal genus $g \geq 2$ curves with the proof of Theorem 1.2 in Section 5.3. One first establishes Theorem 1.4 under the assumption that each $(\Sigma_i, \mathfrak{j}_i)$ is a smooth connected Riemann surface of genus $g \geq 1$ (the $g = 0$ case is treated by Theorem 1.2). If $g = 1$, we add a marked point to each domain $(\Sigma_i, \mathfrak{j}_i)$ and take a subsequence converging in $\overline{\mathcal{M}}_{1,1}$ to the equivalence class of some stable nodal elliptic curve $(\Sigma'_\infty, \mathfrak{j}'_\infty, z'_\infty)$. If $g \geq 2$, we take a subsequence of $(\Sigma_i, \mathfrak{j}_i)$ converging in $\overline{\mathcal{M}}_g$ to the equivalence class of some stable nodal genus g curve $(\Sigma'_\infty, \mathfrak{j}'_\infty)$. This ensures the existence of a flat family of deformations $\pi' : \mathcal{U}' \longrightarrow \Delta'$ of $(\Sigma'_\infty, \mathfrak{j}'_\infty)$, of a sequence $\lambda'_i \in \Delta'$ converging to $0 \in \Delta'$, and of equivalences $h_i : \pi'^{-1}(\lambda'_i) \longrightarrow (\Sigma_i, \mathfrak{j}_i)$. The associated neighborhood $\mathcal{U}'^*$ of Σ'^*_∞ in $\mathcal{U}'$ can be chosen so that $\pi'^{-1}(\lambda)$–$\mathcal{U}'^*$ consists of finitely many circles for every $\lambda' \in \Delta'$ sufficiently small. The complement of the image of the associated identifications

$$\psi'_\lambda : \Sigma'^*_\infty \longrightarrow \pi'^{-1}(\lambda) \cap \mathcal{U}'^*$$

in $\pi'^{-1}(\lambda)$ has the same property.

One then applies the construction in the proof of Theorem 1.2 to the sequence of J-holomorphic maps

$$u_i \circ h_i' \colon \Sigma_\infty'^* \longrightarrow X$$

to obtain a J-holomorphic map $\widetilde{u}_\infty'$ from the normalization $\widetilde{\Sigma}_\infty'$ of Σ_∞ and finitely J-holomorphic maps from trees of $\mathbb{P}^1$. Each of these trees will have one or two special points that are associated with points of $\widetilde{\Sigma}_\infty'$ (the latter happens if bubbling occurs at a preimage of a node of Σ_∞' in $\widetilde{\Sigma}_\infty'$). Identifying these trees with the corresponding points of $\widetilde{\Sigma}_\infty'$ as in the proof of Theorem 1.2, we obtain a J-holomorphic map $(\Sigma_\infty, \mathfrak{j}_\infty, u_\infty)$ satisfying the requirements of Definition 1.3. It is necessarily stable if $g \geq 2$, or Σ_∞' is smooth, or Σ_∞ contains a separating node. Otherwise, the identifications h_i' may first need to be reparametrized to ensure that either the limiting map $\widetilde{u}_\infty'$ is not constant or the sequence $u_i \circ h_i$ produces a bubble at least one smooth point of $\widetilde{\Sigma}_\infty'$.

A k-marked Riemann surface is a tuple $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k)$ such that $(\Sigma, \mathfrak{j})$ is a Riemann surface and $z_1, \ldots, z_k \in \Sigma^*$ are distinct points. If (X, J) is an almost complex manifold, a k-marked J-holomorphic map into X is a tuple $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k, u)$, where $(\Sigma, \mathfrak{j}, z_1, \ldots, z_k)$ is k-marked Riemann surface and $(\Sigma, \mathfrak{j}, u)$ is a J-holomorphic map into X. The degree of such a map is the homology class

$$A = u_*[\Sigma] \in H_2(X; \mathbb{Z}) .$$

The notions of equivalence, stability, and convergence as in Definition 1.3 and the above convergence argument for smooth domains $(\Sigma_i, \mathfrak{j}_i)$ readily extend to k-marked J-holomorphic maps. The general case of Theorem 1.4, including its extension to stable marked maps, is then obtained by

- passing to a subsequence of $(\Sigma_i, \mathfrak{j}_i, u_i)$ with the same topological structure of the domain,
- viewing it as a sequence of tuples of J-holomorphic maps with smooth domains with an additional marked point for each preimage of the nodes in the normalization, and
- applying the conclusion of the above argument to each component of the tuple.

1.3 Moduli spaces

The natural extension of Definition 1.3 to marked J-holomorphic maps topologizes the moduli space $\overline{\mathfrak{M}}_{g,k}(X, A; J)$ of equivalence classes of stable degree A k-marked genus g J-holomorphic maps into X for each $A \in H_2(X; \mathbb{Z})$. The evaluation maps

$$\mathrm{ev}_i \colon \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow X, \quad (\Sigma, \mathfrak{j}, z_1, \ldots, z_k, u) \longrightarrow u(z_i),$$

are continuous with respect to this topology. If $2g+k \geq 3$, there is a continuous map

$$\mathfrak{f} \colon \overline{\mathfrak{M}}_{g,k}(X, A; J) \longrightarrow \overline{\mathcal{M}}_{g,k}$$

to the Deligne-Mumford moduli space of stable k-marked genus g nodal curves obtained by forgetting the map u and then contracting the unstable components of the domain.

There is a continuous map

$$\mathfrak{f}_{k+1} \colon \overline{\mathfrak{M}}_{g,k+1}(X, A; J) \longrightarrow \overline{\mathfrak{M}}_{g,k}(X, A; J) \qquad (1.3)$$

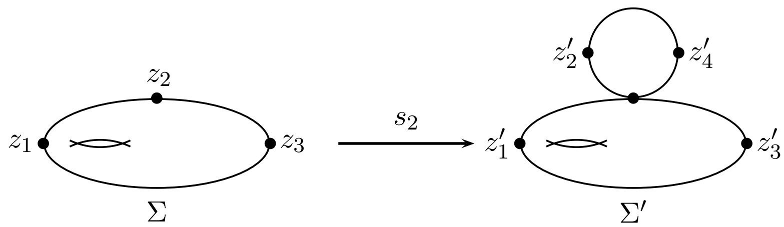


Figure 4: Section s_2 of the fibration (1.3) with $k=3$

obtained by forgetting the last marked point z_{k+1} and then contracting the components of the domain to stabilize the resulting k-marked J-holomorphic map. For each $i=1,\ldots,k$, this fibration has a natural continuous section

$$s_i\colon \overline{\mathfrak{M}}_{g,k}(X,A;J) \longrightarrow \overline{\mathfrak{M}}_{g,k+1}(X,A;J)$$

described as follows. For a k-marked nodal Riemann surface $(\Sigma,\mathfrak{j},z_1,\ldots,z_k)$, let $(\Sigma',\mathfrak{j}',z_1,\ldots,z_{k+1})$ be the $(k+1)$-marked nodal Riemann surface so that $(\Sigma',\mathfrak{j}')$ consists of $(\Sigma,\mathfrak{j})$ with $\mathbb{P}^1$ attached at z_i, $z'_1, z'_i \in \mathbb{P}^1$, and $z'_j = z_j \in \Sigma$ for all $j=1,\ldots,k$ different from k; see Figure 4. We define

$$s_i\big([\Sigma,\mathfrak{j},z_1,\ldots,z_k,u]\big) = \big[\Sigma',\mathfrak{j}',z'_1,\ldots,z'_{k+1},u'\big],$$

with $(\Sigma',\mathfrak{j}',z'_1,\ldots,z'_{k+1})$ as described and u' extending u over the extra $\mathbb{P}^1$ by the constant map with value $u(z_i)$. The pullback

$$L_i \longrightarrow \overline{\mathfrak{M}}_{g,k}(X,A;J)$$

of the vertical tangent line bundle of (1.3) by s_i is called the universal tangent line bundle at the i-th marked point. Let $\psi_i = c_1(L_i^*)$ be the i-th descendant class.

A remarkable property of Gromov's topology which lies behind most of its applications is that the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ is Hausdorff and has a particularly nice deformation-obstruction theory. In the algebraic-geometry category, the latter is known as a perfect two-term deformation-obstruction theory. In the almost complex category, this is reflected in the existence of an atlas of finite-dimensional approximations in the terminology of [10] or of an atlas of Kuranishi charts in the terminology of [10].

If (X,J) is an almost complex manifold and J is tamed by a symplectic form ω, then the energy $E_g(u)$ of degree A J-holomorphic map u with respect to the metric g determined by J and ω is $\omega(A)$; see (2.7). In particular, it is the same for all elements of the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$. If in addition X is compact, then Theorem 1.4 implies that this moduli space is also compact. Combining this with the remarkable property of the previous paragraph, the constructions of [1, 9, 10, 3] endow $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ with a virtual fundamental class. It depends only on ω, in a suitable sense, and not an almost complex structure J tamed by ω. This class in turn gives rise to Gromov-Witten invariants of (X,ω):

$$\big\langle \tau_{a_1}\alpha_1,\ldots,\tau_{a_k}\alpha_k\big\rangle_{g,A}^X \equiv \big\langle \big(\psi_1^{a_1}\mathrm{ev}_1^*\alpha_1\big)\ldots\big(\psi_k^{a_k}\mathrm{ev}_k^*\alpha_k\big), \big[\overline{\mathfrak{M}}_{g,k}(X,A;J)\big]^{\mathrm{vir}}\big\rangle \in \mathbb{Q}$$

for all $a_i \in \mathbb{Z}^{\geq 0}$ and $\alpha_i \in H^*(X;\mathbb{Q})$.

2 Preliminaries

An outline of these notes with an informal description of the key statements appears in Section 2.1; Figure 5 indicates primary connections between these statements. Sections 2.2 introduces the most frequently used notation and terminology and makes some basic observations.

2.1 Overview of the main statements

The main technical statement of Section 3 of these notes and of Chapter 2 in [12] is the Carleman Similarity Principle; see Proposition 3.1. It yields a number of geometric conclusions about the local behavior of a J-holomorphic map $u\colon \Sigma \longrightarrow X$ from a Riemann surface $(\Sigma, \mathfrak{j})$ into an almost complex manifold (X, J). For example, for every $z \in \Sigma$ contained in a component of Σ on which u is not constant, the ℓ-th derivative of u at z in a chart around $u(z)$ does not vanish for some $\ell \in \mathbb{Z}^+$; see Corollary 3.3. We denote by $\mathrm{ord}_z u \in \mathbb{Z}^+$ the minimum of such integers ℓ and call it the order of u at z; it is independent of the choice of a chart around $u(z)$. If u is constant on the component of Σ on containing z, we set $\mathrm{ord}_z u = 0$. A point $z \in u$ is singular, i.e. $\mathrm{d}_z u = 0$, if and only if $\mathrm{ord}_z u \neq 1$.

If u is not constant on every connected component of Σ, the singular points of u and the preimages of a point $x \in X$ are discrete subsets of Σ; see Corollary 3.4. In the case Σ is compact, the second statement of Corollary 3.4 implies that

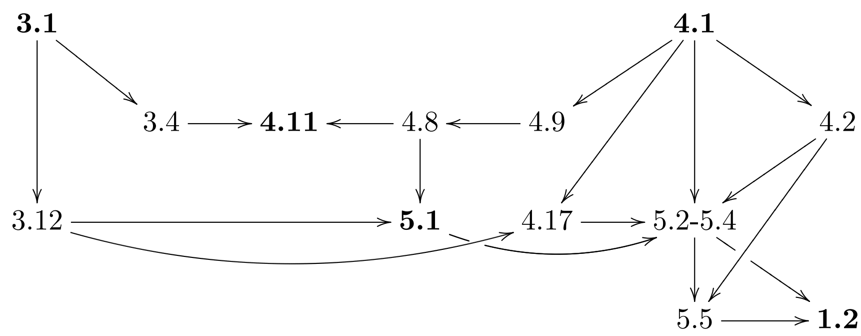
$$\mathrm{ord}_x u \equiv \sum_{z \in u^{-1}(x)} \mathrm{ord}_z u \quad \in \mathbb{Z}^{\geq 0} \qquad \forall\, x \in X; \tag{2.1}$$

we call this number the order of u at x. If $x \notin \mathrm{Im}(u)$, then $\mathrm{ord}_x u = 0$. By Corollary 3.11, the number (2.1) is seen by the behavior of the energy (2.5) of u and its restrictions to open subsets of Σ. This observation underpins the Monotonicity Lemma for J-holomorphic maps, which bounds below the energy required to "escape" from a small ball in X; see Proposition 3.12.

The main technical statement of Section 4 of these notes and of Chapter 4 in [12] is the Mean Value Inequality. It bounds the pointwise differentials $\mathrm{d}_z u$ of a J-holomorphic map u from $(\Sigma, \mathfrak{j})$ into (X, J) of sufficiently small energy $E_g(u)$ by $E_g(u)$, i.e. by the L^2-norm of $\mathrm{d}u$, from above and immediately yields a bound on the energy of non-constant J-holomorphic maps from S^2 into (X, J) from below; see Proposition 4.1 and Corollary 4.2, respectively. The Mean Value Inequality also implies that the energy of a J-holomorphic map u from a cylinder $[-R, R] \times S^1$ carried by $[-R+T, R-T] \times S^1$ and the diameter of the image of this middle segment decay at least exponentially with T, provided the overall energy of u is sufficiently small. As shown in the proof of Proposition 5.5, this technical implication ensures that the energy is preserved under Gromov's convergence and the resulting bubbles connect.

Another important implication of Proposition 4.1 is that a continuous map from a Riemann surface $(\Sigma, \mathfrak{j})$ into an almost complex manifold (X, J) which is holomorphic outside of a discrete collection of points and has bounded energy is in fact holomorphic on all of Σ; see Proposition 4.8. This conclusion plays a central role in the proof of Lemma 5.4. Theorem 1.2 is deduced from Lemma 5.4 and Proposition 5.5 in Section 5.3.

Combined with Proposition 3.1 and some of its corollaries, Proposition 4.1 implies that every non-constant J-holomorphic map from a compact Riemann surface $(\Sigma, \mathfrak{j})$ factors through a somewhere



3.1 Carleman Similarity Principle
4.1 Mean Value Inequality
4.8 Regularity of J-holomorphic maps
4.11 Global structure of J-holomorphic maps
5.1 Removal of Singularity
5.5 Gromov's convergence
3.12 Monotonicity Lemma
4.2 Lower Energy Bound
4.9 Isoperimetric Inequality
4.17 Bounds on long cylinders
5.2-5.4 Bubbling

Figure 5: Connections between the main statements leading to Theorem 1.2 and Proposition 4.11

injective J-holomorphic map from a compact Riemann surface $(\Sigma', \mathfrak{j}')$; see Proposition 4.11. The proof of this statement with X compact appears in Chapter 2 of [12], but uses the Removal Singularities Theorem proved in Chapter 4 of [12]. Proposition 4.1 is the key technical step in establishing transversality for the moduli spaces of simple J-holomorphic maps and constructing pseudocycles out of these spaces; see [19].

2.2 Notation and terminology

Let $(\Sigma, \mathfrak{j})$ be a Riemann surface, V be a vector bundle over Σ, and

$$\mu, \eta \in \Gamma\big(\Sigma; T^*\Sigma \otimes_{\mathbb{R}} V\big) \qquad \text{and} \qquad g \in \Gamma\big(\Sigma; T^*\Sigma^{\otimes 2} \otimes_{\mathbb{R}} V\big).$$

For a local coordinate $z = s + \mathfrak{i}t$, define

$$\begin{aligned} g(\mu \otimes_{\mathfrak{j}} \eta) &= \big(g\big(\mu(\partial_s), \eta(\partial_s)\big) + g\big(\mu(\partial_t), \eta(\partial_t)\big)\big) \mathrm{d}s \wedge \mathrm{d}t\,, \\ g(\mu \wedge_{\mathfrak{j}} \eta) &= \big(g\big(\mu(\partial_s), \eta(\partial_t)\big) - g\big(\mu(\partial_t), \eta(\partial_s)\big)\big) \mathrm{d}s \wedge \mathrm{d}t\,. \end{aligned} \tag{2.2}$$

By a direct computation, the 2-forms $g(\mu \otimes_{\mathfrak{j}} \eta)$ and $g(\mu \wedge_{\mathfrak{j}} \eta)$ are independent of the choice of local coordinate $z = s + \mathfrak{i}t$. Thus, (2.2) determines global 2-forms on Σ (which depend on the choice of $\mathfrak{j}$).

We denote by $\mathfrak{i}$ the standard complex structure on $\mathbb{C}$ and by $J_{\mathbb{C}^n}$ the standard complex structures on $\mathbb{C}^n$ and $T\mathbb{C}^n$. For an almost complex structure J and a 2-form ω on a manifold X, we define a 2-tensor and a 2-form on X by

$$\begin{aligned} g_J(v, v') &= \frac{1}{2}\big(\omega(v, Jv') - \omega(Jv, v')\big), \\ \omega_J(v, v') &= \frac{1}{2}\big(\omega(Jv, Jv') - \omega(v, v')\big) \end{aligned} \qquad \forall\, v, v' \in T_xX,\ x \in X. \tag{2.3}$$

We note that

$$g_J(v,v)+g_J(v',v')=2\omega(v,v')+g_J(v+Jv',v+Jv')+2\omega_J(v,v') \quad \forall\, v,v'\in T_xX,\ x\in X. \tag{2.4}$$

The 2-form ω tames J if $g(v,v)>0$ for all $v\in TX$ nonzero; in such a case, ω is nondegenerate and g_J is a metric. The almost complex structure J is ω-compatible if ω tames J and $\omega_J=0$.

Let X be a manifold, $(\Sigma,\mathfrak{j})$ be a Riemann surface, and $f:\Sigma\longrightarrow X$ be a smooth map. We denote the pullbacks of a 2-tensor g and a 2-form ω on X to the vector bundle f^*TX over Σ also by g and ω. If g is a Riemannian metric on X and $U\subset\Sigma$ is an open subset, let

$$E_g(f)\equiv\frac{1}{2}\int_\Sigma g(\mathrm{d}f\otimes_{\mathfrak{j}}\mathrm{d}f)\in[0,\infty] \qquad\text{and}\qquad E_g(f;U)\equiv E_g\big(f|_U\big) \tag{2.5}$$

be the energy of f and of its restriction to U. By the first equation in (2.2),

$$E_g(f)=\frac{1}{2}\int_\Sigma|\mathrm{d}f|^2_{g_\Sigma,g} \tag{2.6}$$

is the square of the L^2-norm of $\mathrm{d}f$ with respect to the metric g on X and a metric g_Σ compatible with $\mathfrak{j}$. In particular, the right-hand side of (2.6) depends on the metric g on X and on the complex structure $\mathfrak{j}$ on Σ, but *not* on the metric g_Σ on Σ compatible with $\mathfrak{j}$.

Let J be an almost complex structure on a manifold X and $(\Sigma,\mathfrak{j})$ be a Riemann surface. For a smooth map $f:\Sigma\longrightarrow X$, define

$$\bar\partial_J f=\frac{1}{2}\big(\mathrm{d}f+J\circ\mathrm{d}f\circ\mathfrak{j}\big)\in\Gamma\big(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}f^*(TX,J)\big).$$

If ω is a 2-form on X taming J and $u:\Sigma\longrightarrow X$ is J-holomorphic, then

$$E_{g_J}(f)=\int_\Sigma\big(f^*\omega+2g_J(\bar\partial_Jf\otimes_{\mathfrak{j}}\bar\partial_Jf)+f^*\omega_J\big) \tag{2.7}$$

by (2.5) and (2.4). If J is ω-compatible, the last term above vanishes. A smooth map $u:\Sigma\longrightarrow X$ is J-holomorphic if $\bar\partial_J u=0$. For such a map, the last two terms in (2.7) vanish.

For each $R\in\mathbb{R}^+$, denote by $B_R\subset\mathbb{C}$ the open ball of radius R around the origin and let

$$B_R^*=B_R-\{0\}.$$

If in addition (X,g) is a Riemannian manifold and $x\in X$, let $B_\delta^g(x)\subset X$ be the ball of radius δ around x in X with respect to the metric g.

Let (X,J) be an almost complex manifold and $(\Sigma,\mathfrak{j})$ be a Riemann surface. A smooth map $u:\Sigma\longrightarrow X$ is called

- somewhere injective if there exists $z\in\Sigma$ such that $u^{-1}(u(z))=\{z\}$ and $\mathrm{d}_zu\neq0$,
- multiply covered if $u=u'\circ h$ for some smooth connected orientable surface Σ', branched cover $h:\Sigma\longrightarrow\Sigma'$ of degree different from ±1, and a smooth map $u':\Sigma'\longrightarrow X$,
- simple if it is not multiply covered.

By Proposition 4.11, every J-holomorphic map from a compact Riemann surface is simple if and only if it is somewhere injective (the *if* implication is trivial).

3 Local Properties

We begin by studying local properties of J-holomorphic maps u from Riemann surfaces $(\Sigma,\mathfrak{j})$ into almost complex manifolds (X,J) that resemble standard properties of holomorphic maps. None of the statements in Section 3 depending on X being compact; very few depend on Σ being compact.

3.1 Carleman Similarity Principle

Carleman Similarity Principle, i.e. Proposition 3.1 below, is a local description of solutions of a non-linear differential equation which generalizes the equation $\bar\partial_J u=0$. It states that such solutions look similar to holomorphic maps and implies that they exhibit many local properties one would expect of holomorphic maps.

