On Asymptotic Behavior of GW-Invariants

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1 Summary

Claim 1 ([FI, Proposition 3]) *If $N_{0,d}$ is the number of degree d rational curves through $3d-1$ general points in $\mathbb{P}^2$, then*

$$\lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{0,d}}{(3d-1)!}}$$

exists (and is nonzero).

Claim 1 is needed to establish [FI, Proposition 3], but is **not** proved in [FI]. It is shown in [FI] that the numbers $\sqrt[d]{\frac{N_{0,d}}{(3d-1)!}}$ are bounded (above and below away from 0), but not that they converge; see Section 2 for details. *Mathematica* suggests that these numbers are increasing (after the first few terms), but it is not clear how this can be proved.

Conjecture 2 ([FI, Footnote 2]) *If $N_{g,d}$ is the number of degree d genus g curves through $3d-1+g$ general points in $\mathbb{P}^2$, then*

$$\frac{N_{g,d}}{(3d-1+g)!}=a_g b^d d^{-1-\frac{5}{2}(1-g)}\big(1+o(1)\big),$$

for some $b\in\mathbb{R}^+$ independent of g and for some $a_g\in\mathbb{R}^+$.

The $g=0$ case of this conjecture is [FI, Proposition 3]. Along with the Eguchi-Hori-Xiong recursion for $N_{1,d}$ (proved in [P]), it almost implies the $g=1$ case; see Section 3.

Proposition 3 *If Claim 1 is true, then*

$$\lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{1,d}}{(3d)!}}=\lim_{d\longrightarrow\infty}\sqrt[d]{\frac{N_{0,d}}{(3d-1)!}}\,.$$

For $\mathbb{P}^3$, *Mathematica* suggests the following conjecture; it is based on the numbers up to $d=200$ (the computation of these numbers already takes a long time). As the convergence appears to be very slow (for $N_{0,d}$, it is still going noticeably even for $d=1000$), it is feasible that the limit below is even independent of the slope chosen, but the numbers so far do not suggest this.

Conjecture 4 *If $N_{0,d}(p)$ is the number of degree d rational curves through $2d-p$ points and $2p$ lines in general position in $\mathbb{P}^3$, then*

$$\lim_{d\longrightarrow\infty} \sqrt[\alpha d]{\frac{N_{0,\alpha d}(\beta d)}{((2\alpha+\beta)d)!}}$$

exists for $\alpha,\beta\in\mathbb{Z}^+$.

An upper bound on the sequences in Conjecture 4 can be obtained from a two-variable version of the approach used in the proof of [FI, Proposition 3]; it also follows immediately from [Z, Theorem 1]. A lower bound appears more elusive, since the recursion of [RT, Theorem 10.4] for $\mathbb{P}^3$ involves negative coefficients; see Section 4.

2 On the proof of [FI, Proposition 3]

Let $n_1, n_2, \ldots$ be a sequence of numbers satisfying

$$n_d = a \sum_{\substack{d_1+d_2=d \\ d_1,d_2\ge 1}} n_{d_1} n_{d_2} \qquad \forall\, d\ge 2,$$

for some $a>0$. The generating function

$$\Phi(q) \equiv \sum_{d=1}^{\infty} n_d q^d$$

then satisfies $\Phi(q) = n_1 q + a\Phi(q)^2$. Thus,

$$\Phi(q) = \frac{1-\sqrt{1-4an_1q}}{2a} = -\frac{1}{2a}\sum_{d=1}^{\infty}\binom{1/2}{d}(-4an_1q)^d = \sum_{d=1}^{\infty}\frac{(2d-2)!}{d!(d-1)!}a^{d-1}n_1^d q^d\,;$$

the middle equality above is the Binomial Theorem.