Proposition 3.1 (Carleman Similarity Principle, [2, Theorem 2.2])**.** *Suppose $n\in\mathbb{Z}^+$, $p,\epsilon\in\mathbb{R}^+$ with $p>2$, $J\in L^p_1(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$, $C\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$, and $u\in L^p_1(B_\epsilon;\mathbb{C}^n)$ are such that*

$$u(0)=0, \qquad J(z)^2=-\mathrm{Id}_{\mathbb{C}^n}, \quad u_s(z)+J(z)u_t(z)+C(z)u(z)=0 \;\;\forall\, z=s+\mathfrak{i}t\in B_\epsilon\,. \tag{3.1}$$

Then, there exist $\delta\in(0,\epsilon)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$\sigma(0)=0, \qquad J(z)\Phi(z)=\Phi(z)J_{\mathbb{C}^n}, \quad u(z)=\Phi(z)\sigma(z) \;\;\forall\, z\in B_\delta\,. \tag{3.2}$$

By the Sobolev Embedding Theorem [18, Corollary 4.3], the assumption $p>2$ implies that u is a continuous function. In particular, all equations in (3.1) and in (3.2) make sense. This assumption also implies that the left-hand sides of the third equation in (3.1) and of the second equation in (3.2) and the right-hand side of the third equations in (3.2) lie in L^p_1.

Example 3.2. Let $\mathfrak{c}\colon\mathbb{C}\longrightarrow\mathbb{C}$ denote the usual conjugation. Define

$$\widehat{J}(z_1,z_2)=\begin{pmatrix}\mathfrak{i}&0\\-2\mathfrak{i}s_1\mathfrak{c}&\mathfrak{i}\end{pmatrix}=\begin{pmatrix}1&0\\s_1\mathfrak{c}&1\end{pmatrix}J_{\mathbb{C}^2}\begin{pmatrix}1&0\\s_1\mathfrak{c}&1\end{pmatrix}^{-1}\colon\mathbb{C}^2\longrightarrow\mathbb{C}^2 \quad \forall\, z_i=s_i+\mathfrak{i}t_i,$$
$$u\colon\mathbb{C}\longrightarrow\mathbb{C}^2, \qquad u(s+\mathfrak{i}t)=\big(z,s^2\big).$$

Thus, $\widehat{J}$ is an almost complex structure on $\mathbb{C}^2$ and u is a $\widehat{J}$-holomorphic map, i.e. it satisfies the last condition in (3.1) with $J(z)=\widehat{J}(u(z))$ and $C(z)=0$. The functions

$$\sigma\colon\mathbb{C}\longrightarrow\mathbb{C}^2, \quad \sigma(z)=(z,0), \qquad \Phi\colon\mathbb{C}\longrightarrow\mathrm{GL}_4\mathbb{R}, \quad \Phi(s+\mathfrak{i}t)=\begin{pmatrix}1&0\\s\mathfrak{c}+\frac{\mathfrak{i}st}{z}&1\end{pmatrix},$$

satisfy (3.2).

Corollary 3.3. *Let n, p, ϵ, J, C, and u be as in Proposition 3.1. If in addition $J_0=J_{\mathbb{C}^n}$ and u does not vanish to infinite order 0, then there exist $\ell\in\mathbb{Z}^+$ and $\alpha\in\mathbb{C}^n-0$ such that*

$$\lim_{z\longrightarrow0}\frac{u(z)-\alpha z^\ell}{z^\ell}=0\,.$$

Proof. This follows from (3.2) and from the existence of such ℓ and α for σ. □

Corollary 3.4. *Suppose (X,J) is an almost complex manifold, $(\Sigma,\mathfrak{j})$ is a Riemann surface, and $u\colon \Sigma\longrightarrow X$ is a J-holomorphic map. If u is not constant on every connected component of Σ, then the subset*

$$u^{-1}\big(\{u(z)\colon\, z\in\Sigma,\ \mathrm{d}_z u=0\}\big)\subset\Sigma$$

is discrete. If in addition $x\in X$, the subset $u^{-1}(x)\subset\Sigma$ is also discrete.

Proof. The first and third equations in (3.2) immediately imply the second claim (but not the first, since Φ may not be in C^1). The first claim follows from Corollary 3.3 and Taylor's formula for u (as well as from Corollary 3.6). □

Before establishing the full statement of Proposition 3.1, we consider a special case.

Lemma 3.5. *Suppose $n\in\mathbb{Z}^+$ and $p,\epsilon\in\mathbb{R}^+$ are as in Proposition 3.1, $A\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$, and $u\in L^p_1(B_\epsilon;\mathbb{C}^n)$ are such that*

$$u(0)=0, \qquad u_s+J_{\mathbb{C}^n}u_t(z)+A(z)u(z)=0 \ \ \forall\ z=s+\mathfrak{i}t\in B_\epsilon\,. \tag{3.3}$$

Then, there exist $\delta\in(0,\epsilon)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_n\mathbb{C})$, a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$\sigma(0)=0, \qquad \Phi(0)=\mathrm{Id}_{\mathbb{C}^n}, \qquad u(z)=\Phi(z)\sigma(z) \ \ \forall\ z\in B_\delta\,. \tag{3.4}$$

Proof. For each $\delta\in[0,\epsilon]$, we define

$$A_\delta\in L^p(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad A_\delta(z)=\begin{cases}A(z), & \text{if } z\in B_\delta;\\ 0, & \text{otherwise};\end{cases}$$

$$D_\delta: L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad D_\delta\Theta=\big(\Theta_s+J_{\mathbb{C}^n}\Theta_t+A_\delta\Theta\big)\mathrm{d}\bar{z}\,.$$

Since the cokernel of $D_0=2\bar\partial$ is isomorphic $H^1(S^2;\mathbb{C})\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n$, D_0 is surjective and the homomorphism

$$\widetilde{D}_0\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus\mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow\big(D_0\Theta,\Theta(0)\big),$$

is an isomorphism. Since

$$\big\|D_\delta\Theta-D_0\Theta\big\|_{L^p}\le\|A_\delta\|_{L^p}\|\Theta\|_{C^0}\le C\|A_\delta\|_{L^p}\|\Theta\|_{L^p_1} \qquad \forall\ \Theta\in L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$$

and $\|A_\delta\|_{L^p}\longrightarrow0$ as $\delta\longrightarrow0$, the homomorphism

$$\widetilde{D}_\delta\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus\mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow\big(D_\delta\Theta,\Theta(0)\big),$$

is also an isomorphism for $\delta>0$ sufficient small. Let $\Theta_\delta=D_\delta^{-1}(0,\mathrm{Id}_{\mathbb{C}^n})$. Since D_δ is an isomorphism,

$$\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{C^0}\le C\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{L^p_1}\le C'\big\|D_\delta(\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n})\big\|_{L^p}=C'\big\|A_\delta\big\|_{L^p}\,.$$

Since $\|A_\delta\|_{L^p}\longrightarrow0$ as $\delta\longrightarrow0$, $\Theta_\delta\in L^p_1(B_\delta;\mathrm{GL}_n\mathbb{C})$. By (3.3) and $D_\delta\Theta_\delta=0$, the function $\sigma\equiv\Theta_\delta^{-1}u$ satisfies

$$\sigma(0)=0, \qquad \sigma_s+J_{\mathbb{C}^n}\sigma_t=0 \quad \forall\ z\in B_\delta,$$

i.e. σ is $J_{\mathbb{C}^n}$-holomorphic, as required. □

Proof of Proposition 3.1. (1) Since B_ϵ is contractible, the complex vector bundles $u^*(T\mathbb{C}^n, J_{\mathbb{C}^n})$ and $u^*(T\mathbb{C}^n, J)$ over B_ϵ are isomorphic. Thus, there exists

$$\Psi \in L^p_1(B_\epsilon; \mathrm{GL}_{2n}\mathbb{R}) \qquad \text{s.t.} \quad J(z)\Psi(z) = \Psi(z)J_{\mathbb{C}^n} \quad \forall\, z\in B_\epsilon\,.$$

Let $v=\Psi^{-1}u$. By the assumptions on u, $v\in L^p_1(B_\epsilon;\mathbb{C}^n)$ and

$$\begin{gathered} v(0)=0, \qquad v_s(z)+J_{\mathbb{C}^n}v_t(z)+\widetilde{C}(z)v(z)=0 \quad \forall\, z=s+\mathfrak{i}t\in B_\epsilon, \\ \text{where} \quad \widetilde{C}=\Psi^{-1}\cdot\big(\Psi_s+J\Psi_t+C\Psi\big)\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)\,. \end{gathered} \tag{3.5}$$

Thus, we have reduced the problem to the case $J=J_{\mathbb{C}^n}$.

(2) Let $\widetilde{C}^{\pm}=\frac{1}{2}(\widetilde{C}\mp J_{\mathbb{C}^n}\widetilde{C}J_{\mathbb{C}^n})$ be the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of $\widetilde{C}$, i.e. $\widetilde{C}^{\pm}J_{\mathbb{C}^n}=\pm J_{\mathbb{C}^n}\widetilde{C}^{\pm}$. With $\langle\cdot,\cdot\rangle$ denoting the Hermitian inner-product on $\mathbb{C}^n$ which is $\mathbb{C}$-antilinear in the second input, define

$$D\in L^\infty(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n), \quad D(z)w=\begin{cases}|v(z)|^{-2}\langle v(z),w\rangle v(z), & \text{if } v(z)\neq 0;\\ 0, & \text{otherwise;}\end{cases} \quad A=\widetilde{C}^{+}+\widetilde{C}^{-}D\,.$$

Since $DJ_{\mathbb{C}^n}=-J_{\mathbb{C}^n}D$ and $Dv=v$, $A\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$ and $Av=\widetilde{C}v$. Thus, by (3.5),

$$v_s+J_{\mathbb{C}^n}v_t+Av=0\,.$$

The claim now follows from Lemma 3.5. □

Corollary 3.6. *Suppose $n\in\mathbb{Z}^+$, $\epsilon\in\mathbb{R}^+$, J is a smooth almost complex structure on $\mathbb{C}^n$ with $J_0=J_{\mathbb{C}^n}$, and $u: B_\epsilon\longrightarrow\mathbb{C}^n$ is a J-holomorphic map with $u(0)=0$. Then, there exist $\delta\in(0,\epsilon)$, $C\in\mathbb{R}^+$, $\Phi\in C^0(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma: B_\delta\longrightarrow\mathbb{C}^n$ such that Φ is smooth on B^*_δ,*

$$\sigma(0)=0, \quad \Phi(0)=\mathrm{Id}_{\mathbb{C}^n}, \quad J(u(z))\Phi(z)=\Phi(z)J_{\mathbb{C}^n}, \quad u(z)=\Phi(z)\sigma(z), \quad \big|\mathrm{d}_z\Phi\big|\le C\ \forall\, z\in B^*_\delta\,.$$

Proof. We can assume that u is not identically 0 on some neighborhood of $0\in B_\epsilon$. Similarly to (1) in the proof of Proposition 3.1, there exists

$$\Psi\in C^\infty(\mathbb{C}^n;\mathrm{GL}_{2n}\mathbb{R}) \qquad \text{s.t.} \qquad \Psi(0)=\mathrm{Id}_{\mathbb{C}^n}, \quad J(x)\Psi(x)=\Psi(x)J_{\mathbb{C}^n} \quad \forall\, x\in\mathbb{C}^n\,.$$

Let $v(z)=\Psi(u(z))^{-1}u(z)$. By Corollary 3.3, we can choose complex linear coordinates on $\mathbb{C}^n$ so that

$$v(z)=\big(f(z),g(z)\big)h(z)\in\mathbb{C}\oplus\mathbb{C}^{n-1} \qquad \forall\, z\in B_{\epsilon'}$$

for some $\epsilon'\in(0,\epsilon)$, holomorphic function h on $B_{\epsilon'}$ with $h(0)=0$, and continuous functions f and g on $B_{\epsilon'}$ with $f(0)=1$ and $g(0)=0$. By Lemma 3.7 below applied with f above and with each component of g separately, there exists $\delta\in(0,\epsilon')$ so that the function

$$\Phi\colon B_\delta\longrightarrow\mathrm{GL}_{2n}\mathbb{R}, \qquad \Phi(z)=\Psi\big(u(z)\big)\begin{pmatrix} f(z) & 0\\ g(z) & 1\end{pmatrix},$$

is continuous on B_δ and smooth on $B_\delta-0$ with $|\mathrm{d}_z\Phi|$ uniformly bounded on $B_\delta-0$. Taking $\sigma(z)=(h(z),0)$, we conclude the proof. □

Lemma 3.7. *Suppose $\epsilon\in\mathbb{R}^+$, and $f,h\colon B_\epsilon\longrightarrow\mathbb{C}$ are continuous functions such that h is holomorphic, $h(z)\neq 0$ for some $z\in B_\epsilon$, and the function*

$$B_\epsilon\longrightarrow\mathbb{C},\qquad z\longrightarrow f(z)h(z), \tag{3.6}$$

is smooth. Then there exist $\delta\in(0,\epsilon)$ and $C\in\mathbb{R}^+$ such that f is differentiable on $B_\epsilon-0$ and

$$\big|\mathrm{d}_z f\big|\le C\qquad\forall\ z\in B_\delta-0\,. \tag{3.7}$$

Proof. After a holomorphic change of coordinate on $B_{2\delta}\subset B_\epsilon$, we can assume that $h(z)=z^\ell$ for some $\ell\in\mathbb{Z}^{\ge 0}$. Define

$$g\colon B_{2\delta}\longrightarrow\mathbb{C},\qquad g(z)=f(z)z^\ell-f(0)z^\ell\,.$$

By Taylor's Theorem and the smoothness of the function (3.6), there exists $C>0$ such that the smooth function g satisfies

$$\big|g(z)\big|\le C|z|^{\ell+1}\qquad\forall\ z\in B_\delta\,.$$

Dividing g by z^ℓ, we thus obtain (3.7). □

Remark 3.8. Corollary 3.6 refines the conclusion of Proposition 3.1 for J-holomorphic maps. In contrast to the output (Φ,σ) of Proposition 3.1, the output of Corollary 3.6 does not depend continuously on the input u with respect to the L^p_1-norms. This makes Corollary 3.6 less suitable for applications in settings involving families of J-holomorphic maps.

3.2 Local structure of J-holomorphic maps

We now obtain three corollaries from Proposition 3.1. They underpin important geometric statements established later in these notes, such as Propositions 3.12 and 4.11 and Lemma 5.4.

Corollary 3.9 (Unique Continuation)**.** *Suppose (X,J) is an almost complex manifold, $(\Sigma,\mathfrak{j})$ is a connected Riemann surface, and*

$$u,u'\colon(\Sigma,\mathfrak{j})\longrightarrow(X,J)$$

are J-holomorphic maps. If u_0 and u'_0 agree to infinite order at $z_0\in\Sigma$, then $u'=u'$.

Proof. Since the subset of the points of Σ at which u and u' agree is closed to infinite order, it is enough to show that $u=u'$ on some neighborhood of z_0. By the continuity of u, we can assume that $X=\mathbb{C}^n$, $\Sigma=B_1$, $z_0=0$, and $u(0),u'(0)=0$. Let

$$w=u'-u:B_\epsilon\longrightarrow\mathbb{C}^n\,.$$

Since J is C^1,

$$J(x+y)=J(x)+\int_0^1\frac{\mathrm{d}J(x+ty)}{\mathrm{d}t}\mathrm{d}t=J(x)+\sum_{i=1}^n y_i\int_0^1\frac{\partial J}{\partial y_i}\bigg|_{x+ty}\mathrm{d}t\,. \tag{3.8}$$

Since u and u' are J-holomorphic, (3.8) implies that

$$\partial_s w+J\big(u(z)\big)\partial_t w+C(z)w(z)=0,\qquad\text{where}\quad C\in L^p\big(B_1;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n\big),$$
$$C(z)y=\sum_{i=1}^n y_i\bigg(\int_0^1\frac{\partial J}{\partial y_i}\bigg|_{v(z)+tw(z)}\mathrm{d}t\bigg)\partial_t w|_z\,.$$

By Proposition 3.1, there thus exist $\delta\in(0,1)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and holomorphic map $\widetilde{w}\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$w(z)=\Phi(z)\widetilde{w}(z)\qquad\forall\ z\in B_\delta\,.$$

Since w vanishes to infinite order at 0, it follows that $\widetilde{w}(z)=0$ for all $z\in B_\delta$ (otherwise, w would satisfy the conclusion of Corollary 3.3) and thus $w(z)=0$ for all $z\in B_\delta$. □

Corollary 3.10. *Suppose (X,J) is an almost complex manifold,*

$$u,u'\colon(\Sigma,\mathrm{j}),(\Sigma',\mathrm{j}')\longrightarrow(X,J)$$

are J-holomorphic maps, $z_0\in\Sigma$ is such that $\mathrm{d}_{z_0}u\neq0$, and $z_0'\in\Sigma'$ is such that $u'(z_0')=u(z_0)$. If there exist sequences $z_i\in\Sigma-z_0$ and $z_i'\in\Sigma'-z_0'$ such that

$$\lim_{i\longrightarrow\infty}z_i=z_0\,,\qquad\lim_{i\longrightarrow\infty}z_i'=z_0'\,,\quad\text{and}\quad u(z_i)=u'(z_i)\ \ \forall\ i\in\mathbb{Z}^+\,,$$

then there exists a holomorphic map $\sigma\colon U'\longrightarrow\Sigma$ from a neighborhood of z_0' in Σ' such that $\sigma(z_0')=z_0$ and $u'|_{U'}=u\circ\sigma$.

Proof. It can be assumed that $(\Sigma,\mathrm{j},z_0),(\Sigma',\mathrm{j}',z_0')=(B_1,\mathrm{j}_0,0)$, where $B_1\subset\mathbb{C}$ is the unit ball with the standard complex structure. Since $\mathrm{d}_{z_0}u\neq0$ and u is J-holomorphic, u is an embedding near $0\in B_1$ and so is a slice in a coordinate system. Thus, we can assume that

$$u,u'\equiv(v,w)\colon(B_1,0)\longrightarrow(\mathbb{C}\times\mathbb{C}^{n-1},0),\qquad u(z)=(z,0)\in\mathbb{C}\times\mathbb{C}^{n-1}\,,$$

and u,u' are J-holomorphic with respect to some almost complex structure

$$J(x,y)=\begin{pmatrix}J_{11}(x,y)&J_{12}(x,y)\\J_{21}(x,y)&J_{22}(x,y)\end{pmatrix}\colon\mathbb{C}\times\mathbb{C}^{n-1}\longrightarrow\mathbb{C}\times\mathbb{C}^{n-1}\,,\qquad(x,y)\in\mathbb{C}\times\mathbb{C}^{n-1}\,.$$

Since J is C^1,

$$J_{ij}(x,y)=J_{ij}(x,0)+\int_0^1\frac{\mathrm{d}J_{ij}(x,ty)}{\mathrm{d}t}\mathrm{d}t=J_{ij}(x,0)+\sum_{i=1}^{n-1}y_i\int_0^1\frac{\partial J_{ij}}{\partial y_i}\bigg|_{(x,ty)}\mathrm{d}t\,.\tag{3.9}$$

Since u is J-holomorphic,

$$J_{21}(x,0)=0,\quad J_{22}(x,0)^2=-\mathrm{Id}\qquad\forall\ x\in B_1\subset\mathbb{C}.\tag{3.10}$$

Since u' is J-holomorphic,

$$\partial_sw+J_{22}\big(v(z),w(z)\big)\partial_tw+J_{21}\big(v(z),w(z)\big)\partial_tv=0.$$

Combining this with (3.9) and the first equation in (3.10), we find that

$$\partial_sw+J_{22}\big(v(z),0\big)\partial_tw+C(z)w(z)=0,\qquad\text{where}\quad C\in L^p\big(B_1;\mathrm{End}_{\mathbb{R}}\mathbb{C}^{n-1}\big),$$
$$C(z)y=\sum_{i=1}^{n-1}y_i\Bigg(\bigg(\int_0^1\frac{\partial J_{22}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_tw|_z+\bigg(\int_0^1\frac{\partial J_{21}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_tv|_z\Bigg)\,.$$

By Proposition 3.1 and the second identity in (3.10), there thus exist $\delta\in(0,1)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n-2}\mathbb{R})$, and holomorphic map $\widetilde{w}\colon B_\delta\longrightarrow\mathbb{C}^{n-1}$ such that

$$w(z)=\Phi(z)\widetilde{w}(z)\qquad\forall\ z\in B_\delta\,.$$

Since $u'(z_i')=u(z_i)$, $\widetilde{w}(z_i')=0$ for all $i\in\mathbb{Z}^+$. Since $z_i'\longrightarrow0$ and $z_i'\neq0$, it follows that $w=0$. This implies the claim with $U'=B_\delta$ and $\sigma=v$. □

Corollary 3.11. *Let (X,J) be an almost complex manifold with a Riemannian metric g and $x\in X$ be such that g is compatible with J at x. If $u:\Sigma\longrightarrow X$ is a J-holomorphic map from a compact Riemann surface with boundary, then*

$$\lim_{\delta\longrightarrow 0}\frac{E_g(u;u^{-1}(B_\delta^g(x)))}{\pi\delta^2}=\mathrm{ord}_x u\,.$$

Proof. By the continuity of u, we can assume that $X=\mathbb{C}^n$, J agrees with the standard complex structure $J_{\mathbb{C}^n}$ at the origin, g agrees with the standard metric $g_{\mathbb{C}^n}$ at the origin, $\Sigma=\overline{B_R}$ for some $R\in\mathbb{R}^+$, and $u(0)=0$. In particular, there exists $C\geq 1$ such that

$$\big|J_x-J_{\mathbb{C}^n}\big|\leq C|x|,\quad \big|g_x-g_{\mathbb{C}^n}\big|\leq C|x| \qquad \forall\, x\in\mathbb{C}^n \text{ s.t. } |x|\leq 1, \tag{3.11}$$

where $|\cdot|$ denotes the usual norm of x (i.e. the distance to the origin with respect to $g_{\mathbb{C}^n}$).

Let $\ell\equiv\mathrm{ord}_0 u$ and $\alpha\in\mathbb{C}^{n-1}-0$ be as in Corollary 3.3, where $0\in B_R$ is the origin in the domain of u. Thus, there exist $\epsilon\in(0,1)$ and $C\in\mathbb{R}^+$ such that

$$u(z)=\alpha\cdot\big(z^\ell+f(z)\big),\quad \big|f(z)\big|\leq C|z|^{\ell+1}\qquad \forall\, z\in B_\epsilon\,. \tag{3.12}$$

Let $z=s+\mathfrak{i}t$ as before. By (3.12), there exists $C\in\mathbb{R}^+$ such that

$$u_s(z)=\alpha\cdot\big(\ell z^{\ell-1}+f_s(z)\big),\quad u_s(z)=\alpha\cdot\big(\ell\mathfrak{i}z^{\ell-1}+f_t(z)\big),\quad \big|f_s(z)\big|,\big|f_t(z)\big|\leq C|z|^\ell\quad \forall\, z\in B_\epsilon\,. \tag{3.13}$$

We can also assume that the three constants C in (3.11), (3.12), and (3.13) are the same, $C\geq 1$,

$$C_\alpha\epsilon\equiv\big(C+C|\alpha|+C^2|\alpha|\big)\epsilon\leq 1\,,$$

and $|u(z)|\leq 1$ for all $z\in B_\epsilon$. By (3.11)-(3.13),

$$\begin{split}\left|\frac{|u(z)|_g}{|\alpha||z|^\ell}-1\right|,\left|\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}}-1\right|,\left|\frac{|u_t(z)|_g}{|\alpha|\ell|z|^{\ell-1}}-1\right|&\leq C|z|+C|\alpha||z|^\ell+C^2|\alpha||z|^{\ell+1}\\ &\leq C_\alpha|z|\quad\forall\, z\in B_\epsilon,\end{split} \tag{3.14}$$

where $|\cdot|_g$ denotes the distance to the origin in $\mathbb{C}^n$ with respect to the metric g and the corresponding norm on $T\mathbb{C}^n$.

Given $r\in(0,1)$, let $\delta_r\in(0,\epsilon)$ be such that

$$C_\alpha\bigg(\frac{2\delta_r}{(1-r)|\alpha|}\bigg)^{1/\ell}\leq r\,. \tag{3.15}$$

For any $\delta\in[0,\delta_r]$, (3.14) and (3.15) give

$$\begin{aligned} |z|\leq\bigg(\frac{\delta}{(1+r)|\alpha|}\bigg)^{1/\ell} &\qquad\Longrightarrow\qquad u(z)\in B_\delta^g(0)\,,\\ u(z)\in B_\delta^g(0) &\qquad\Longrightarrow\qquad |z|\leq\bigg(\frac{\delta}{(1-r)|\alpha|}\bigg)^{1/\ell},\\ |z|\leq\bigg(\frac{\delta}{(1-r)|\alpha|}\bigg)^{1/\ell} &\qquad\Longrightarrow\qquad 1-r\leq\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}},\frac{|u_t(z)|_g}{|\alpha|\ell|z|^{\ell-1}}\leq 1+r. \end{aligned}$$

Combining these, we obtain

$$\int_{|z|\le\left(\frac{\delta}{(1+r)|\alpha|}\right)^{\frac{1}{\ell}}}(1-r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2 \le \frac{1}{2}\int_{u^{-1}(B^g_\delta(0))}\big(|u_s|^2_g+|u_t|^2_g\big)$$
$$\le \int_{|z|\le\left(\frac{\delta}{(1-r)|\alpha|}\right)^{\frac{1}{\ell}}}(1+r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2 .$$

Evaluating the outer integrals, we find that

$$\left(\frac{1-r}{1+r}\right)^2\ell\pi\delta^2 \le E_g\big(u;u^{-1}(B^g_\delta(0))\big) \le \left(\frac{1+r}{1-r}\right)^2\ell\pi\delta^2 .$$

These inequalities hold for all $r\in(0,1)$ and $\delta\in(0,\delta_r)$; the claim is obtained by sending $r\longrightarrow 0$. $\square$

3.3 The Monotonicity Lemma

Proposition 3.12 below is a key step in the continuity part of the proof of the Removal of Singularity Proposition 5.1. The precise nature of the lower energy bound on the right hand-side of (3.16) does not matter, as long as it is positive for $\delta>0$.

Proposition 3.12 (Monotonicity Lemma). *If (X,J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $C_{g,J}: X\longrightarrow\mathbb{R}^+$ with the following property. If $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary, $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map, $x\in X$, and $\delta\in\mathbb{R}^+$ are such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$, then*

$$E_g(u) \ge \big(\mathrm{ord}_x u\big)\frac{\pi\delta^2}{(1+C_{g,J}(x)\delta)^4} . \tag{3.16}$$

If $\omega(\cdot,\cdot)\equiv g(J\cdot,\cdot)$ is a symplectic form on X, then the above fraction can be replaced by the product $\pi\delta^2\mathrm{e}^{-C_{g,J}(x)\delta^2}$.

According to this proposition, "completely getting out" of the ball $B_\delta(x)$ via a J-holomorphic map requires an energy bounded below by a little less than $\pi\delta^2$. Thus, the L^2_1-norm of a J-holomorphic map u exerts some control over the C^0-norm of u. If $p>2$, the L^p_1-norm of any smooth map f from a two-dimensional manifold controls the C^0-norm of f. However, this is not the case of the L^2_1-norm, as illustrated by the example of [12, Lemma 10.4.1]: the function

$$f_\epsilon\colon\mathbb{R}^2\longrightarrow[0,1], \qquad f_\epsilon(z)=\begin{cases}1, & \text{if } |z|\le\epsilon;\\ \frac{\ln|z|}{\ln\epsilon}, & \text{if } \epsilon\le|z|\le 1;\\ 0, & \text{if } |z|\ge 1;\end{cases}$$

with any $\epsilon\in(0,1)$ is continuous and satisfies

$$\int_{\mathbb{R}^2}|\mathrm{d}f_\epsilon|^2_g = -\frac{2\pi}{\ln\epsilon} .$$

It is arbitrarily close in the L^2_1-norm to a smooth function $\widetilde{f}_\epsilon$. Thus, it is possible to "completely get out" of $B^g_\delta(x)$ using a smooth function with arbitrarily small energy ($\widetilde{f}_\delta$ does this for the ball $B_1(1)$ in $\mathbb{R}$).

By (2.7), the holomorphic maps are the local minima of the functional

$$C^{\infty}(\Sigma;X)\longrightarrow\mathbb{R},\qquad f\longrightarrow E_g(f)-\int_{\Sigma}f^*\omega_J\,,$$

for every compact Riemann surface $(\Sigma,\mathfrak{j})$ without boundary. This fact underlines Lemma 3.16, the key ingredient in the proof of the Monotonicity Lemma. Lemma 3.16 implies that the ratio of $E_g(u;u^{-1}(B_\delta^g(x)))$ and the fraction on the right-hand side (3.16) is a non-decreasing function of δ, as long as $u(\partial\Sigma)\cap B_\delta^g(x)=\emptyset$. By Corollary 3.11, this ratio approaches $\mathrm{ord}_x u$ as δ approaches 0. These two statements imply Proposition 3.12.

We first make some general Riemannian geometry observations. Let (X,g) be a Riemannian manifold. Denote by $\exp\colon\mathcal{W}_g\longrightarrow X$, the exponential map from a neighborhood of X in TX with respect to the Levi-Civita connection ∇ of g. For each $v\in TX$, we denote by

$$\gamma_v\colon[0,1]\longrightarrow X,\qquad \gamma_v(\tau)=\exp_x(\tau v),$$

the geodesic with $\gamma_v'(0)=v$. Let

$$r_g\colon X\longrightarrow\mathbb{R}^+\qquad\text{and}\qquad d_g\colon X\times X\longrightarrow\mathbb{R}^{\ge0}$$

be the injectivity radius of exp and the distance function. For each $x\in X$, define

$$\zeta_x\in\Gamma\big(B^g_{r_g(x)}(x);TX\big)\quad\text{by}\quad \exp_y\big(\zeta_x(y)\big)=x,\;\; g\big(\zeta_x(y),\zeta_x(y)\big)<r_g(x)^2\quad\forall\,y\in B^g_{r_g(x)}(x).$$

Lemma 3.13. *Let (X,g) be a Riemannian manifold and $x\in X$. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow X$ is a smooth curve such that $\alpha(0)\in B^g_{r_g(x)}(x)$, then*

$$\frac12\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\bigg|_{\tau=0}=-g\big(\alpha'(0),\zeta_x(\alpha(0))\big)\,.$$

Proof. If $\beta(\tau)=\exp_x^{-1}\alpha(\tau)$, then

$$\frac12\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\bigg|_{\tau=0}=\frac12\frac{\mathrm{d}}{\mathrm{d}\tau}|\beta(\tau)|^2\bigg|_{\tau=0}=g\big(\beta'(0),\beta(0)\big)\,.$$

By Gauss's Lemma,

$$g\big(\beta'(0),\beta(0)\big)=g\big(\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta'(0)),\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta(0))\big)=g\big(\alpha'(0),-\zeta_x(\alpha(0))\big)\,.$$

This establishes the claim. ∎

Lemma 3.14. *If (X,g) is a Riemannian manifold, there exists a continuous function $C_g\colon X\longrightarrow\mathbb{R}^+$ with the following property. If $x\in X$, $v\in T_xX$ with $|v|_g<\frac12 r_g(x)$, and $\tau\longrightarrow J(\tau)$ is a Jacobi vector field along the geodesic γ_v with $J(0)=0$, then*

$$\big|J'(1)-J(1)\big|_g\le C_g(x)|v|_g^2\big|J(1)\big|_g\,.$$

Proof. Let R_g be the Riemann curvature tensor of g and $f(\tau)=|\tau J'(\tau)-J(\tau)|_g$. Then, $f(0)=0$ and

$$f(\tau)f'(\tau)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}f(\tau)^2=g\big(\tau J''(\tau),\tau J'(\tau)-J(\tau)\big)=\tau g\big(R(\gamma'(\tau),J(\tau))\gamma'(\tau),\tau J'(\tau)-J(\tau)\big)$$
$$\le C_g(x)|v|_g^2|J(\tau)|_g\tau f(\tau).$$

If C_g is sufficiently large, then $|J(\tau)|_g\le C_g(x)|J(1)|_g$. Thus,

$$f(\tau)f'(\tau)\le C_g(x)|v|_g^2|J_v(\tau)|_g\tau f(\tau)\le C_g(x)^2|v|_g^2|J(1)|_g\tau f(\tau),\quad f'(\tau)\le C_g(x)^2|v|_g^2|J(1)|_g\tau.$$

The claim follows from the last inequality. □

Corollary 3.15. *If (X,g) is a Riemannian manifold, there exists a continuous function $C_g\colon X\longrightarrow\mathbb{R}^+$ with the following property. If $x\in X$, then*

$$\big|\nabla_w\zeta_x|_y+w\big|_g\le C_g(x)d_g(x,y)^2|w|_g\qquad\forall\ w\in T_yX,\ y\in B^g_{r_g(x)/2}(x).$$

Proof. Let $\tau\longrightarrow u(s,\tau)$ be a family of geodesics such that

$$u(s,0)=x,\qquad u(0,1)=y,\qquad \frac{\mathrm{d}}{\mathrm{d}s}u(s,1)\bigg|_{s=0}=w.$$

Since $\tau\longrightarrow u(s,\tau)$ is a geodesic,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u(s,\tau)\bigg|_{\tau=1}=\big\{\mathrm{d}_{u_\tau(s,0)}\exp_x\big\}\big(u_\tau(s,0)\big)=-\zeta_x\big(u(s,1)\big),$$
$$\frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}s}\bigg|_{(s,\tau)=(0,1)}=\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}\tau}\bigg|_{(s,\tau)=(0,1)}=-\nabla_w\zeta_x|_y\,.$$

Furthermore, $J(\tau)\equiv\frac{\mathrm{d}}{\mathrm{d}s}u(s,\tau)\big|_{s=0}$ is a Jacobi vector field along the geodesic $\tau\longrightarrow u(0,\tau)$ with

$$J(0)=0,\quad J(1)=w,\quad J'(1)=\frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}s}\bigg|_{(s,\tau)=(0,1)}=-\nabla_w\zeta_x|_y\,.$$

Thus, the claim follows from Lemma 3.14. □

Lemma 3.16. *Suppose (X,ω) is a symplectic manifold, J is an almost complex structure on X tamed by ω, and ∇ is the Levi-Civita connection of the metric g_J. If $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary and $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map, then*

$$\int_\Sigma g_J\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi\big)=\int_\Sigma\big(u^*\{\nabla_\xi\omega_J\}+\omega_J(\mathrm{d}u\wedge_{\mathfrak{j}}\nabla\xi)\big)\qquad\forall\ \xi\in\Gamma(\Sigma;u^*TX)\ s.t.\ \xi|_{\partial\Sigma}=0.$$

Proof. For $\tau\in\mathbb{R}$ sufficiently close to 0, define

$$u_\tau\colon\Sigma\longrightarrow X,\qquad u_\tau(z)=\exp_{u(z)}(\tau\xi(z)).$$

Since $\xi|_{\partial\Sigma}=0$, $u_\tau|_{\partial\Sigma}=u|_{\partial\Sigma}$. Denote by $\widehat\Sigma$ the closed oriented surface obtained by gluing two copies of Σ along the common boundary and reversing the orientation on the second copy. Let

$$\widehat u_\tau\colon\widehat\Sigma\longrightarrow X$$

be the map restricting to u_τ on the first copy of Σ and to u on the second.

By (2.7),

$$E(\tau)\equiv E_{g_J}(u_\tau)-\int_\Sigma u_\tau^*\omega_J-E_{g_J}(u)=\int_{\widehat\Sigma}\widehat u_\tau^*\omega+2\int_\Sigma g_J\big(\bar\partial u_\tau\otimes_{\mathfrak j}\bar\partial u_\tau\big)\ge 0\quad\forall\tau.$$

Since ω is closed and $\widehat u_*$ represents the zero class in $H_2(X;\mathbb Z)$, the first integral on the right-hand side above vanishes. Thus, the function $\tau\longrightarrow E(\tau)$ is minimized at $\tau=0$ (when it equals 0) and so

$$\begin{aligned}0=E'(0)&=\frac{\mathrm d}{\mathrm d\tau}\bigg(E_{g_J}(u_\tau)-\int_\Sigma u_\tau^*\omega_J\bigg)\bigg|_{\tau=0}\\&=\frac{\mathrm d}{\mathrm d\tau}\bigg(\frac12\int_\Sigma g_J(\mathrm du_\tau\otimes_{\mathfrak j}\mathrm du_\tau)-\int_\Sigma u_\tau^*\omega_J\bigg)\bigg|_{\tau=0};\end{aligned}\tag{3.17}$$

the last equality above uses the definition of $E(u_\tau)$ in (2.5).

Let $z=s+\mathfrak it$ be a local coordinate on $(\Sigma,\mathfrak j)$. Since ∇ is torsion-free,

$$\frac{D}{\mathrm d\tau}(u_\tau)_s\Big|_{\tau=0}\equiv\frac{D}{\mathrm d\tau}\frac{\mathrm du_\tau}{\mathrm ds}\Big|_{\tau=0}=\frac{D}{\mathrm ds}\frac{\mathrm du_\tau}{\mathrm d\tau}\Big|_{\tau=0}=\frac{D}{\mathrm ds}\xi\equiv\nabla_s\xi,\qquad\frac{D}{\mathrm d\tau}(u_\tau)_t\Big|_{\tau=0}=\nabla_t\xi\,.$$

Since ∇ is also g-compatible,

$$\begin{aligned}\frac12\frac{\mathrm d}{\mathrm d\tau}g_J(\mathrm du_\tau\otimes_{\mathfrak j}\mathrm du_\tau)\Big|_{\tau=0}&=\bigg(g_J\Big(u_s,\frac{D}{\mathrm d\tau}(u_\tau)_s\Big|_{\tau=0}\Big)+g_J\Big(u_t,\frac{D}{\mathrm d\tau}(u_\tau)_t\Big|_{\tau=0}\Big)\bigg)\mathrm ds\wedge\mathrm dt\\&=g_J(u_s,\nabla_s\xi)+g_J(u_t,\nabla_t\xi)=g_J\big(\mathrm du\otimes_{\mathfrak j}\nabla\xi\big)\,,\\\frac{\mathrm d}{\mathrm d\tau}u_\tau^*\omega_J\Big|_{\tau=0}&=\bigg(\big\{\nabla_\xi\omega_J\big\}(u_s,u_t)+\omega_J\Big(\frac{D}{\mathrm d\tau}(u_\tau)_s\Big|_{\tau=0},u_t\Big)+\omega_J\Big(u_s,\frac{D}{\mathrm d\tau}(u_\tau)_t\Big|_{\tau=0}\Big)\bigg)\mathrm ds\wedge\mathrm dt\\&=u^*\{\nabla_\xi\omega_J\}+\omega_J\big(\mathrm du\wedge_{\mathfrak j}\nabla\xi\big)\,.\end{aligned}$$

Combining this with (3.17), we obtain the claim. □

Proof of Proposition 3.12. Let $\delta_g:X\longrightarrow\mathbb R^+$ be a continuous function such that for every $x\in X$ there exists a symplectic form ω_x on $B^g_{2\delta_g(x)}(x)$ so that J is tamed by ω_x on $B^g_{2\delta_g(x)}(x)$ and compatible with ω_x at x. We assume that $2\delta_g(x)\le r_g(x)$ for every $x\in X$. It is sufficient to establish the proposition for each $x\in X$ and each $\delta\le\delta_g(x)$ under the assumption that the metric g is determined by J and ω_x on $B^g_{\delta_g(x)}(x)$.

Choose a C^∞-function $\eta:\mathbb R\longrightarrow[0,1]$ such that

$$\eta(\tau)=\begin{cases}1,&\text{if }\tau\le\frac12;\\0,&\text{if }\tau\ge1;\end{cases}\qquad\eta'(\tau)\le0.\tag{3.18}$$

For a compact Riemann surface with boundary $(\Sigma,\mathfrak j)$, a smooth map $u:\Sigma\longrightarrow X$, $x\in X$, and $\delta\in\mathbb R^+$, define

$$\eta_{u,x,\delta}\in C^\infty(\Sigma;\mathbb R),\qquad\eta_{u,x,\delta}(z)=\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg),$$
$$E_{u,x,\eta}(\delta)=\frac12\int_\Sigma\eta_{u,x,\delta}(z)g\big(\mathrm du\otimes_{\mathfrak j}\mathrm du\big)\,,\quad E_{u,x}(\delta)=E_g\big(u;u^{-1}(B^g_\delta(x))\big).$$

We show in the remainder of this proof that there exists a continuous function $C_{g,J}\colon X\longrightarrow\mathbb{R}^+$ such that

$$-\delta E'_{u,x,\eta}(\delta)+2E_{u,x,\eta}(\delta)\le 2C_{g,J}(x)\delta E_{u,x,\eta}(\delta)+C_{g,J}(x)\delta^2E'_{u,x,\eta}(\delta) \tag{3.19}$$

for every compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, J-holomorphic map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$ such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. This inequality is equivalent to

$$\bigg(E_{u,x,\eta}(\delta)\Big/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}\bigg)'\ge 0.$$

By Lebesgue's Dominated Convergence Theorem, $E_{u,x,\eta}(\delta)$ approaches $E_{u,x}(\delta)$ from below as η approaches the characteristic function $\chi_{(-\infty,1)}$ of $(-\infty,1)$. Thus, the function

$$\delta\longrightarrow E_{u,x}(\delta)\Big/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}$$

is non-decreasing as long as $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. By Corollary 3.11,

$$\lim_{\delta\longrightarrow0}\bigg(E_{u,x}(\delta)\Big/\frac{\delta^2}{(1+C_{g,J}(x)\delta)^4}\bigg)=\lim_{\delta\longrightarrow0}\frac{E_{u,x}(\delta)}{\delta^2}=\big(\mathrm{ord}_xu\big)\pi.$$

This implies the first claim.

Fix $x\in X$. We note that

$$E'_{u,x,\eta}(\delta)=-\frac12\int_\Sigma\eta'\Big(\frac{d_g(x,u(z))}{\delta}\Big)\frac{d_g(x,u(z))}{\delta^2}g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\mathrm{d}u\big). \tag{3.20}$$

For a compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, a smooth map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$, let

$$\xi_{u,x,\delta}\in\Gamma(\Sigma;u^*TX),\qquad \xi_{u,x,\delta}(z)=-\eta_{u,x,\delta}(z)\zeta_x\big(u(z)\big);$$

the vanishing assumption in (3.18) implies that $\xi_{u,x,\delta}$ is well-defined. If $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$, then $\xi_{u,x,\delta}|_{\partial\Sigma}=0$. By Lemma 3.13,

$$\nabla\xi_{u,x,\delta}|_z=\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{1}{\delta\,d_g(x,u(z))}g\big(\mathrm{d}_zu,\zeta_x(u(z))\big)\zeta_x(u(z))-\eta_{u,x,\delta}(z)\nabla\zeta_x\circ\mathrm{d}_zu. \tag{3.21}$$

Along with Corollary 3.15, (3.20), and the last assumption in (3.18), this implies that

$$\int_\Sigma d_g(x,u(z))\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\le 2\delta^2E'_{u,x,\eta}(\delta)+2\big(1+C_g(x)\delta^2\big)\delta E_{u,x,\eta}(\delta). \tag{3.22}$$

By the ω_x-compatibility assumption on J at x, there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\int_\Sigma\big|(\omega_x)_J(\mathrm{d}u\wedge_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\le C(x)\int_\Sigma d_g\big(x,u(z)\big)\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|$$

for all u and δ as above. Along with this, Lemma 3.16 implies that there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\bigg|\int_\Sigma g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta}\big)\bigg|\le C(x)\int_\Sigma\Big(g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\mathrm{d}u\big)|\xi_{u,x,\delta}|+d_g(x,u(z))\big|g(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta})\big|\Big)$$

for every compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, J-holomorphic map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$ such that $u(\partial\Sigma)\cap B_\delta^g(x)=\emptyset$. Combining this with (3.22), we conclude that there exists a continuous function $C\colon X\longrightarrow\mathbb{R}^+$ such that

$$\bigg|\int_\Sigma g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta}\big)\bigg|\le C(x)\big(\delta E_{u,x,\eta}(\delta)+\delta^2E'_{u,x,\eta}(\delta)\big) \tag{3.23}$$

for all u and δ as above.

Suppose $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary, $u:\Sigma\longrightarrow X$ is a smooth map, and $\delta\in(0,\delta_g(x))$. Let $z=s+\mathfrak{i}t$ be a coordinate on $(\Sigma,\mathfrak{j})$. By (3.21),

$$\begin{split} g\big(u_s,\nabla_s\xi_{u,x,\delta}\big)=\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{1}{\delta\,d_g(x,u(z))}g\big(u_s,\zeta_x(u(z))\big)^2\\ +\eta_{u,x,\delta}(z)g\big(u_s,\nabla_s(-\zeta_x)|_z\big). \end{split} \tag{3.24}$$

By Corollary 3.15,

$$|u_s|^2\le g\big(u_s,\nabla_s(-\zeta_x)|_z\big)+C_g(x)d_g(x,u(z))^2|u_s|^2\quad\forall\,z\in u^{-1}\big(B^g_{\delta_g(x)}(x)\big)\,. \tag{3.25}$$

If u is J-holomorphic, then $|u_s|=|u_t|$, $\langle u_s,u_t\rangle=0$, and

$$\frac12\big(|u_s|^2+|u_t|^2\big)d_g(x,u(z))^2=|u_s|^2|\zeta_x(u(z))|^2\ge g\big(u_s,\zeta_x(u(z))\big)^2+g\big(u_t,\zeta_x(u(z))\big)^2\,. \tag{3.26}$$

Since $\eta'\le0$, (3.24)-(3.26) give

$$\begin{split} \frac12\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{d_g(x,u(z))}{\delta}\big(|u_s|^2+|u_t|^2\big)+\eta_{u,x,\delta}(z)\big(|u_s|^2+|u_t|^2\big)\\ \le g\big(u_s,\nabla_s\xi_{u,x,\delta}\big)+g\big(u_t,\nabla_t\xi_{u,x,\delta}\big)+C_g(x)\eta_{u,x,\delta}(z)d_g(x,u(z))^2\big(|u_s|^2+|u_t|^2\big). \end{split} \tag{3.27}$$

Along with (3.20), this implies that

$$-\delta E'_{u,x,\eta}(\delta)+2E_{u,x,\eta}(\delta)\le\int_\Sigma g\big(\mathrm{d}u\otimes_{\mathfrak{j}}\nabla\xi_{u,x,\delta}\big)+2C_g(x)\delta^2E_{u,x,\eta}(\delta) \tag{3.28}$$

for every compact Riemann surface with boundary $(\Sigma,\mathfrak{j})$, J-holomorphic map $u\colon\Sigma\longrightarrow X$, and $\delta\in(0,\delta_g(x))$. Combining this inequality with (3.23), we obtain (3.19).

Suppose $\omega\equiv g(J\cdot,\cdot)$ is a symplectic form on X. By Lemma 3.16, the left-hand side of (3.23) then vanishes. From (3.28), we thus obtain

$$-\delta E'_{u,x,\eta}(\delta)+2E_{u,x,\eta}(\delta)\le2C_{g,J}(x)\delta^2E_{u,x,\eta}(\delta)\,.$$

The reasoning below (3.19) now yields the second claim. □

4 Mean Value Inequality and applications

We now move to properties of J-holomorphic maps u from Riemann surfaces $(\Sigma,\mathfrak{j})$ into almost complex manifolds (X,J) that are of a more global nature. They generally concern the distribution of the energy of such a map over its domain and are consequences of the Mean Value Inequality for J-holomorphic maps. These fairly technical properties lead to geometric conclusions such as Propositions 4.3 and 5.1.

4.1 Statement and proof

According to Cauchy's Integral Formula, a holomorphic map $u\colon B_R \longrightarrow \mathbb{C}^n$ satisfies

$$u'(0) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=r} \frac{u(z)}{z^2} \mathrm{d}z \qquad \forall\, r \in (0,R).$$

This immediately implies that a bounded holomorphic function defined on all of $\mathbb{C}$ is constant. The Mean Value Inequality of Proposition 4.1 bounds the norms of the differentials of J-holomorphic maps of sufficiently small energy away from the boundary of the domain "uniformly" by their L^2-norms. In general, one would not expect the value of a function to be bounded by its integral. The Mean Value Inequality implies that a J-holomorphic map which is defined on all of $\mathbb{C}$ and has sufficiently small energy is in fact constant; see Corollary 4.2.

Proposition 4.1 (Mean Value Inequality)**.** *If (X,J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $\hbar_{J,g}\colon X\times\mathbb{R} \longrightarrow \mathbb{R}^+$ with the following property. If $u\colon B_R \longrightarrow X$ is a J-holomorphic map such that*

$$u(B_R) \subset B_r^g(x) \qquad \text{and} \qquad E_g(u) < \hbar_{J,g}(x,r)$$

for some $x \in X$ and $r \in \mathbb{R}$, then

$$\big|\mathrm{d}_0 u\big|_g^2 < \frac{16}{\pi R^2} E_g(u)\,. \tag{4.1}$$

Proof. Let $\phi(z) = \frac{1}{2}|\mathrm{d}_z u|_g^2$. By Lemma 4.7 below, $\Delta\phi \ge -A_{J,g}\phi^2$ with $A_{J,g}\colon X\times\mathbb{R} \longrightarrow \mathbb{R}^+$ determined by (X,J,g). The claim with $\hbar_{J,g} = \pi/8A_{J,g}$ thus follows from Proposition 4.6. $\square$

Corollary 4.2 (Lower Energy Bound)**.** *If (X,J) is a compact almost complex manifold and g is a Riemannian metric on X, then there exists $\hbar_{J,g} \in \mathbb{R}^+$ such that $E_g(u) \ge \hbar_{J,g}$ for every non-constant J-holomorphic map $u\colon S^2 \longrightarrow X$.*

Proof. By the compactness of X, we can assume that g is compatible with J. Let $\hbar_{J,g} > 0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 4.1 on the compact space $X \times [0, \mathrm{diam}_g(X)]$. If $u\colon S^2 \longrightarrow X$ is J-holomorphic map with $E_g(u) < \hbar_{J,g}$,

$$\big|\mathrm{d}_z u\big|_g^2 < \frac{16}{\pi R^2} E_g\big(u; B_R(z)\big) \le \frac{16}{\pi R^2} E_g(u) \qquad \forall\, z \in \mathbb{C},\ R \in \mathbb{R}^+$$

by Proposition 4.1, since $B_R(z) \subset \mathbb{C}$ as Riemann surfaces. Thus, $\mathrm{d}_z u = 0$ for all $z \in \mathbb{C}$, and so u is constant. $\square$

If $\phi\colon U \longrightarrow \mathbb{R}$ is a C^2-function on an open subset of $\mathbb{R}^2$, let

$$\Delta\phi = \frac{\partial^2\phi}{\partial s^2} + \frac{\partial^2\phi}{\partial t^2} \equiv \phi_{ss} + \phi_{tt}$$

denote the Laplacian of ϕ.

Exercise 4.3. Show that in the polar coordinates (r,θ) on $\mathbb{R}^2$,

$$\Delta\phi = \phi_{rr} + r^{-1}\phi_r + r^{-2}\phi_{\theta\theta}\,. \tag{4.2}$$

Lemma 4.4. *If* $\phi\colon \overline{B_R}\longrightarrow\mathbb{R}$ *is* C^2*, then*

$$2\pi R\,\phi(0) = -R\int_{(r,\theta)\in B_R}(\ln R-\ln r)\Delta\phi + \int_{\partial B_R}\phi\,. \tag{4.3}$$

Proof. By Stokes' Theorem applied to $\phi\mathrm{d}\theta$ on $\overline{B_R}-B_\delta$,

$$\begin{aligned}\oint_{\partial B_R}\phi\,\mathrm{d}\theta-\oint_{\partial B_\delta}\phi\,\mathrm{d}\theta &= \int_{\overline{B_R-B_\delta}}\phi_r\,\mathrm{d}r\wedge\mathrm{d}\theta = \int_0^{2\pi}\!\!\int_\delta^R(r\phi_r)r^{-1}\,\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}(\ln R-\ln\delta)\delta\,\phi_r(\delta,\theta)\mathrm{d}\theta + \int_0^{2\pi}\!\!\int_\delta^R(\ln R-\ln r)(\phi_{rr}+r^{-1}\phi_r)r\,\mathrm{d}r\mathrm{d}\theta\,;\end{aligned}$$

the last equality above is obtained by applying integration by parts to the functions $\ln r-\ln R$ and $r\phi_r$. Sending $\delta\longrightarrow 0$ and using (4.2), we obtain

$$\frac{1}{R}\int_{\partial B_R}\phi - 2\pi\,\phi(0) = 0 + \int_{(r,\theta)\in B_R}(\ln R-\ln r)\Delta\phi\,,$$

which is equivalent to (4.3). □

Corollary 4.5. *If* $\phi\colon \overline{B_R}\longrightarrow\mathbb{R}$ *is* C^2 *and* $\Delta\phi\ge -C$ *for some* $C\in\mathbb{R}^+$*, then*

$$\phi(0)\le \frac{1}{8}CR^2 + \frac{1}{\pi R^2}\int_{B_R}\phi\,. \tag{4.4}$$

Proof. By (4.3),

$$2\pi r\,\phi(0)\le Cr\int_0^{2\pi}\!\!\int_0^r(\ln r-\ln\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_{\partial B_r}\phi = Cr\cdot 2\pi\cdot\frac{r^2}{4}+\int_{\partial B_r}\phi \qquad \forall\, r\in(0,R).$$

Integrating the above in $r\in(0,R)$, we obtain

$$2\pi\phi(0)\cdot\frac{R^2}{2}\le 2\pi C\cdot\frac{R^4}{16}+\int_{B_R}\phi.$$

This inequality is equivalent to (4.4). □

Proposition 4.6. *If* $\phi: B_R\longrightarrow\mathbb{R}^{\ge 0}$ *is* C^2 *and there exists* $A\in\mathbb{R}^+$ *such that* $\Delta\phi\ge -A\phi^2$ *and* $\int_{B_R}\phi<\frac{\pi}{8A}$*, then*

$$\phi(0)\le\frac{8}{\pi R^2}\int_{B_R}\phi\,. \tag{4.5}$$

Proof. Replacing A by $\widetilde{A}=R^2A$ and ϕ by

$$\widetilde{\phi}\colon B_1\longrightarrow\mathbb{R}, \qquad \widetilde{\phi}(z)=\phi(Rz),$$

we can assume that $R=1$, as well as that ϕ is defined on $\overline{B_1}$.

(1) Define

$$f:[0,1)\longrightarrow\mathbb{R}^{\ge 0} \qquad \text{by} \qquad f(r)=(1-r)^2\max_{\overline{B_r}}\phi\,.$$

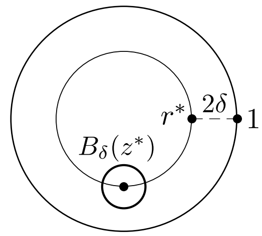


Figure 6: Setup for the proof of Proposition 4.6

In particular, $f(0)=\phi(0)$ and $f(1)=0$. Choose $r^*\in[0,1)$ and $z^*\in B_{r^*}$ such that

$$f(r^*)=\sup f \qquad \text{and} \qquad \phi(z^*)=\sup_{B_{r^*}}\phi\equiv c^*\,.$$

Let $\delta=\frac{1}{2}(1-r^*)>0$; see Figure 6. Thus,

$$\sup_{B_\delta(z^*)}\phi\le\sup_{B_{r^*+\delta}}\phi=\frac{f(r^*+\delta)}{(1-(r^*+\delta))^2}\le\frac{f(r^*)}{\frac{1}{4}(1-r^*)^2}=4\phi(z^*)=4c^*\,.$$

In particular, $\Delta\phi\ge -A\phi^2\ge -16Ac^{*2}$ on $B_\delta(z^*)$.

(2) Using Corollary 4.5, we thus find that

$$c^*=\phi(z^*)\le\frac{1}{8}\cdot 16Ac^{*2}\cdot\rho^2+\frac{1}{\pi\rho^2}\int_{B_\rho(z^*)}\phi\le 2Ac^{*2}\rho^2+\frac{1}{\pi\rho^2}\int_{B_1}\phi \qquad \forall\,\rho\in[0,\delta]\,. \tag{4.6}$$

If $2Ac^*\delta^2\le\frac{1}{2}$, the $\rho=\delta$ case of the above inequality gives

$$\frac{1}{2}c^*\le\frac{1}{\pi\delta^2}\int_{B_1}\phi\,,\qquad \phi(0)=f(0)\le f(r^*)=4c^*\cdot\delta^2\le\frac{8}{\pi}\int_{B_1}\phi\,,$$

as claimed. If $2Ac^*\delta^2\ge\frac{1}{2}$, $\rho\equiv(4Ac^*)^{-\frac{1}{2}}\le\delta$ and (4.6) gives

$$c^*\le 2Ac^{*2}\cdot\frac{1}{4Ac^*}+\frac{4Ac^*}{\pi}\int_{B_1}\phi\,.$$

Thus, $\dfrac{\pi}{8A}\le\displaystyle\int_{B_1}\phi$, contrary to the assumption. □

Lemma 4.7. *If (X,J) is an almost complex manifold and g is a Riemannian metric on X compatible with J, there exists a continuous function $A_{J,g}\colon X\times\mathbb{R}\longrightarrow\mathbb{R}^+$ with the following property. If $\Omega\subset\mathbb{C}$ is an open subset, $u\colon\Omega\longrightarrow X$ is a J-holomorphic map, and $u(\Omega)\subset B_r^g(x)$ for some $x\in X$ and $r\in\mathbb{R}$, then the function $\phi(z)\equiv\frac{1}{2}|\mathrm{d}_zu|_g^2$ satisfies $\Delta\phi\ge -A_{J,g}(x,r)\phi^2$.*

Proof. Let $z=s+\mathrm{i}t$ be the standard coordinate on $\mathbb{C}$. Denote by u_s and u_t the s and t-partials of u, respectively. Since u is J-holomorphic, i.e. $u_s=-Ju_t$, and g is J-compatible, i.e. $g(J\cdot,J\cdot)=g(\cdot,\cdot)$, $|u_s|_g^2=|u_t|_g^2$. Since the Levi-Civita connection ∇ of g is g-compatible and torsion-free,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2t}|u_s|_g^2=|\nabla_tu_s|_g^2+\big\langle\nabla_t\nabla_tu_s,u_t\big\rangle_g=|\nabla_tu_s|_g^2+\big\langle\nabla_t\nabla_su_t,u_s\big\rangle_g\,. \tag{4.7}$$

Similarly,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2 s}|u_t|_g^2 = \big|\nabla_s u_t\big|_g^2 + \big\langle \nabla_s\nabla_t u_s, u_t\big\rangle_g. \tag{4.8}$$

Since $u_s = -Ju_t$,

$$\begin{aligned}\langle \nabla_s\nabla_t u_s, u_t\rangle_g &= -\big\langle \nabla_s\nabla_t(Ju_t), u_t\big\rangle_g \\ &= -\big\langle J\nabla_s\nabla_t u_t, u_t\big\rangle_g - \big\langle (\nabla_s J)\nabla_t u_t, u_t\big\rangle_g - \big\langle \nabla_s((\nabla_t J)u_t), u_t\big\rangle_g \\ &= -\big\langle \nabla_s\nabla_t u_t, u_s\big\rangle_g - \big\langle (\nabla_s J)\nabla_t u_t, u_t\big\rangle_g - \big\langle \nabla_s((\nabla_t J)u_t), u_t\big\rangle_g\,.\end{aligned} \tag{4.9}$$

Putting (4.7)-(4.9), we find that

$$\frac{1}{2}\Delta\phi = \big|\nabla_t u_s\big|_g^2 + \big|\nabla_s u_t\big|_g^2 + \big\langle R_g(u_t,u_s)u_t, u_s\big\rangle_g - \big\langle (\nabla_s J)\nabla_t u_t, u_t\big\rangle_g - \big\langle \nabla_s((\nabla_t J)u_t), u_t\big\rangle_g\,, \tag{4.10}$$

where R_g is the curvature tensor of the connection ∇. Since $u(\Omega)\subset B_r^g(x)$,

$$\begin{aligned}\big|\langle R_g(u_t,u_s)u_t,u_s\rangle_g\big| &\le C_g(x,r)|u_s|_g^2|u_t|_g^2\,, \\ \big|\langle (\nabla_s J)\nabla_t u_t, u_t\rangle_g\big| &\le C_{J,g}(x,r)|u_s|_g|u_t|_g\big|\nabla_t(Ju_s)\big|_g \le C_{J,g}(x,r)|u_s|_g|u_t|_g\big(|u_s|_g|u_t|_g+|\nabla_t u_s|_g\big) \\ &\le (C_{J,g}(x,r)+C_{J,g}(x,r)^2)|u_s|_g^2|u_t|_g^2 + |\nabla_t u_s|_g^2\,, \\ \big|\langle \nabla_s((\nabla_t J)u_t), u_t\rangle_g\big| &\le C_{J,g}(x,r)|u_t|_g^2\big(|u_s|_g|u_t|_g+|\nabla_s u_t|_g\big) \\ &\le C_{J,g}(x,r)|u_s|_g|u_t|_g^3 + C_{J,g}(x,r)^2|u_t|_g^4 + |\nabla_s u_t|_g^2.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we find that

$$\frac{1}{2}\Delta\phi \ge -C(x,r)\big(|u_s|_g^2|u_t|_g^2+|u_s|_g|u_t|_g^3+|u_t|_g^4\big) \ge -3C(x,r)\phi^2,$$

as claimed. □

4.2 Regularity of J-holomorphic maps

By Cauchy's Integral Formula, a continuous extension of a holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow\mathbb{C}^n$ over the origin is necessarily holomorphic. By Proposition 4.8 below, the same is the case for a J-holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow X$ of bounded energy.

Proposition 4.8. *Let (X,J) be an almost complex manifold and g be a Riemannian metric on X. If $R\in\mathbb{R}^+$ and $u: B_R\longrightarrow X$ is a continuous map such that $u|_{B_R^*}$ is a J-holomorphic map and $E_g(u;B_R^*)<\infty$, then u is smooth and J-holomorphic on B_R.*

For a smooth loop $\gamma\colon S^1\longrightarrow X$, define

$$\gamma'(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}\gamma\big(\mathrm{e}^{\mathrm{i}\theta}\big) \in T_{\gamma(\mathrm{e}^{\mathrm{i}\theta})}X \qquad\text{and}\qquad \ell_g(\gamma) = \int_0^{2\pi}\big|\gamma'(\theta)\big|_g\mathrm{d}\theta \in \mathbb{R}^{\ge 0}$$

to be the velocity of γ and the length of γ, respectively.

Lemma 4.9 (Isoperimetric Inequality)**.** *Let (X,J,g), R, and u be as in Proposition 4.8 and*

$$\gamma_r\colon S^1\longrightarrow X, \quad \gamma_r\big(\mathrm{e}^{\mathrm{i}\theta}\big) = u\big(r\mathrm{e}^{\mathrm{i}\theta}\big) \qquad \forall\, r\in(0,R).$$

There exist $\delta\in(0,R)$ and $C\in\mathbb{R}^+$ such that

$$E_g\big(u;B_r^*\big) \le C\ell_g(\gamma_r)^2 \qquad \forall\, r\in(0,\delta). \tag{4.12}$$

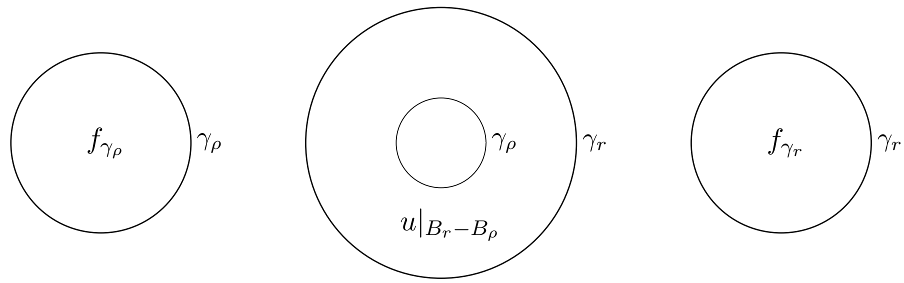


Figure 7: The maps from an annulus and two disks glued together to form the map $F_{\rho;r}\colon S^2 \longrightarrow X$ in the proof of Lemma 4.9

Proof. Let exp be as above the statement of Lemma 3.13, δ_g and ω_x be as in the first two sentences in the proof of Proposition 3.12,

$$x_0 = u(0), \quad \delta_0 = \delta_g(x_0), \quad \omega_0 = \omega_{x_0}, \qquad E\colon (0,R) \longrightarrow \mathbb{R}, \quad E(r) = E_g(u; B_r^*).$$

We can assume that the metric g is determined by J and ω_0 on $B^g_{\delta_0}(x_0)$.

For a smooth loop $\gamma\colon S^1 \longrightarrow B^g_{\delta_0}(x_0)$, define

$$\begin{aligned} \xi_\gamma\colon S^1 &\longrightarrow T_{x_0}X \qquad \text{by} \quad \exp_{x_0}\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta}) = \gamma(\mathrm{e}^{\mathrm{i}\theta}), \quad \big|\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big| < \delta_0, \\ f_\gamma\colon B_1 &\longrightarrow X, \qquad f_\gamma\big(r\mathrm{e}^{\mathrm{i}\theta}\big) = \exp_{x_0}\!\big(r\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big). \end{aligned}$$

In particular,

$$\big|\partial_r f_\gamma(\rho\mathrm{e}^{\mathrm{i}\theta})\big|_g = \big|\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})\big|_g \le \ell_g(\gamma)/2, \quad \big|r^{-1}\partial_\theta f_\gamma(r\mathrm{e}^{\mathrm{i}\theta})\big|_g = \big|\mathrm{d}_{r\xi_\gamma(\mathrm{e}^{\mathrm{i}\theta})}(\xi'_\gamma(\theta))\big|_g \le C\big|\gamma'(\theta)\big|_g$$

for some $C\in\mathbb{R}^+$ determined by x_0. Thus,

$$\begin{aligned} \bigg|\int_{B_1} f_\gamma^*\omega_0\bigg| &\le C\!\int_0^{2\pi}\!\!\int_0^1 \big|\partial_r f_\gamma(\rho\mathrm{e}^{\mathrm{i}\theta})\big|_g \big|r^{-1}\partial_\theta f_\gamma(r\mathrm{e}^{\mathrm{i}\theta})\big|_g r\,\mathrm{d}r\mathrm{d}\theta \\ &\le C'\ell_g(\gamma)\!\int_0^{2\pi}\!\!\int_0^r \big|\gamma'(\theta)\big|_g r\,\mathrm{d}r\mathrm{d}\theta = \frac{1}{2}C'\ell_g(\gamma)^2 \end{aligned} \tag{4.13}$$

for some $C, C'\in\mathbb{R}^+$ determined by x_0 and ω_0.

By Proposition 4.1 and the finiteness assumption on $E(u;B_R^*)$, there exists $\delta\in(0,R/2)$ such that

$$\big|\gamma_r'(\theta)\big|_g^2 \equiv \big|\partial_\theta u(r\mathrm{e}^{\mathrm{i}\theta})\big|_g^2 = r^2\big|\partial_r u(\mathrm{e}^{\mathrm{i}\theta})\big|_g^2 \le \frac{32}{\pi}E(2r) \qquad \forall\ r\in(0,\delta), \tag{4.14}$$

$$\ell_g(\gamma_r)^2 = 128\pi E(2r) \qquad \forall\ r\in(0,\delta). \tag{4.15}$$

By the continuity of u, we can assume that $u(B_{2\delta})\subset B^g_{\delta_0}(x_0)$. For $r\in(0,\delta)$ and $\rho\in(0,r)$, define

$$F_{\rho;r}\colon S^2 \longrightarrow X$$

to be the map obtained from $u|_{B_r-B_\rho}$ by attaching disks to the boundary components ∂B_r and ∂B_ρ and letting $F_{\rho;r}$ be given by f_{γ_r} and f_{γ_ρ} on these two disks, respectively; see Figure 7. Since $F_{\rho;r}$ is homotopic to a constant map and ω_0 is closed,

$$0 = \int_{S^2} F_{\rho;r}^*\omega_0 = E_g\big(u; B_r - B_\rho\big) + \int_{B_1} f_{\gamma_\rho}^*\omega_0 - \int_{B_1} f_{\gamma_r}^*\omega_0\,.$$

Combining this with (4.13) and (4.15), we obtain

$$E_g\big(u;B_r-B_\rho\big)\le C\ell_g(\gamma_r)^2+CE(2\rho) \tag{4.16}$$

for some $C\in\mathbb{R}^+$ independent of r and ρ as above. Since $E_g(u;B_R^*)<0$, $E(2\rho)\longrightarrow 0$ as $\rho\longrightarrow 0$. Taking the limit of (4.16) as $\rho\longrightarrow 0$, we thus obtain (4.12). □

Corollary 4.10. *If (X,J,g), R, and u are as in Proposition 4.8, there exist $\delta\in(0,R)$ and $\mu,C\in\mathbb{R}^+$ such that*

$$\big|\mathrm{d}_{re^{i\theta}}u\big|_g\le Cr^{\mu-1}\qquad\forall\,r\in(0,\delta). \tag{4.17}$$

Proof. Let γ_r, δ, C, and $E(r)$ be as in the statement and proof of Lemma 4.9. Thus,

$$\begin{aligned}E(r)\equiv\frac12\int_0^{2\pi}\!\!\int_0^r\big|\mathrm{d}_{\rho e^{i\theta}}u\big|_g^2\rho\,\mathrm{d}\rho\,\mathrm{d}\theta\le C\ell_g(\gamma_r)^2&=\frac12 Cr^2\bigg(\int_0^{2\pi}\big|\mathrm{d}_{re^{i\theta}}u\big|_g\mathrm{d}\theta\bigg)^2\\&\le C\pi r^2\!\int_0^{2\pi}\big|\mathrm{d}_{re^{i\theta}}u\big|_g^2\mathrm{d}\theta=2C\pi rE'(r)\qquad\forall\,r\in(0,\delta).\end{aligned}$$

This implies that

$$\big(r^{-1/2C\pi}E(r)\big)'\ge 0,\quad E(r)\le\delta^{-1/2C\pi}E(\delta)\cdot r^{1/2C\pi}\equiv C'r^{2\mu}\qquad\forall\,r\in(0,\delta).$$

Combining this with (4.14), we obtain (4.17) with δ replaced by $\delta/2$. □

Proof of Proposition 4.8. With μ as in Corollary 4.10, let $p\in\mathbb{R}^+$ be such that $p>2$ and $(1-\mu)p<2$. In particular,

$$u|_{B_{R/2}}\in L_1^p\big(B_{R/2};X\big),\qquad\bar\partial_J u|_{B_{R/2}}=0\in L^p\big(B_{R/2};X\big).$$

By elliptic regularity, this implies that u is smooth; see [12, Theorem B.4.1]. By the continuity of $\bar\partial_J u$, u is then J-holomorphic on all of B_R. □

4.3 Global structure of J-holomorphic maps

We next combine the local statement of Proposition 3.1 and some of its implications with the regularity statement of Proposition 4.8 to obtain a global description of J-holomorphic maps.

Proposition 4.11. *Let (X,J) be an almost complex manifold, $(\Sigma,\mathfrak{j})$ be a compact Riemann surface, $u\colon\Sigma\longrightarrow X$ be a J-holomorphic map. If u is simple, then u is somewhere injective and all limit points of the set*

$$\big\{z\in\Sigma\colon\,|u^{-1}(u(z))|>1\big\} \tag{4.18}$$

are critical points of u.

Suppose (X,J) is an almost complex manifold, $(\Sigma,\mathfrak{j})$ is a Riemann surface, and $u\colon\Sigma\longrightarrow X$ is a J-holomorphic map. Let

$$\Sigma_u^*=\Sigma-u^{-1}\big(u\big(\{z\in\Sigma:\mathrm{d}_z u=0\}\big)\big) \tag{4.19}$$

be the preimage of the regular values of u and

$$\mathcal{R}_u^*\subset\Sigma_u^*\times\Sigma_u^*$$

be the subset of pairs (z,z') such that there exists a diffeomorphism $\varphi_{z'z}: U_z \longrightarrow U_{z'}$ between neighborhoods of z and z' in Σ satisfying

$$\varphi_{z'z}(z) = z' \qquad \text{and} \qquad u|_{U_z} = u\circ\varphi_{z'z}. \tag{4.20}$$

Denote by $\mathcal{R}_u \subset \Sigma\times\Sigma$ the closure of $\mathcal{R}_u^*$.

It is immediate that $\mathcal{R}_u^*$ is an equivalence relation on Σ and $u(z)=u(z')$ whenever $(z,z')\in\mathcal{R}_u^*$. Thus, $\mathcal{R}_u$ is also a reflexive and symmetric relation and $u(z)=u(z')$ whenever $(z,z')\in\mathcal{R}_u$. By Lemma 4.14 below, $\mathcal{R}_u$ is transitive as well. We denote this equivalence relation by $\sim_u$. Let

$$h_u: \Sigma \longrightarrow \Sigma' \equiv \Sigma/\!\sim_u \qquad \text{and} \qquad u': \Sigma' \longrightarrow X \tag{4.21}$$

be the quotient map and the continuous map induced by u, respectively. In particular,

$$u = u'\circ h_u: \Sigma \longrightarrow X.$$

In the case Σ is compact, we will show that Σ' inherits a Riemann surface structure $\mathfrak{j}'$ from $\mathfrak{j}$ so that the maps h_u and u' are $\mathfrak{j}'$- and J-holomorphic, respectively. If the degree of h is 1, we will show that all limit points of the set (4.18) are critical points of u.

Lemma 4.12. *Suppose (X,J) is an almost complex manifold, $R\in\mathbb{R}^+$, and $u: B_R \longrightarrow X$ is a non-constant J-holomorphic map such that $\mathrm{d}_z u\neq 0$ for all $z\in B_R^*$. Then there exist $m\in\mathbb{Z}^+$ and a neighborhood U_0 of 0 in B_R such that*

$$h_u: U_0\cap B_R^* \longrightarrow h_u\big(U_0\cap B_R^*\big) \subset B_R' \tag{4.22}$$

is a covering projection of degree m.

Proof. By the continuity of u, we can assume that $X=\mathbb{C}^n$, $u(0)=0$, and $J_0=J_{\mathbb{C}^n}$. As shown in the proof of Corollary 3.11, there exist $\epsilon\in(0,R)$ and $\delta\in(0,\epsilon/2)$ such that

$$U_0 \equiv u^{-1}\big(u(B_\delta)\big)\cap B_\epsilon \subset B_{2\delta}.$$

By Proposition 3.1 and the compactness of $\overline{B_{2\delta}}\subset B_R$, the number

$$m(z) \equiv \big|h_u^{-1}(h_u(z))\cap U_0\big|$$

is finite for every $z\in U_0\cap B_R^*$.

Suppose $z_i\in B_\delta^*$ and $z_i'\in U_0$ are sequences such that z_i converges to some $z_0\in B_\delta^*$ with $z_i\neq z_0$ for all i and $h_u(z_i)=h_u(z_i')$ for all i. Passing to a subsequence, we can assume that z_i' converges to some $z_0'\in\overline{B_{2\delta}}$. By the continuity of u, $u(z_0')=u(z_0)$ and so $z_0'\in U_0$. Corollary 3.10 then implies that $h_u(z_0')=h_u(z_0)$. Since B_δ^* is connected, this implies that the number $m(z)$ is independent of $z\in U_0\cap B_R^*$; we denote it by m.

Suppose $z\in U_0\cap B_R^*$ and

$$h_u^{-1}\big(h_u(z)\big)\cap U_0 = \big\{z_1,\ldots,z_m\big\}.$$

Let $\varphi_i: U_1 \longrightarrow U_i$ for $i=1,\ldots,m$ be diffeomorphisms between neighborhoods of z_1 and z_i in $U_0\cap B_R^*$ such that

$$\varphi_i(z_1) = z_i, \quad u = u\circ\varphi_i \;\; \forall\, i, \qquad U_i\cap U_j = \emptyset \;\; \forall\, i\neq j,$$

and $u\colon U_1 \longrightarrow X$ is injective. Then $h_u(U_1)\subset B'_R$ is an open neighborhood of $h_u(z)$,

$$h_u^{-1}\big(h_u(U_1)\big)\cap U_0=\bigsqcup_{i=1}^{m}U_i\,,$$

and $h_u\colon U_i \longrightarrow h_u(U_1)$ is a homeomorphism. Thus, (4.22) is a covering projection of degree m. □

Lemma 4.13. *Suppose (X,J), R, and u are as in Lemma 4.12. Then there exists a neighborhood U_0 of 0 in B_R such that*

$$\Psi_0\colon h_u(U_0)\longrightarrow \mathbb{C}, \qquad h_u(z)=\prod_{z'\in h_u^{-1}(h_u(z))\cap U_0} z', \tag{4.23}$$

is a homeomorphism from an open neighborhood of $h_u(0)$ in B'_R to an open neighborhood of 0 in $\mathbb{C}$ and $\Psi_0\circ h_u|_{U_0}$ is a holomorphic map.

Proof. By Lemma 4.12, there exists a neighborhood U_0 of 0 in B_R so that (4.22) is a covering projection of some degree $m\in\mathbb{Z}^+$. Since the restriction of u to B_R^* is a J-holomorphic immersion, the diffeomorphisms φ_i as in the proof of Lemma 4.12 are holomorphic. Thus, the map

$$\Psi_0\circ h_u|_{U_0\cap B_R^*}\colon U_0\cap B_R^*\longrightarrow \mathbb{C}, \qquad z\longrightarrow \prod_{z'\in h_u^{-1}(h_u(z))\cap U_0} z'$$

is holomorphic. Since it is also bounded, it extends to a holomorphic map

$$\widetilde{\Psi}_0\colon U_0\longrightarrow \mathbb{C}.$$

This extension is non-constant and vanishes at 0.

After possibly shrinking U_0, we can assume that there exist $k\in\mathbb{Z}^+$ and $C\in\mathbb{R}^+$ such that

$$C^{-k}|z|^k\le \big|\widetilde{\Psi}_0(z)\big|\le C^k|z|^k \qquad \forall\, z\in U_0. \tag{4.24}$$

Since $\widetilde{\Psi}_0(z')=\widetilde{\Psi}_0(z)$ for all $z'\in h_u^{-1}(h_u(z))\cap U_0$, it follows that

$$\begin{gathered} C^{-2}|z|\le|z'|\le C^2|z| \qquad \forall\; z'\in h_u^{-1}(h_u(z))\cap U_0,\; z\in U_0,\\ C^{-2m}|z|^m\le \big|\widetilde{\Psi}_0(z)\big|\le C^{2m}|z|^m \qquad \forall\, z\in U_0. \end{gathered}$$

Along with (4.24), the last estimate implies that $k=m$ and that $\widetilde{\Phi}_0$ has a zero of order precisely m at $z=0$. Thus, shrinking δ in the proof of Lemma 4.12 if necessary, we can assume that $\widetilde{\Phi}_0$ is $m\!:\!1$ over $\overline{U_0}\cap B_R^*$. This implies that the map (4.23) and its extension over the closure of $h_u(U_0)$ in B'_R are continuous and injective. Since the closure of $h_u(U_0)$ is compact and $\mathbb{C}$ is Hausdorff, we conclude that (4.23) is a homeomorphism onto an open subset of $\mathbb{C}$. □

Lemma 4.14. *Suppose (X,J), $(\Sigma,\mathfrak{j})$, and u are as in Proposition 4.11 and $(x,y)\in\mathcal{R}_u$. For every neighborhood U_x of x in Σ, the image of the projection*

$$\mathcal{R}_u\cap(U_x\times X)\longrightarrow X$$

to the second component contains a neighborhood U_y of y in Σ.

Proof. By Corollary 3.4, the last set in (4.19) is finite. By the same reasoning as in the last part of the proof of Lemma 4.12,

$$h_u\colon \Sigma_u^* \longrightarrow h_u(\Sigma_u^*) \subset \Sigma' \tag{4.25}$$

is a local homeomorphism. Since $u(z)=u(z')$ for all $(z,z')\in\mathcal{R}_u^*$, the definition of Σ_u^* thus implies that (4.25) is a finite-degree covering projection over each topological component of $h_u(\Sigma_u^*)$. Since the complement of finitely many points in a connected Riemann surface is connected, the degree of this covering over a point $h_u(z)$ depends only on the topological component of Σ containing z. For any point $z\in\Sigma$, not necessarily in Σ_u^*, we denote this degree by $d(z)$.

By Corollary 3.4, the set

$$S \equiv u^{-1}\big(u(x)\big) \subset \Sigma$$

is finite. Let $W\subset X$ be a neighborhood of $u(x)$ such that the topological components Σ_s of $u^{-1}(W)$ containing the points $s\in S$ are pairwise disjoint (if U is a union of disjoint balls around the points of S, then

$$W \equiv X - u(\Sigma - U)$$

works). By Lemma 4.12, for each $s\in S$ there exists a neighborhood U_s' of s in Σ_s such that

$$h_u\colon U_s'-\{s\} \longrightarrow h_u\big(U_s'-\{s\}\big) \subset \Sigma'$$

is a covering projection of some degree $m_s\in\mathbb{Z}^+$; we can assume that $U_x'\subset U_x$. Along with the compactness of Σ, the former implies that

$$\begin{aligned} &\big|h_u^{-1}\big(h_u(y')\big)\cap U_s'\big| \in \{0, m_s\} && \forall\, y'\in U_{s'}'\cap\Sigma_u^*,\ s,s'\in S,\\ &\sum_{s\in S}\big|h_u^{-1}\big(h_u(y')\big)\cap U_s'\big| = d(s') && \forall\ y'\in U_{s'}'\cap\Sigma_u^*,\ s'\in S. \end{aligned} \tag{4.26}$$

Define

$$\mathcal{P}_y(S) = \Big\{S'\subset S\colon \sum_{s\in S'} m_s = d(y)\Big\}.$$

Let $U_y''\subset U_y'$ be a connected neighborhood of y. For each $S'\in\mathcal{P}_y(S)$, define

$$U_{y;S'}'' = \big\{y'\in U_y''\cap\Sigma_u^*\colon \{s\in S\colon h_u^{-1}(h_u(y'))\cap U_s'\neq\emptyset\} = S'\big\}.$$

By (4.26), these sets partition $U_y''\cap\Sigma_u^*$. Since each set

$$\big\{y'\in U_y''\cap\Sigma_u^*\colon h_u^{-1}(h_u(y'))\cap U_s'\neq\emptyset\big\}$$

is open, (4.26) also implies that each set $U_{y;S'}''$ is open. Since the set $U_y''\cap\Sigma_u^*$ is connected, it follows that $U_y''\cap\Sigma_u^* = U_{y;S_y}''$ for some $S_y\in\mathcal{P}_y(S)$. Since $(x,y)\in\mathcal{R}_u$, $x\in S_y$. Thus, the image of the projection

$$\mathcal{R}_u\cap(U_x'\times X) \longrightarrow X$$

to the second component contains the neighborhood U_y'' of y in Σ. □

Corollary 4.15. *Suppose (X,J), $(\Sigma,\mathfrak{j})$, and u are as in Proposition 4.11. The quotient map h_u in (4.21) is open and closed.*

Proof. The openness of h_u is immediate from Lemma 4.14. Suppose $A\subset\Sigma$ is a closed subset and $y_i\in h_u^{-1}(h_u(A))$ is a sequence converging to some $y\in\Sigma$. Let $x_i\in A$ be such that $h_u(x_i)=h_u(y_i)$. Passing to a subsequence, we can assume that the sequence x_i converges to some $x\in A$. Since $\Sigma-\Sigma_u^*$ consists of isolated points, we can also assume that $y_i\in\Sigma_u^*$ and so $(x_i,y_i)\in\mathcal{R}_u^*$. Thus, $(x,y)\in\mathcal{R}_u$ and so $y\in h_u^{-1}(h_u(A))$. We conclude that h_u is a closed map. □

Proof of Proposition 4.11. Let Σ', h_u, and u' be as in (4.21). By the second statement in Corollary 4.15 and [13, Lemma 73.3], Σ' is a Hausdorff topological space. Fix a Riemannian metric g on X.

For $(z,z')\in\mathcal{R}_u^*$ with $z\neq z'$, the neighborhoods U_z and $U_{z'}$ as in (4.20) can be chosen so that they are disjoint and $u|_{U_z}$ is an embedding. Since u is J-holomorphic, $\varphi_{z'z}$ is then a biholomorphic map and $h_u|_{U_z}$ is a homeomorphism onto $h_u(U_z)\subset\Sigma'$. Thus, the Riemann surface structure $\mathfrak{j}$ on Σ determines a Riemann surface structure $\mathfrak{j}'$ on $h_u(\Sigma_u^*)$ so that $h_u|_{\Sigma_u^*}$ is a holomorphic covering projection of $h_u(\Sigma_u^*)$ and $u'|_{h_u(\Sigma_u^*)}$ is a J-holomorphic map with

$$E_g\big(u';h_u(\Sigma_u^*)\big)\le E_g(u). \tag{4.27}$$

By Corollary 3.4, $\Sigma'_u-h_u(\Sigma_u^*)$ consists of finitely many points. By the first statement in Corollary 4.15 and by Lemma 4.13, $\mathfrak{j}'$ extends over these points to a Riemann surface structure on Σ'; we denote the extension also by $\mathfrak{j}'$. Since the continuous map h_u is $\mathfrak{j}'$-holomorphic outside of the finitely many points of $\Sigma-\Sigma_u^*$, it is holomorphic everywhere. Since the continuous map u' is J-holomorphic on $h_u(\Sigma_u^*)$, (4.27) and Proposition 4.8 imply that it is J-holomorphic everywhere.

Suppose $z\in\Sigma$ and $z_i,z_i'\in\Sigma$ with $i\in\mathbb{Z}^+$ are such that

$$\mathrm{d}_z u\neq 0,\qquad z_i\neq z_i',\ u(z_i)=u(z_i')\ \ \forall\,i,\qquad \lim_{i\longrightarrow\infty}z_i=z.$$

Passing to a subsequence, we can assume that the sequence z_i' converges to some point $z'\in\Sigma$ with $u(z')=u(z)$. Since the restriction of u to a neighborhood of z is an embedding, $z'\neq z$. By Corollary 3.10, there exists a diffeomorphism $\varphi_{z'z}$ as in (4.20). Thus, $h_u(z)=h_u(z')$, the map h_u is not injective, and u is not simple. □

4.4 Energy bound on long cylinders

Proposition 4.16 and Corollary 4.17 below concern J-holomorphic maps from long cylinders. Their substance is that most of the energy and variation of such maps are concentrated near the ends. These technical statements are used to obtain important geometric conclusions in Sections 5.2 and 5.3.

Proposition 4.16. *If (X,J) is an almost complex manifold and g is a Riemannian metric on X, then there exist continuous functions $\delta_{J,g},\hbar_{J,g},C_{J,g}:X\longrightarrow\mathbb{R}^+$ with the following properties. If $u\colon[-R,R]\times S^1\longrightarrow X$ is a J-holomorphic map such that $\mathrm{Im}\,u\subset B^g_{\delta_{J,g}(u(0,1))}(u(0,1))$, then*

$$E_g\big(u;[-R+T,R-T]\times S^1\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T}E_g(u)\qquad\forall\,T\ge0\,. \tag{4.28}$$

If in addition $E_g(u)<\hbar_{J,g}\big(u(0,1)\big)$, then

$$\mathrm{diam}_g\big(u([-R+T,R-T]\times S^1)\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T/2}\sqrt{E_g(u)}\quad\forall\,T\ge1\,. \tag{4.29}$$

Corollary 4.17. *If (X,J) is a compact almost complex manifold and g is a Riemannian metric on X, there exist $\hbar_{J,g}, C_{J,g} \in \mathbb{R}^+$ with the following property. If $u: [-R,R]\times S^1 \longrightarrow X$ is a J-holomorphic map such that $E_g(u)<\hbar_{J,g}$, then*

$$\begin{aligned} E_g\big(u;[-R{+}T,R{-}T]\times S^1\big) &\le C_{J,g}\mathrm{e}^{-T}E_g(u) && \forall\, T\ge 1,\\ \mathrm{diam}_g\big(u([-R{+}T,R{-}T]\times S^1)\big) &\le C_{J,g}\mathrm{e}^{-T/2}\sqrt{E_g(u)} && \forall\, T\ge 2\,. \end{aligned}$$

As an example, the energy of the injective map

$$[-R,R]\times S^1 \longrightarrow \mathbb{C}, \qquad (s,\theta)\longrightarrow s\mathrm{e}^{\mathrm{i}\theta},$$

is the area of its image, i.e. $\pi(\mathrm{e}^{2R}–\mathrm{e}^{-2R})$. Thus, the exponent e^{-T} in (4.28) can be replaced by e^{-2T} in this case. The proof of Proposition 4.16 shows that in general the exponent can be taken to be $\mathrm{e}^{-\mu T}$ with μ arbitrarily close to 2, but at the cost of increasing $C_{J,g}$ and reducing $\delta_{J,g}$.

Lemma 4.18 (Poincare Inequality)**.** *If $f: S^1\longrightarrow\mathbb{R}^n$ is a smooth function such that $\int_0^{2\pi}f(\theta)\mathrm{d}\theta=0$, then*

$$\int_0^{2\pi}|f(\theta)|^2\mathrm{d}\theta \le \int_0^{2\pi}|f'(\theta)|^2\mathrm{d}\theta.$$

Proof: We can write $f(\theta)=\sum\limits_{k>-\infty}^{k<\infty}a_k\mathrm{e}^{\mathrm{i}k\theta}$. Since $\int_0^{2\pi}f(\theta)\mathrm{d}\theta=0$, $a_0=0$. Thus,

$$\int_0^{2\pi}|f(\theta)|^2\mathrm{d}\theta = \sum_{k>-\infty}^{k<\infty}|a_k|^2 \le \sum_{k>-\infty}^{k<\infty}|ka_k|^2 = \int_0^{2\pi}|f'(\theta)|^2\mathrm{d}\theta.$$

Proof of Proposition 4.16. It is sufficient to establish the first statement under the assumption that (X,g) is $\mathbb{C}^n$ with the standard Riemannian metric, J agrees with the standard complex structure $J_{\mathbb{C}^n}$ at $0\in\mathbb{C}^n$, and $u(0,1)=0$. Let

$$\bar\partial u = \frac{1}{2}\big(u_t + J_{\mathbb{C}^n}u_\theta\big)\,.$$

By our assumptions, there exist $\delta', C>0$ (dependent on $u(0,1)$) such that

$$\big|\bar\partial_z u\big| \le C\delta\big|\mathrm{d}_z u\big| \qquad \forall\, z\in u^{-1}\big(B_\delta(0)\big),\ \delta\le\delta'\,. \tag{4.30}$$

Write $u=f+\mathrm{i}g$, with f,g taking values in $\mathbb{R}^n$ and assume that $\mathrm{Im}\,u\subset B_\delta(0)$. By (2.4),

$$|\mathrm{d}u|^2 = 4\big|\bar\partial u\big|^2 + 2\mathrm{d}(f\cdot\mathrm{d}g).$$

Combining this with (4.30) and Stokes' Theorem, we obtain

$$\int_{[-t,t]\times S^1}|\mathrm{d}u|^2 \le 4C^2\delta^2\int_{[-t,t]\times S^1}|\mathrm{d}u|^2 + 2\int_{\{t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta - 2\int_{\{-t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta\,. \tag{4.31}$$

Let $\widetilde f=f-\frac{1}{2\pi}\int_0^{2\pi}f\mathrm{d}\theta$. By Hölder's inequality and Lemma 4.18,

$$\begin{aligned} \bigg|\int_{\{\pm t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta\bigg| &= \bigg|\int_{\{\pm t\}\times S^1}\widetilde f\cdot g_\theta\,\mathrm{d}\theta\bigg| \le \bigg(\int_{\{\pm t\}\times S^1}|\widetilde f|^2\mathrm{d}\theta\bigg)^{\frac12}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac12}\\ &\le \bigg(\int_{\{\pm t\}\times S^1}|\widetilde f_\theta|^2\mathrm{d}\theta\bigg)^{\frac12}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac12} \le \frac12\int_{\{\pm t\}\times S^1}|u_\theta|^2\mathrm{d}\theta\,. \end{aligned} \tag{4.32}$$

Since

$$3|u_\theta|^2 = 2|u_\theta|^2 + \big|u_t - 2\bar{\partial}u\big|^2 \le 2|\mathrm{d}u|^2 + 8\big|\bar{\partial}u\big|^2,$$

the inequalities (4.30)-(4.32) give

$$\big(1-4C^2\delta^2\big)\int_{[-t,t]\times S^1}|\mathrm{d}u|^2 \le \frac{2}{3}\big(1+4C^2\delta^2\big)\bigg(\int_{\{t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta + \int_{\{-t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\bigg).$$

Thus, the function

$$\varepsilon(T) \equiv E_g\big(u;[-R+T,R-T]\big) \equiv \frac{1}{2}\int_{[-R+T,R-T]\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\mathrm{d}t$$

satisfies $\varepsilon(T)\le-\varepsilon'(T)$ for all $T\in[-R,R]$, if δ is sufficiently small (depending on C). This implies (4.28).

Let $h_{J,g}(x)=(x,\delta_{J,g}(x))$, with $h_{J,g}(\cdot,\cdot)$ as in Proposition 4.1 and $\delta_{J,g}(\cdot)$ as provided by the previous paragraph. Suppose u also satisfies the last condition in Proposition 4.16. By Proposition 4.1 and (4.28),

$$|\mathrm{d}_{(t,\theta)}u| \le 3\sqrt{E_g(u;[-|t|-1,|t|+1]\times S^1)} \le 3\sqrt{C_{J,g}(u(0,1))}\mathrm{e}^{(1+|t|-R)/2}\sqrt{E_g(u)}$$

for all $t\in[-R+1,R-1]$ and $\theta\in S^1$. Thus, for all $t_1,t_2\in[-R+T,R-T]$ with $T\ge1$ and $\theta_1,\theta_2\in S^1$,

$$\begin{aligned}d_g\big(u(t_1,\theta_1),u(t_2,\theta_2)\big) &\le 3\sqrt{C_{J,g}(u(0,1))E_g(u)}\bigg(\pi\mathrm{e}^{(1+|t_1|-R)/2}+\int_{t_1}^{t_2}\mathrm{e}^{(1+|t|-R)/2}\mathrm{d}t\bigg)\\ &\le \big(3\pi+12\big)\sqrt{C_{J,g}(u(0,1))}\,\mathrm{e}^{(1-T)/2}\sqrt{E_g(u)}\,.\end{aligned}$$

This establishes (4.29). □

Lemma 4.19. *If (X,J) is a compact almost complex manifold and g is a Riemannian metric on X, there exists a continuous function $\epsilon_{J,g}\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$ with the following property. If $\delta\in\mathbb{R}^+$ and $u\colon(-R,R)\times S^1\longrightarrow X$ is a J-holomorphic map with $E_g(u)<\epsilon_{J,g}(\delta)$, then*

$$\mathrm{diam}_g\big(u([-R+1,R-1]\times S^1)\big) \le \delta\,.$$

Proof. By Proposition 3.12 and the compactness of X, there exists $c_{J,g}\in\mathbb{R}^+$ with the following property. If $(\Sigma,\mathfrak{j})$ is a compact connected Riemann surface with boundary, $u\colon\Sigma\longrightarrow X$ is a non-constant J-holomorphic map, $x\in X$, and $\delta\in\mathbb{R}^+$ are such that $u(\partial\Sigma)\cap B_\delta^g(x)=\emptyset$, then

$$E_g(u) \ge c_{J,g}\delta^2\,. \tag{4.33}$$

Let $\hbar_{J,g}>0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 4.1 on the compact space $X\times[0,\mathrm{diam}_g(X)]$.

Suppose $u\colon(-R,R)\times S^1\longrightarrow X$ is a J-holomorphic map with $E_g(u)<\hbar_{J,g}$ and

$$\delta_u \equiv \mathrm{diam}_g\big(u([-R+1,R-1]\times S^1)\big) > 32\sqrt{E_g(u)}.$$

By the first condition on u,

$$\begin{aligned} \big|\mathrm{d}_z u\big|_g^2 &\le \frac{16}{\pi} E_g(u) \qquad \forall\, z\in[-R+1,R-1]\times S^1,\\ \mathrm{diam}_g\big(u(r\times S^1)\big) &\le 8\sqrt{E_g(u)} \quad \forall\, r\in[-R+1,R-1]. \end{aligned} \tag{4.34}$$

Let $r_-, r_0, r_+\in[-R+1,R-1]$ and $\theta_-,\theta_0,\theta_+\in S^1$ be such that

$$r_- < r_0 < r_+, \quad d_g\big(u(r_0,\theta_0), u(r_\pm,\theta_\pm)\big) \ge \frac{1}{2}\delta_u\,.$$

By (4.34), we can apply (4.33) with

$$\Sigma = [r_-,r_+]\times S^1, \qquad x = u(r_0,\theta_0), \qquad \delta = \frac{1}{4}\delta_u,$$

and u replaced by its restriction to Σ. We conclude that

$$E_g(u) \ge \frac{c_{J,g}}{16}\delta_u^2\,.$$

It follows that the function

$$\epsilon_{J,g}\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \qquad \epsilon_{J,g}(\delta) = \min\left(\hbar_{J,g}, \frac{\delta^2}{32^2}, \frac{c_{J,g}}{16}\delta^2\right),$$

has the desired property. □

Proof of Corollary 4.17. Let $\delta\in\mathbb{R}^+$ be the minimum of the function $\delta_{J,g}$ in Proposition 4.16 and $\varepsilon_{J,g}(\cdot)$ be as in Lemma 4.19. Take $C_{J,g}$ to be the maximum of the function $C_{J,g}$ in Proposition 4.16 times e and $\hbar_{J,g}\in\mathbb{R}^+$ to be smaller than the minimum of the function $\hbar_{J,g}$ in Proposition 4.16 and the number $\varepsilon_{J,g}(\delta)$. □

5 Limiting behavior of J-holomorphic maps

This section studies the limiting behavior of sequences of J-holomorphic maps from Riemann surfaces into a compact almost complex manifold (X,J). The compactness of X plays an essential role in the statements below, in contrast to nearly all statements in Sections 3 and 4.

5.1 Removal of Singularity

By Cauchy's Integral Formula, a bounded holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow\mathbb{C}^n$ extends over the origin. By Proposition 5.1 below, the same is the case for a J-holomorphic map $u\colon B_{\mathbb{R}}^*\longrightarrow X$ of bounded energy if X is compact.

Proposition 5.1 (Removal of Singularity)**.** *Let (X,J) be a compact almost complex manifold and $u\colon B_R^*\longrightarrow X$ be a J-holomorphic map. If the energy $E_g(u)$ of u, with respect to any metric g on X, is finite, then u extends to a J-holomorphic map $\widetilde{u}\colon B_R\longrightarrow X$.*

A basic example of a holomorphic function $u\colon\mathbb{C}^*\longrightarrow\mathbb{C}$ that does not extend over the origin $0\in\mathbb{C}$ is $z\longrightarrow 1/z$. The energy of $u|_{B_R^*}$ with respect to the standard metric on $\mathbb{C}$ is given by

$$E\big(u;B_R^*\big) = \frac{1}{2}\int_{B_R}|\mathrm{d}u|^2 = \int_{B_R}\frac{1}{|z|^2} = \int_0^{2\pi}\!\!\int_0^R r^{-1}\mathrm{d}r\mathrm{d}\theta \not< \infty.$$

The above integral would have been finite if $|\mathrm{d}u|^2$ were replaced by $|\mathrm{d}u|^{2-\epsilon}$ for *any* $\epsilon>0$. This observation illustrates the crucial role played by the energy in the theory of J-holomorphic maps.

By Cauchy's Integral Formula, the conclusion of Proposition 5.1 holds if J is an integrable almost complex structure and $u(B_\delta^*)$ is contained in a complex coordinate chart for some $\delta\in(0,R)$. We will use the Monotonicity Lemma to show that the latter is the case if the energy of u is finite; the integrability of J turns out to be irrelevant here.

Proof of Proposition 5.1. In light of Proposition 4.8, it is to sufficient to show that u extends continuously over the origin. Let $c_{J,g},\hbar_{J,g}\in\mathbb{R}^+$ be as in the proof of Lemma 4.19. We can assume that $R=1$ and u is non-constant. Define

$$v\colon \mathbb{R}^-\times S^1\longrightarrow X, \qquad v\big(r,\mathrm{e}^{\mathrm{i}\theta}\big)=u\big(\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

This map is J-holomorphic and satisfies $E_g(v)=E_g(u)<\infty$.

Since $E_g(u)<\infty$,

$$\lim_{r\longrightarrow-\infty}E_g\big(v;(-\infty,r)\times S^1\big)=\lim_{r\longrightarrow-\infty}E_g\big(u;B_{e^r}^*\big)=0. \tag{5.1}$$

In particular, there exists $R\in\mathbb{R}^-$ such that

$$E_g\big(v;(-\infty,r)\times S^1\big)<\hbar_{J,g}\qquad\forall\ r<R.$$

By Proposition 4.1 and our choice of $\hbar_{J,g}$, this implies that

$$\begin{gathered}\big|\mathrm{d}_zv\big|_g^2\le\frac{16}{\pi}E_g\big(v;(-\infty,r+1)\times S^1\big)\qquad\forall\ z\in(-\infty,r)\times S^1,\ r<R-1,\\ \operatorname{diam}_g\big(v(\{r\}\times S^1)\big)\le4\sqrt{\pi}\sqrt{E_g(v;(-\infty,r+1)\times S^1)}\quad\forall\ r<R-1.\end{gathered}$$

Combining the last bound with (5.1), we obtain

$$\lim_{r\longrightarrow-\infty}\operatorname{diam}_g\big(v(\{r\}\times S^1)\big)=0.$$

Thus, it remains to show that $\lim\limits_{r\longrightarrow-\infty}v(r,1)$ exists.

Since X is compact, every sequence in X has a convergent subsequence. Suppose there exist

$$\begin{gathered}\delta\in\mathbb{R}^+,\quad x,y\in X,\quad i_k,r_k\in\mathbb{R}^-\qquad\text{s.t.}\\ d_g(x,y)>3\delta,\quad r_{k+1}<i_k<r_k,\quad v\big(\{i_k\}\times S^1\big)\subset B_\delta(x),\quad v\big(\{r_k\}\times S^1\big)\subset B_\delta(y).\end{gathered}$$

We thus can apply (4.33) with Σ, x, and u replaced by

$$\Sigma_k\equiv[r_{k+1},r_k]\times S^1,\qquad x_k\equiv u(i_k,1),\quad\text{and}\quad v_k\equiv v|_{\Sigma_k},$$

respectively. We conclude that

$$E_g(v)\ge\sum_kE_g\big(v;\Sigma_k\big)=\sum_kE_g(v_k)\ge\sum_kc_{J,g}\delta^2=\infty\,.$$

However, this contradicts the assumption that $E_g(v)=E_g(u)<\infty$. $\square$

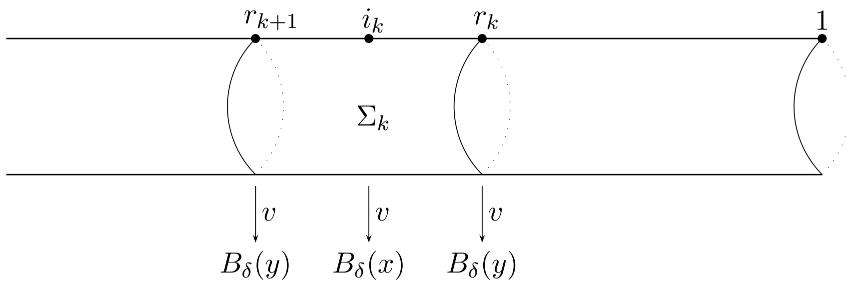


Figure 8: Setup for the proof of Proposition 5.1

5.2 Bubbling

The next three statements are used in Section 5.2 to show that no energy is lost under Gromov's convergence procedure, the resulting bubbles connect, and their number is finite.

Lemma 5.2. *Suppose (X,J) is an almost complex manifold with a Riemannian metric g and $u_i\colon B_1\longrightarrow X$ is a sequence of J-holomorphic maps converging uniformly in the C^∞-topology on compact subsets of B_1^* to a J-holomorphic map $u\colon B_1\longrightarrow X$ such that the limit*

$$\mathfrak{m}\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta) \tag{5.2}$$

exists and is nonzero.

(1) The limit $\mathfrak{m}(\delta)\equiv\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta)$ exists and is a continuous, non-decreasing function of δ.

(2) For every sequence $z_i\in B_1$ converging to 0, $\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta(z_i))=\mathfrak{m}(\delta)$.

(3) For every sequence $z_i\in B_1$ converging to 0, $\mu\in(0,\mathfrak{m})$, and $i\in\mathbb{Z}^+$ sufficiently large, there exists a unique $\delta_i(\mu)\in(0,1-|z_i|)$ such that $E_g(u_i;B_{\delta_i(\mu)}(z_i))=\mu$. Furthermore,

$$\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{R\delta}(z_i)-B_{\delta_i(\mu)}(z_i)\big)=\mathfrak{m}-\mu. \tag{5.3}$$

Proof. (1) Since du_i converges uniformly to du on compact subsets of B_1^*,

$$\begin{aligned}\mathfrak{m}(\delta)\equiv\lim_{i\longrightarrow\infty}E_g\big(u_i;B_\delta\big)&=\lim_{\delta'\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{\delta'}\big)+\lim_{\delta'\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_\delta-B_{\delta'}\big)\\&=\mathfrak{m}+\lim_{\delta'\longrightarrow0}E_g\big(u;B_\delta-B_{\delta'}\big)=\mathfrak{m}+E_g(u;B_\delta).\end{aligned}$$

Since $E_g(u;B_\delta)$ is a continuous, non-decreasing function of δ, so is $\mathfrak{m}(\delta)$.

(2) For all $\delta,\delta'\in\mathbb{R}^+$ and $z_i\in B_{\delta'}$, $B_{\delta-\delta'}\subset B_\delta(z_i)\subset B_{\delta+\delta'}$. Thus,

$$E_g\big(u_i;B_{\delta-\delta'}\big)\le E_g\big(u_i;B_\delta(z_i)\big)\le E_g\big(u_i;B_{\delta+\delta'}\big)$$

for all $i\in\mathbb{Z}^+$ sufficiently large and

$$\lim_{\delta'\longrightarrow0}\mathfrak{m}(\delta-\delta')\le\lim_{\delta'\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i;B_\delta(z_i)\big)\le\lim_{\delta'\longrightarrow0}\mathfrak{m}(\delta+\delta')\qquad\forall\ \delta'\in\mathbb{R}^+.$$

The claim now follows from (1).

(3) By (2), (1), and (5.2),

$$\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) = \mathfrak{m}(\delta) \ge \mathfrak{m}\,.$$

Thus, there exists $i(\mu)\in\mathbb{Z}^+$ such that

$$E_g(u_i; B_\delta(z_i)) > \mu \qquad \forall\ i \ge i(\mu).$$

Since $E_g(u_i; B_\delta(z_i))$ is a continuous, increasing function of δ which vanishes at $\delta = 0$, for every $i\ge i(\mu)$ there exists a unique $\delta_i(\mu)\in(0,\delta)$ such that $E_g(u_i; B_{\delta_i(\mu)}(z_i))=\mu$.

By (2), (1), and (5.2),

$$\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_{R\delta}(z_i)\big) = \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\mathfrak{m}(R\delta) = \lim_{R\longrightarrow\infty}\mathfrak{m} = \mathfrak{m}.$$

Combining this with the definition of $\delta_i(\mu)$, we obtain (5.3). □

Corollary 5.3. *If (X,J) is a compact almost complex manifold with a Riemannian metric g, then there exists $\hbar_{J,g}\in\mathbb{R}^+$ with the following properties. If $u_i\colon B_1\longrightarrow X$ is a sequence of J-holomorphic maps converging uniformly in the C^∞-topology on compact subsets of B_1^* to a J-holomorphic map $u\colon B_1\longrightarrow X$ such that*

$$\lim_{i\longrightarrow\infty}\max_{\overline{B_{1/2}}}\big|\mathrm{d}u_i\big|_g = \infty$$

and the limit (5.2) exists, then

(1) $\mathfrak{m}\ge\hbar_{J,g}$;

(2) for every sequence $z_i\in B_\delta$ converging to 0 and $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$, the numbers $\delta_i(\mu)\in(0,1-|z_i|)$ of Lemma 5.2(3) satisfy

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g(u_i; B_{R\delta_i(\mu)}(z_i)) = \mathfrak{m}, \tag{5.4}$$

$$\lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}\mathrm{diam}_g\big(u_i(B_\delta(z_i)-B_{R\delta_i(\mu)}(z_i))\big) = 0. \tag{5.5}$$

Proof. Let $\hbar_{J,g}$ be the smaller of the constants $\hbar_{J,g}$ in Corollaries 4.2 and 4.17. Let u_i, u, and $\mathfrak{m}$ be as in the statement of the corollary.

(1) For each $i\in\mathbb{Z}^+$, let

$$M_i = \max_{\overline{B_{1/2}}}\big|\mathrm{d}_z u_i\big|_g \in \mathbb{R}^+$$

and $z_i\in\overline{B_{1/2}}$ be such that $|\mathrm{d}_{z_i}u_i|_g=M_i$. Since $M_i\longrightarrow\infty$ as $i\longrightarrow\infty$ and u_i converges uniformly in the C^∞-topology on compact subsets of B_1^* to u, $z_i\longrightarrow 0$. For $i\in\mathbb{Z}^+$ such that $|z_i|+1/\sqrt{M_i}<1/2$, define

$$v_i\colon B_{\sqrt{M_i}}\longrightarrow X, \qquad v_i(z) = u_i\big(z_i+z/M_i\big).$$

Thus, v_i is a J-holomorphic map with

$$\sup\big|\mathrm{d}v_i\big|_g = \big|\mathrm{d}_0 v_i\big|_g = 1, \qquad E_g(v_i) = E_g\big(u_i; B_{1/\sqrt{M_i}}(z_i)\big) \le E_g\big(u_i; B_{|z_i|+1/\sqrt{M_i}}\big)\,. \tag{5.6}$$

By the first statement in (5.6) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of v_i converges uniformly in the C^∞-topology on compact subsets of $\mathbb{C}$ to a non-constant J-holomorphic map $v\colon \mathbb{C}\longrightarrow X$. By the second statement in (5.6) and Lemma 5.2(1),

$$E_g(v) \le \limsup_{i\longrightarrow\infty} E_g\big(u_i; B_{1/\sqrt{M_i}}(z_i)\big) \le \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta\big) = \mathfrak{m}. \tag{5.7}$$

By Proposition 5.1, v thus extends to a J-holomorphic map $\widetilde{v}\colon \mathbb{P}^1\longrightarrow X$. By Corollary 4.2,

$$E_g(v) = E_g(\widetilde{v}) \ge \hbar_{J,g}\,.$$

Combining this with (5.7), we obtain the first claim.

(2) By the first two statements in Lemma 5.2 and (5.2),

$$\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) = \lim_{\delta\longrightarrow 0}\mathfrak{m}(\delta) = \mathfrak{m}. \tag{5.8}$$

After passing to a subsequence of u_i, we can thus assume that there exists a sequence $\delta_i \longrightarrow 0$ such that

$$\lim_{i\longrightarrow\infty} E_g(u_i; B_{\delta_i}(z_i)) = \mathfrak{m}. \tag{5.9}$$

Since $\delta_i\longrightarrow 0$, (5.8) and (5.9) imply that

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g(u_i; B_{R\delta_i}(z_i)) = \mathfrak{m}. \tag{5.10}$$

Suppose $\mu\in(\mathfrak{m}-\hbar_{J,g},\mathfrak{m})$. By (5.10) and the definition of $\delta_i(\mu)$,

$$\lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u; B_{R\delta_i}(z_i)-B_{\delta_i(\mu)}(z_i)\big) = \mathfrak{m}-\mu < \hbar_{J,g}\,.$$

Thus, Corollary 4.17 applies with (R,T) replaced by $(\frac{1}{2}\ln(R\delta_i/\delta_i(\mu)),\ln R)$ and u replaced by the J-holomorphic map

$$v\colon (-R,R)\times S^1 \longrightarrow X, \qquad v\big(r,\mathrm{e}^{\mathrm{i}\theta}\big) = u\big(z_i + \sqrt{R\delta_i\delta_i(\mu)}\,\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

By the first statement of Corollary 4.17,

$$E_g\big(u; B_{\delta_i}(z_i)\big) - E_g\big(u; B_{R\delta_i(\mu)}(z_i)\big) = E_g\big(u; B_{\delta_i}(z_i)-B_{R\delta_i(\mu)}(z_i)\big) \le \frac{C_{J,g}}{R}E_g(u)$$

for all i sufficiently large (depending on R); see Figure 9. Combining this with (5.9), we obtain (5.4).

It remains to establish (5.5). By (5.3), for all $R>0$ and sufficiently small $\delta>0$ (depending on R) there exists $i(R,\delta)\in\mathbb{Z}^+$ such that

$$E_g\big(u_i; B_{R\delta}(z_i)-B_{\delta_i(\mu)}(z_i)\big) < \hbar_{J,g} \qquad \forall\ i > i(R,\delta).$$

Thus, Corollary 4.17 applies with (R,T) replaced by $(\frac{1}{2}\ln(R\delta/\delta_i(\mu)),\ln R)$ and u replaced by the J-holomorphic map

$$v\colon (-R,R)\times S^1 \longrightarrow X, \qquad v\big(r,\mathrm{e}^{\mathrm{i}\theta}\big) = u\big(z_i + \sqrt{R\delta\delta_i(\mu)}\,\mathrm{e}^{r+\mathrm{i}\theta}\big).$$

By the second statement of Corollary 4.17,

$$\mathrm{diam}_g\big(u_i(B_\delta(z_i)-B_{R\delta_i(\mu)}(z_i))\big) \le \frac{C_{J,g}}{\sqrt{R}}\hbar_{J,g} \qquad \forall\ i > i(R,\delta).$$

This gives (5.5). $\square$

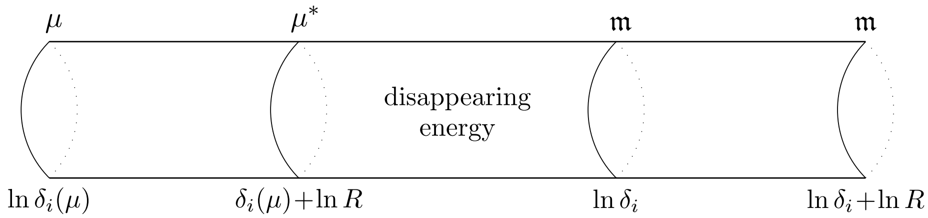


Figure 9: Illustration for the proof of (5.4)

Lemma 5.4. *If (X,J) is a compact almost complex manifold with a Riemannian metric g, then there exists a function $N:\mathbb{R}\longrightarrow\mathbb{Z}$ with the following property. If $(\Sigma,\mathfrak{j})$ is compact Riemann surface, $S_0\subset\Sigma$ is a finite subset, and $u_i:U_i\longrightarrow X$ is a sequence of J-holomorphic maps from open subsets of Σ with*

$$U_i\subset U_{i+1}, \qquad \Sigma-S_0=\bigcup_{i=1}^{\infty}U_i, \quad \textit{and} \quad E\equiv\liminf_{i\longrightarrow\infty}E_g(u_i)<\infty, \tag{5.11}$$

then there exist a subset $S\subset\Sigma$ with $|S|\le N(E)+|S_0|$ and a subsequence of u_i converging uniformly in the C^∞-topology on compact subsets of $\Sigma-S$ to a J-holomorphic map $u:\Sigma\longrightarrow X$.

Proof. Let $\hbar_{J,g}$ be the minimal value of the function provided by Proposition 4.1. For $E\in\mathbb{R}^+$, let $N(E)\in\mathbb{Z}^{\ge0}$ be the smallest integer such that $E\le N(E)\hbar_{J,g}$.

Let Σ, S_0, u_i, and E be as in the statement of the lemma and $N=N(E)+|S_0|$. Fix a Riemannian metric g_Σ on Σ. For $z\in\Sigma$ and $\delta\in\Sigma$, let $B_\delta(z)\subset\Sigma$ denote the ball of radius δ around z. By Proposition 4.1, there exists $C\in\mathbb{R}^+$ with the following property. If $u:\Sigma\longrightarrow X$ is a J-holomorphic map, $z\in\Sigma$, and $\delta\in\mathbb{R}^+$, then

$$E_g\big(u;B_\delta(z)\big)<\hbar_{J,g} \qquad \Longrightarrow \qquad \big|\mathrm{d}_zu\big|_g\le C/\delta^2\,. \tag{5.12}$$

For every pair $i,j\in\mathbb{Z}^+$, let $\{z_{ij}^k\}_{k=1}^N$ be a subset of points of Σ containing S_0 such that

$$z\in\Sigma_{ij}^*\equiv\Sigma-\bigcup_{k=1}^{N}B_{2/j}\big(z_{ij}^k\big) \qquad \Longrightarrow \qquad E_g\big(u_i;B_{1/j}(z)\cap U_i\big)<\hbar_{J,g}\,. \tag{5.13}$$

By (5.12) and (5.13),

$$\big|\mathrm{d}_zu_i\big|_g\le Cj^2 \qquad \forall\,z\in\Sigma_{ij}^* \text{ s.t. } B_{1/j}(z)\subset U_i\,. \tag{5.14}$$

After passing to a subsequence of $\{u_i\}$, we can assume that the sequence $E_g(u_i)$ converges to E and that the sequence $\{z_{ij}^k\}_{i\in\mathbb{Z}^+}$ converges to some $z_j^k\in\Sigma$ for every $k=1,\ldots,N$ and $j\in\mathbb{Z}^+$. Along with (5.14) and the first two assumptions in (5.11), this implies that

$$\limsup_{i\longrightarrow\infty}\big|\mathrm{d}_zu_i\big|_g\le Cj^2 \qquad \forall\,z\in\Sigma_{ij}^*\,. \tag{5.15}$$

After passing to another subsequence of $\{u_i\}$, we can assume that the sequence $\{z_j^k\}_{j\in\mathbb{Z}^+}$ converges to some $z^k\in\Sigma$ for every $k=1,\ldots,N$.

By (5.15) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of u_i converges uniformly in the C^∞-topology on compact subsets of Σ_1^* to a J-holomorphic map $v_1\colon \Sigma_1^* \longrightarrow X$. By (5.15) and the ellipticity of the $\bar{\partial}$-operator, a subsequence of this subsequence in turn converges uniformly in the C^∞-topology on compact subsets of Σ_2^* to a J-holomorphic map $v_2\colon \Sigma_2^* \longrightarrow X$. Continuing in this way, we obtain a subsequence of u_i converging uniformly in the C^∞-topology on compact subsets of Σ_j^* to a J-holomorphic map $v_j\colon \Sigma_j^* \longrightarrow X$ for every $j \in \mathbb{Z}^+$. The limiting maps satisfy

$$v_j|_{\Sigma_j \cap \Sigma_{j'}^*} = v_{j'}|_{\Sigma_j^* \cap \Sigma_{j'}^*} \qquad \forall\, j, j' \in \mathbb{Z}^+ .$$

Thus, the map

$$u\colon \Sigma^* \equiv \Sigma^* - \{z^k\} \longrightarrow X, \qquad u(z) = v_j(z) \;\; \forall\, z \in \Sigma_j^*,$$

is well-defined and J-holomorphic.

By construction, the final subsequence of u_i converges uniformly in the C^∞-topology on compact subsets of Σ^* to u. This implies that

$$E_g(u) \le \liminf_{i \longrightarrow \infty} E_g(u_i) = E\,.$$

By Proposition 5.1, u thus extends to a J-holomorphic map $\Sigma \longrightarrow X$. □

5.3 Gromov's convergence

We next show that a sequence of maps as in Corollary 5.3 gives rise to a continuous map from a tree of spheres attached at $0 \in B_1$, i.e. a connected union of spheres that have a distinguished, base component and no loops; the distinguished component will be attached at $\infty \in S^2$ to $0 \in B_1$. The combinatorial structure of such a tree is described by a finite rooted linearly ordered set, i.e. a partially ordered set $(I, \prec)$ such that

(RS1) there is a minimal element (root) $i_0 \in I$, i.e. $i_0 \prec h$ for every $h \in I - \{i_0\}$, and

(RS2) for all $h_1, h_2, i \in I$ with $h_1, h_2 \prec i$, either $h_1 = h_2$, or $h_1 \prec h_2$, or $h_2 \prec h_1$.

For each $i \in I - \{i_0\}$, let $p(i) \in I$ denote the immediate predecessor of i, i.e. $p(i) \in I$ such that $h \prec p(i) \prec i$ for all $h \in I - \{p(i)\}$ such that $h \prec i$. Such $p(i) \in I$ exists by (RS1) and is unique by (RS2). In the first diagram in Figure 10, the vertices (dots) represent the elements of a rooted linearly ordered set $(I, \prec)$ and the edges run from $i \in I - \{i_0\}$ down to $p(i)$.

Given a finite rooted linearly ordered set $(I, \prec)$ with minimal element i_0 and a function

$$z\colon I - \{i_0\} \longrightarrow \mathbb{C}, \quad i \longrightarrow z_i, \quad \text{s.t.} \quad \big(p(i_1), z_{i_1}\big) \neq \big(p(i_2), z_{i_2}\big) \;\; \forall\, i_1, i_2 \in I - \{i_0\},\; i_1 \neq i_2, \quad (5.16)$$

let

$$\Sigma = \Big(\bigsqcup_{i \in I} \{i\} \times S^2\Big)\Big/\!\sim, \qquad (i, \infty) \sim \big(p(i), z_i\big) \quad \forall\, i \in I - \{i_0\};$$

see the second diagram in Figure 10. Thus, the tree of spheres Σ is obtained by attaching ∞ in the sphere indexed by i to z_i in the sphere indexed by $p(i)$. The last condition in (5.16) insures that Σ is a nodal Riemann surface, i.e. each non-smooth point (node) has only two local branches (pieces homeomorphic to $\mathbb{C}$).

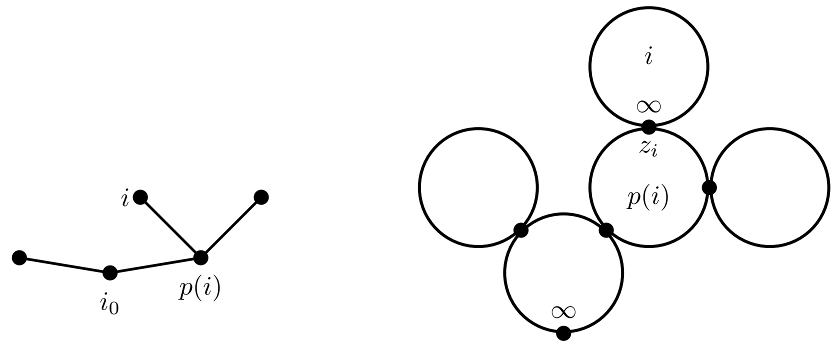


Figure 10: A rooted linearly ordered set and an associated tree of spheres

Proposition 5.5. *Let (X,J) be a compact almost complex manifold with a Riemannian metric g and $u_i\colon B_1\longrightarrow X$ be a sequence of J-holomorphic maps converging uniformly in the C^∞-topology on compact subsets of B_1^* to a J-holomorphic map $u\colon B_1\longrightarrow X$. If the limit*

$$\mathfrak{m}\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta) \tag{5.17}$$

exists and is nonzero, then there exist

(1) a nodal Riemann surface $(\Sigma_\infty,\mathfrak{j}_\infty)$ consisting of B_1 with a tree of Riemann spheres $\mathbb{P}^1$ attached at $0\in B_1$,

(2) a J-holomorphic map $u_\infty\colon\Sigma_\infty\longrightarrow X$,

(3) a subsequence of $\{u_i\}$ still denoted by $\{u_i\}$, and

(4) a biholomorphic map $\psi_i\colon U_i\longrightarrow B_{1/2}$, where $U_i\subset\mathbb{C}$ is an open subset,

such that

(4a) $E_g(u_\infty;\Sigma_\infty-B_1)=\mathfrak{m}$, $U_i\subset U_{i+1}$, and $\mathbb{C}=\bigcup_{i=1}^\infty U_i$,

(4b) $u_i\circ\psi_i$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty,w_1^*,\ldots,w_k^*$ in the sphere $\mathbb{P}^1_0$ attached at $0\in B_1$,

(4c) if $u_\infty|_{\mathbb{P}^1_0}$ is constant, $\mathbb{P}^1_0$ contains at least three nodes of Σ_∞;

(4d) (4) applies with B_1, $(\{u_i\},0)$, and $\mathfrak{m}$ replaced by a neighborhood of w_r^* in $\mathbb{C}$, $(\{u_i\circ\psi_i\},w_r^*)$, and

$$\mathfrak{m}_r'\equiv\lim_{\delta\longrightarrow0}\lim_{i\longrightarrow\infty}E_g\big(u_i\circ\psi_i;B_\delta(w_r^*)\big), \tag{5.18}$$

respectively, for each $r=1,\ldots,k$.

Proof. Let $\hbar_{J,g}$ be the smaller of the numbers $\hbar_{J,g}$ in Corollaries 4.2 and 5.3. In particular, $\mathfrak{m}\geq\hbar_{J,g}$.

For each $i\in\mathbb{Z}^+$ sufficiently large, choose $z_i\in\overline{B_{1/2}}$ so that

$$\max_{z\in\overline{B_{1/2}}}\big|\mathrm{d}u_i\big|_g=\big|\mathrm{d}_{z_i}u_i\big|_g. \tag{5.19}$$

Since u_i converges uniformly in the C^∞-topology on compact subsets of B_1^* to u, $z_i \longrightarrow 0$ as $i \longrightarrow \infty$. Thus, $B_{1/2}(z_i) \subset B_1$ for all $i \in \mathbb{Z}^+$ sufficiently large. By Lemma 5.2(3), for all $i \in \mathbb{Z}^+$ sufficiently large there exists $\delta_i \in (0, 1/2)$ such that

$$E_g\big(u_i; B_{\delta_i}(z_i)\big) = \mathfrak{m} - \frac{\hbar_{J,g}}{2}\,. \tag{5.20}$$

Define

$$\psi_i\colon U_i \equiv \big\{w \in \mathbb{C}\colon z_i + \delta_i w \in B_{1/2}\big\} \longrightarrow B_{1/2} \qquad \text{by} \qquad \psi_i(w) = z_i + \delta_i w\,.$$

Since $\delta_i \longrightarrow 0$, the second property in (4a) holds.

For each $i \in \mathbb{Z}^+$ sufficiently large, let

$$v_i = u_i \circ \psi_i\colon U_i \longrightarrow X.$$

Since u_i is J-holomorphic and ψ_i is biholomorphic onto its image, v_i is a J-holomorphic map with $E_g(v_i) = E_g(u_i; B_{1/2})$. Along with Lemma 5.2(2), this implies that

$$\lim_{i \longrightarrow \infty} E_g(v_i) = \mathfrak{m}(1/2) < \infty\,.$$

By Lemma 5.4, there thus exist a finite collection $w_1^*, \ldots, w_k^* \in \mathbb{C}$ of distinct points and a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, such that v_i converges uniformly in the C^∞-topology on compact subsets of $\mathbb{P}^1 - \{\infty, w_1^*, \ldots, w_\ell^*\}$ to a J-holomorphic map $u\colon \mathbb{P}^1 \longrightarrow X$. In particular, (4b) holds and $|\mathrm{d}v_i|_g$ is uniformly bounded on compact subsets of $\mathbb{P}^1 - \{\infty, w_1^*, \ldots, w_\ell^*\}$. We can also assume that the limit (5.18) exists for every $r = 1, \ldots, k$. We note that

$$\begin{split} E_g(v) + \sum_{r=1}^{k} \mathfrak{m}'_r &= \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(v_i, B_R - \bigcup_{r=1}^{k} B_\delta(w_r^*)\big) + \sum_{r=1}^{k} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(v_i; B_\delta(w_r^*)\big) \\ &= \lim_{R \longrightarrow \infty} \lim_{i \longrightarrow \infty} E_g\big(v_i, B_R\big) = \lim_{R \longrightarrow \infty} \lim_{i \longrightarrow \infty} E_g\big(u_i, B_{R\delta_i}(z_i)\big) = \mathfrak{m}\,; \end{split} \tag{5.21}$$

the last equality holds by (5.4).

Let $\delta_0 \in \mathbb{R}^+$ be such that the balls $B_{\delta_0}(w_r^*)$ are pairwise disjoint. If

$$\limsup_{i \longrightarrow \infty} \max_{B_{\delta_0}(w_r^*)} \big|\mathrm{d}v_i\big| < \infty$$

for some r, then $\{v_i\}$ converges uniformly in the C^∞-topology on $\overline{B_{\delta_0}(w_r^*)}$ to v by the ellipticity of the $\bar\partial$-operator. Thus, we can assume that

$$\lim_{i \longrightarrow \infty} \sup_{B_{\delta_0}(w_r^*)} \big|\mathrm{d}v_i\big| = \infty$$

for every $r = 1, \ldots, k$. In light of Corollary 5.3(1), $\mathfrak{m}'_r \ge \hbar_{J,g}$.

We next show that $u(0) = v(\infty)$, i.e. that the bubble $(\mathbb{P}^1_0, v)$ connects to (B_1, u) at $z = 0$. Note that

$$\begin{split} d_g\big(u(0), v(\infty)\big) &= \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} d_g\big(u(\delta), v(R)\big) = \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} d_g\big(u_i(z_i + \delta), v_i(R)\big) \\ &= \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} d_g\big(u_i(z_i + \delta), u_i(z_i + R\delta_i)\big) \\ &\le \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} \mathrm{diam}_g\big(u_i(B_\delta(z_i) - B_{R\delta_i}(z_i))\big). \end{split}$$

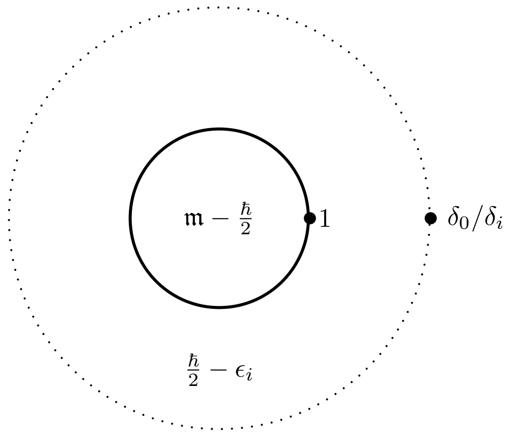


Figure 11: The energy distribution of the rescaled map v_i in the proof of Proposition 5.5

Along with (5.5), this implies that $u(0)=v(\infty)$.

Suppose $v:\mathbb{P}^1\longrightarrow X$ is a constant map. By (5.21), $k\geq 1$ and so there exists $w^*\in\mathbb{C}$ such that $|\mathrm{d}_{w^*}v_i|\longrightarrow\infty$ as $i\longrightarrow\infty$. By (5.19) and the definition of ψ_i, $|\mathrm{d}_0v_i|\geq|\mathrm{d}_wv_i|$ for all $w\in\mathbb{C}$ contained in the domain of v_i and so $|\mathrm{d}_0v_i|\longrightarrow\infty$ as $i\longrightarrow\infty$. By (5.18) and (5.20),

$$\mathfrak{m}_0'\equiv\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty}E_g(v_i)\leq\lim_{i\longrightarrow\infty}E_g\big(v_i;B_1\big)=\lim_{i\longrightarrow\infty}E_g\big(u_i;B_{\delta_i}(z_i)\big)=\mathfrak{m}-\frac{\hbar}{2}<\mathfrak{m},$$

and so $k\geq 2$, as claimed in (4c). Since the amount of energy of v_i contained in $\mathbb{C}-B_1$ approaches $\hbar_{J,g}/2$, as illustrated in Figure 11, there must be in particular a bubble point w_r^* with $|w_r^*|=1$, though this is not material.

The above establishes Proposition 5.5 whenever $k=0$ by taking

$$u_\infty\big|_{B^1}=u\qquad\text{and}\qquad u_\infty\big|_{\mathbb{P}^1_0}=v.$$

Since $\mathfrak{m}_r'\geq\hbar_{J,g}$ for every r, $k=0$ if $\mathfrak{m}<2\hbar_{J,g}$. If $k\geq 1$, $\mathfrak{m}_r'\leq\mathfrak{m}-\hbar_{J,g}$ by (5.21) because $E_g(v)\geq\hbar_{J,g}$ if v is not constant by Corollary 4.2 and $k\geq 2$ otherwise by the above. Thus, by induction on $[\mathfrak{m}/\hbar_{J,g}]\in\mathbb{Z}^+$, we can assume that Proposition 5.5 holds when applied to $\{v_i\}$ on $B_{\delta_0}(w_r^*)\subset\mathbb{C}$ with $r=1,\ldots,k$. This yields a tree Σ_r of Riemann spheres $\mathbb{P}^1$ with a distinguished smooth point ∞ and a J-holomorphic map $v_r:\Sigma_r\longrightarrow X$ such $v_r(\infty)=v(w_r^*)$ and $E_g(v_r)=\mathfrak{m}_r'$. Combining the last equality with (5.21), we obtain

$$E_g(v)+\sum_{r=1}^{k}E_g(v_r)=\mathfrak{m}\,.$$

Identifying ∞ in the base sphere of each Σ_r with $w_r^*\in\mathbb{P}^1_0$, which has been already attached to $0\in B_1^*$, we obtain a J-holomorphic map $u_\infty:\Sigma_\infty\longrightarrow X$ with the desired properties; see Figure 12. □

Proof of Theorem 1.2. Fix a Riemannian metric g_Σ on Σ. For $z\in\Sigma$ and $\delta\in\Sigma$, let $B_\delta(z)\subset\Sigma$ denote the ball of radius δ around z.

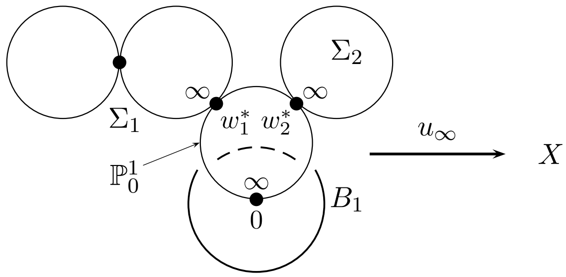


Figure 12: Gromov's limit of a sequence of J-holomorphic maps $u_i\colon B_1 \longrightarrow X$

By Lemma 5.4, there exist a finite collection $z_1^*, \ldots, z_\ell^* \in \Sigma$ of distinct points and a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, such that u_i converges uniformly in the C^∞-topology on compact subsets of $\Sigma - \{z_1^*, \ldots, z_\ell^*\}$ to a J-holomorphic map $u\colon \Sigma \longrightarrow X$. In particular, $|\mathrm{d}u_i|_g$ is uniformly bounded on compact subsets of $\Sigma - \{z_1^*, \ldots, z_\ell^*\}$. We can also assume that the limit

$$\mathfrak{m}_j \equiv \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_\delta(z_j^*)\big)$$

exists for every $j = 1, \ldots, \ell$. We note that

$$\begin{aligned} E_g(u) + \sum_{j=1}^{\ell} \mathfrak{m}_j &= \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u; \Sigma - \bigcup_{j=1}^{\ell} B_\delta(z_j^*)\big) + \sum_{j=1}^{\ell} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_\delta(z_j^*)\big) \\ &= \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g(u_i) = \lim_{i \longrightarrow \infty} E_g(u_i). \end{aligned} \tag{5.22}$$

Let $\delta_0 \in \mathbb{R}^+$ be such that the balls $B_{\delta_0}(z_i^*)$ are pairwise disjoint. If

$$\limsup_{i \longrightarrow \infty} \max_{B_{\delta_0}(z_j^*)} \big|\mathrm{d}u_i\big| < \infty$$

for some j, then $\{u_i\}$ converges uniformly in the C^∞-topology on $\overline{B_{\delta_0}(z_j^*)}$ to u by the ellipticity of the $\bar{\partial}$-operator. Thus, we can assume that

$$\lim_{i \longrightarrow \infty} \sup_{B_{\delta_0}(z_i^*)} \big|\mathrm{d}u_i\big| = \infty$$

for every $j = 1, \ldots, \ell$.

For each $j = 1, \ldots, \ell$, Proposition 5.5 provides a tree Σ_j of Riemann spheres $\mathbb{P}^1$ with a distinguished smooth point ∞ and a J-holomorphic map $v_j : \Sigma_j \longrightarrow X$ such $v_j(\infty) = v(w_r^*)$ and $E_g(v_j) = \mathfrak{m}_j$. Combining the last equality with (5.22), we obtain

$$E_g(v) + \sum_{j=1}^{\ell} E_g(v_j) = \lim_{i \longrightarrow \infty} E_g(u_i)\,.$$

Identifying the distinguished point ∞ of each Σ_j with $z_j^* \in \Sigma$, we obtain a Riemann surface $(\Sigma_\infty, \mathfrak{j}_\infty)$ and a J-holomorphic map $u_\infty\colon \Sigma_\infty \longrightarrow X$ with the desired properties.

If $\Sigma=\mathbb{P}^1$ and the limit map u above is constant, then $\ell\ge 1$ by (5.22). Suppose $\ell\in\{1,2\}$. Let

$$M_i = \sup_{B_{\delta_0}(z_1^*)} |\mathrm{d}u_i|$$

and parametrize $\mathbb{P}^1$ so that $z_1^*=0$. Define

$$h_i\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad h_i(z) = z_i+z/M_i,$$

and apply the preceding argument with u_i replaced by $u_i\circ h_i$. By the proof of Corollary 5.3(1), the limiting map $u|_\Sigma$ is then non-constant and $(\Sigma_\infty, \mathfrak{j}_\infty, u)$ is a stable J-holomorphic map. $\square$

5.4 An example

We now give an example illustrating Gromov's convergence in a classical setting.

Let $n\in\mathbb{Z}^+$, with $n\ge 2$, and $\mathbb{P}^{n-1}=\mathbb{C}\mathbb{P}^{n-1}$. Denote by ℓ the positive generator of $H_2(\mathbb{P}^{n-1};\mathbb{Z})\approx\mathbb{Z}$, i.e. the homology class represented by the standard $\mathbb{P}^1\subset\mathbb{P}^{n-1}$. A degree d map $f\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a continuous map such that $f_*[\mathbb{P}^1]=d\ell$. A holomorphic degree d map $f\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is given by

$$[u,v] \longrightarrow \big[R_1(u,v),\ldots,R_n(u,v)\big]$$

for some degree d homogeneous polynomials $R_1,\ldots,R_d$ on $\mathbb{C}^2$ without a common linear factor. Since the tuple $(\lambda R_1,\ldots,\lambda R_n)$ determines the same map as $(R_1,\ldots,R_n)$ for any $\lambda\in\mathbb{C}^*$, the space of degree d holomorphic maps $f\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a dense open subset of

$$\mathfrak{X}_{n,d} \equiv \big((\mathrm{Sym}^d\mathbb{C}^2)^n - \{0\}\big)/\mathbb{C}^* \approx \mathbb{P}^{(d+1)n-1}\,.$$

Suppose $f_k\colon \mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a sequence of holomorphic degree $d\ge 1$ maps and

$$\mathbf{R}_k = \big[R_{k;1},\ldots,R_{k;n}\big] \in \mathfrak{X}_{n,d}$$

are the associated equivalence classes of n-tuples of homogeneous polynomials without a common linear factor. Passing to a subsequence, we can assume that $[\mathbf{R}_k]$ converges to some

$$\mathbf{R} \equiv \big[(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_1,\ldots,(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_n\big] \in \mathfrak{X}_{n,d}\,, \qquad (5.23)$$

with $d_1,\ldots,d_m\in\mathbb{Z}^+$ and homogeneous polynomials

$$\mathbf{S} \equiv [S_1,\ldots,S_n] \in \mathfrak{X}_{n,d_0}$$

without a common linear factor and with $d_0\in\mathbb{Z}^{\ge 0}$. By (5.23),

$$d_0 + d_1 + \ldots + d_m = d.$$

Rescaling $(R_{k;1},\ldots,R_{k;n})$, we can assume that

$$\lim_{k\longrightarrow\infty} R_{k;i} = (v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_i \qquad \forall\, i=1,\ldots,n. \qquad (5.24)$$

Suppose $z_0\in\mathbb{C}-\{u_1/v_1,\ldots,u_m/v_m\}$. Since the polynomials $S_1,\ldots,S_n$ do not have a common linear factor, $S_{i_0}(z_0,1)\neq 0$ for some $i_0=1,\ldots,n$. This implies that $R_{k;i_0}(z_0,1)\neq 0$ for all k large enough and so

$$\lim_{k\longrightarrow\infty}\frac{R_{k;i}(z,1)}{R_{k;i_0}(z,1)}=\frac{\lim\limits_{k\longrightarrow\infty}R_{k;i}(z,1)}{\lim\limits_{k\longrightarrow\infty}R_{k;i_0}(z,1)}=\frac{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_i(z,1)}{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_{i_0}(z,1)}=\frac{S_i(z,1)}{S_{i_0}(z,1)}$$

for all $i=1,\ldots,n$ and z close to z_0. Furthermore, the convergence is uniform on a neighborhood of z_0. Thus, the sequence f_k C^∞-converges on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$ to the holomorphic degree d_0 map $g\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ determined by $\mathbf{S}$.

Let ω be the Fubini-Study symplectic form on $\mathbb{P}^{n-1}$ normalized so that $\langle\omega,\ell\rangle=1$ and $E(\cdot)$ be the energy of maps into $\mathbb{P}^{n-1}$ with respect to the associated Riemannian metric. For each $\delta>0$ and $j=1,\ldots,m$, denote by $B_\delta([u_j,v_j])$ the ball of radius δ around $[u_j,v_j]$ in $\mathbb{P}^1$ and let

$$\mathbb{P}^1_\delta=\mathbb{P}^1-\bigcup_{j=1}^m B_\delta([u_j,v_j])\,.$$

For each $j=1,\ldots,m$, let

$$\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;B_\delta([u_j,v_j])\big)\in\mathbb{R}^{\ge0}$$

be the energy sinking into the bubble point $[u_j,v_j]$. By Theorem 1.2, the number $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})$ is the value of ω on some element of $H_2(\mathbb{P}^{n-1};\mathbb{Z})$, i.e. an integer. Below we show that $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$.

Since the sequence f_k C^∞-converges to the degree d_0 map $g:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$,

$$d_0=\langle\omega,d_0\ell\rangle=E(g)=\lim_{\delta\longrightarrow0}E_g\big(g;\mathbb{P}^1_\delta\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;\mathbb{P}^1_\delta\big).$$

Thus,

$$\begin{aligned}\sum_{j=1}^m\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)&=\sum_{j=1}^m\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;B_\delta([u_j,v_j])\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k;\bigcup_{j=1}^mB_\delta([u_j,v_j])\big)\\&=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}\big(E_g(f_k)-E_g\big(f_k;\mathbb{P}^1_\delta\big)\big)=d-d_0=d_1+\ldots+d_m\,.\end{aligned}$$

In particular, $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$ if $m=1$, no matter what the "residual" tuple of polynomials $\mathbf{S}$ is. We use this below to establish this energy identity for $m>1$ as well.

By (5.24), for all $k\in\mathbb{Z}^+$ sufficiently large there exist $\lambda_{k;i;j;p}\in\mathbb{C}$ with $i=1,\ldots,n$, $j=1,\ldots,m$, and $p=1,\ldots,d_j$ and tuples

$$\mathbf{S}_k\equiv\big[S_{k;1},\ldots,S_{k;n}\big]\in\mathfrak{X}_{n;d_0}$$

of polynomials without a common linear factor such that

$$\lim_{k\longrightarrow\infty}\mathbf{S}_k=\mathbf{S},\qquad\lim_{k\longrightarrow\infty}\lambda_{k;i;j;p}=1\quad\forall\,i,j,p,$$

$$R_{k;i}(u,v)=\prod_{j=1}^m\prod_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v)\quad\forall\,k,i\,.$$

For each $j_0 = 1, \ldots, m$, let

$$\mathbf{T}_{j_0} \equiv \big[T_{j_0;1}, \ldots, T_{j_0;n}\big] \in \mathfrak{X}_{n;d-d_{j_0}}$$

be a tuple of polynomials without a common linear factor. If in addition, $i = 1, \ldots, n$, $\epsilon \in \mathbb{R}$, and $k \in \mathbb{Z}^+$, let

$$S_{i;j_0;\epsilon}(u,v) \equiv \prod_{j\neq j_0}^{m} (v_j u - u_j v)^{d_j} \cdot S_i(u,v) + \epsilon T_{j_0;i}(u,v)\,, \qquad i = 1, \ldots, n,$$

$$R_{k;i;j_0;\epsilon}(u,v) \equiv R_{k;i}(u,v) + \epsilon \prod_{p=1}^{d_{j_0}} (v_{j_0} u - \lambda_{k;i;j_0;p} u_{j_0} v) \cdot T_{j_0;i}(u,v), \qquad i = 1, \ldots, n.$$

The polynomials within each tuple $(S_{i;j_0;\epsilon})_{i=1,\ldots,n}$ and $(R_{k;i;j_0;\epsilon})_{i=1,\ldots,n}$ have no common linear factor for all $\epsilon \in \mathbb{R}^+$ sufficiently small and k sufficiently large (with the conditions on ϵ and k mutually independent). We denote by

$$f_{k;j_0;\epsilon}\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}$$

the holomorphic degree d map determined by the tuple

$$\mathbf{R}_{k;j_0;\epsilon} \equiv \big[R_{k;1;j_0;\epsilon}, \ldots, R_{k;n;j_0;\epsilon}\big]\,.$$

Since

$$\lim_{k\longrightarrow\infty} \mathbf{R}_{k;j_0;\epsilon} = \big[(v_1 u - u_1 v)^{d_{j_0}} S_{1;j_0;\epsilon}, \ldots, (v_1 u - u_1 v)^{d_{j_0}} S_{n;j_0;\epsilon}\big] \in \mathfrak{X}_{n;d}$$

and the polynomials $S_{1;j_0;\epsilon}, \ldots, S_{n;j_0;\epsilon}$ have no linear factor in common,

$$\lim_{\delta\longrightarrow 0} \lim_{k\longrightarrow\infty} E\big(f_{k;j_0;\epsilon}; B_\delta([u_{j_0}, v_{j_0}])\big) \equiv \mathfrak{m}_{[u_{j_0}, v_{j_0}]}\big(\{f_{k;j_0;\epsilon}\}\big) = d_{j_0} \tag{5.25}$$

by the $m = 1$ case established above.

For $\delta \in \mathbb{R}^+$ sufficiently small, $\epsilon \in \mathbb{R}^+$ sufficiently small, and k sufficiently large,

$$\prod_{j\neq j_0}^{m} \prod_{p=1}^{d_j} (v_j u - \lambda_{k;i;j;p} u_j v) \cdot S_{k;i}(u,v) \neq 0 \qquad \forall\ [u,v] \in B_{2\delta}\big([u_{j_0}, v_{j_0}]\big).$$

Thus, the ratios

$$\frac{R_{k;i;j_0;\epsilon}(u,v)}{R_{k;i}(u,v)} = 1 + \epsilon \frac{T_{j_0;i}(u,v)}{\prod\limits_{j\neq j_0}^{m} \prod\limits_{p=1}^{d_j} (v_j u - \lambda_{k;i;j;p} u_j v) \cdot S_{k;i}(u,v)}$$

converge uniformly to 1 on $B_\delta([u_{j_0}, v_{j_0}])$ as $\epsilon \longrightarrow 0$. Thus, there exists $k^* \in \mathbb{Z}^+$ such that

$$\lim_{\epsilon\longrightarrow 0} \sup_{k\geq k^*} \sup_{z\in B_\delta([u_{j_0}, v_{j_0}])} \left| \frac{|\mathrm{d}_z f_{k;j_0;\epsilon}|}{|\mathrm{d}_z f_k|} - 1 \right| = 0.$$

It follows that

$$\begin{aligned} \mathfrak{m}_{[u_{j_0}, v_{j_0}]}\big(\{f_k\}\big) &\equiv \lim_{\delta\longrightarrow 0} \lim_{k\longrightarrow\infty} E\big(f_k; B_\delta([u_{j_0}, v_{j_0}])\big) = \lim_{\delta\longrightarrow 0} \lim_{k\longrightarrow\infty} \lim_{\epsilon\longrightarrow 0} E\big(f_{k;j_0;\epsilon}; B_\delta([u_{j_0}, v_{j_0}])\big) \\ &= \lim_{\epsilon\longrightarrow 0} \lim_{\delta\longrightarrow 0} \lim_{k\longrightarrow\infty} E\big(f_{k;j_0;\epsilon}; B_\delta([u_{j_0}, v_{j_0}])\big) = \lim_{\epsilon\longrightarrow 0} d_{j_0} = d_{j_0}; \end{aligned}$$

the second-to-last equality above holds by (5.25).

Suppose that either $d_0 \geq 1$ or $m \geq 3$. Otherwise, the maps f_k can be reparametrized so that $d_0 \neq 0$; see the last paragraph of the proof of Theorem 1.2 at the end of Section 5.3. By Theorem 1.2 and the above, a subsequence of $\{f_k\}$ converges to the equivalence class of a holomorphic degree d_0 map $f : \Sigma \longrightarrow \mathbb{P}^{n-1}$, where Σ is a nodal Riemann surface consisting of the component $\Sigma_0 = \mathbb{P}^1$ corresponding to the original $\mathbb{P}^1$ and finitely many trees of $\mathbb{P}^1$'s coming off from Σ_0. The maps on the components in the trees are defined only up reparametrization of the domain. By the above, $f|_{\Sigma_0}$ is the map g determined by the "relatively prime part" $\mathbf{S}$ of the limit $\mathbf{R}$ of the tuples of polynomials. The trees are attached at the roots $[u_j, v_j]$ of the common linear factors $v_j u - u_j v$ of the polynomials in $\mathbf{R}$; the degree of the restriction of f to each tree is the power of the multiplicity d_j of the corresponding common linear factor.

The same reasoning as above applies to the sequence of maps

$$\big(\mathrm{id}_{\mathbb{P}^1}, f_k\big) : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^{n-1},$$

but the condition that either $d_0 \geq 1$ or $m \geq 3$ is no longer necessary for the analogue of the conclusion in the previous paragraph. This implies that the map

$$\mathfrak{M}_{0,0}\big(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d)\big) \longrightarrow \mathfrak{X}_{n,d}, \qquad [f,g] \longrightarrow \big[g \circ f^{-1}\big],$$

from the subspace of $\overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d))$ corresponding to maps from $\mathbb{P}^1$ extends to a continuous surjective map

$$\overline{\mathfrak{M}}_{0,0}\big(\mathbb{P}^1 \times \mathbb{P}^{n-1}, (1,d)\big) \longrightarrow \mathfrak{X}_{n,d}\,. \tag{5.26}$$

In particular, Gromov's moduli spaces refine classical compactifications of spaces of holomorphic maps $\mathbb{P}^1 \longrightarrow \mathbb{P}^{n-1}$. On the other hand, the former are defined for arbitrary almost Kahler manifolds, which makes them naturally suited for applying topological methods. The right-hand side of (5.26) is known as the linear sigma model in the Mirror Symmetry literature. The morphism (5.26) plays a prominent role in the proof of mirror symmetry for the genus 0 Gromov-Witten invariants in [5] and [8]; see [7, Section 30.2].

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