Corollary 5 *If $n_1, n_2, \ldots$ is a sequence of numbers satisfying*

$$n_d = a \sum_{\substack{d_1+d_2=d \\ d_1,d_2\ge 0}} \frac{f(d_1)f(d_2)}{f(d)} n_{d_1} n_{d_2} \qquad \forall\, d\ge 2,$$

for some $a>0$ and $f:\mathbb{Z}^+\longrightarrow\mathbb{R}$, then

$$n_d = \frac{(2d-2)!}{d!(d-1)!}\frac{a^{d-1}}{f(d)}\big(f(1)n_1\big)^d \qquad \forall\, d\ge 1.$$

On the other hand, by Stirling's formula [A, Theorem 15.19],

$$\frac{4^d}{\sqrt{\pi d}}\left(1+\frac{1}{4d}\right)^{-2} \le \frac{(2d)!}{(d!)^2} \le \frac{4^d}{\sqrt{\pi d}}\left(1+\frac{1}{8d}\right). \tag{1}$$

For each $g\in\mathbb{Z}^{\ge 0}$ and $d\in\mathbb{Z}^+$, let

$$n_{g,d} = \frac{N_{g,d}}{(3d-1+g)!}\,.$$

By [RT, Theorem 10.4], the numbers $n_{0,d}$ are described by

$$\begin{gathered} n_{0,1}=\frac{1}{2}, \qquad n_{0,d} = \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}} f(d_1,d_2)\,n_{0,d_1}n_{0,d_2}\,, \qquad \text{where} \\ f(d_1,d_2) = \frac{d_1d_2\big((3d_1-2)(3d_2-2)(d+2)+8(d-1)\big)}{6(3d-3)(3d-2)(3d-1)} = \frac{d_1d_2\big(3d_1d_2(d+2)-2d^2\big)}{2(3d-3)(3d-2)(3d-1)}\,. \end{gathered} \tag{2}$$

Since

$$\begin{aligned} \frac{1}{54}\frac{d_1d_2(3d_1-2)(3d_2-2)}{d(3d-2)} = \frac{d_1d_2(3d_1-2)(3d_2-2)(d-1)}{6(3d-3)(3d-2)3d} &\le f(d_1,d_2) \\ &\le \frac{d_1d_2\cdot 3d_1d_2(d-\frac{2}{3})}{2\frac{3d}{2}(3d-2)\frac{5d}{2}} = \frac{2}{15}\frac{d_1^2d_2^2}{d^2}\,, \end{aligned}$$

Corollary 5 and (1) thus give

$$\frac{8}{5}\left(\frac{1}{27}\right)^d d^{-7/2} \le n_{0,d} \le \frac{45}{16}\left(\frac{4}{15}\right)^d d^{-7/2}\,.$$

This shows that the numbers

$$b_- \equiv \liminf_{d\longrightarrow\infty} \sqrt[d]{n_{0,d}} \qquad \text{and} \qquad b_+ \equiv \limsup_{d\longrightarrow\infty} \sqrt[d]{n_{0,d}}$$

are between $1/27$ and $4/15$, but not that they are the same.

Let

$$F_0(z) = \sum_{d=1}^{\infty} n_{0,d}e^{dz}\,. \tag{3}$$

By the above, there exists $x_0\in\mathbb{R}$ such that this power series converges if $\mathrm{Re}\,z < x_0$ and diverges if $\mathrm{Re}\,z > x_0$. Since $n_{0,d}\in\mathbb{R}^+$ for all d, there is no neighborhood of $z=x_0$ on (all of) which this series converges (otherwise, every point z_0 with $\mathrm{Re}\,z_0=x_0$ would have such a neighborhood). By (2),

$$(9+2F_0'-3F_0'')F_0''' = 2F_0 - 11F_0' + 18F_0'' + (F_0'')^2. \tag{4}$$

Since $0 < F_0(z) < F_0''(z) < F_0''(z) < F_0'''(z)$ for all $z\in(-\infty,x_0)$,

$$3F_0'' - 2F_0' < 9 \qquad \forall\ z\in(-\infty,x_0). \tag{5}$$

In particular, the series for F_0, F_0', and F_0'' converge at $z=x_0$, while the series for F_0''' does not (otherwise, (4) could be used to compute all derivatives of F_0 at $z=x_0$). This implies that

$$\limsup_{d\longrightarrow\infty} \frac{\ln n_{0,d} - d\ln b_+}{\ln d} \in [-4,-3].$$

According to [FI, p16], this also implies that F_0 admits an expansion around $z=x_0$ of the form

$$F_0(x_0+z) = c_0 + c_1 z + \frac{c_2 z^2}{2} + \lambda z^{2+\alpha} + \dots \tag{6}$$

for some $\alpha\in(0,1)$; this is justified in Section 5.[1] By (6) and (4),

$$\big((9+2c_1-3c_2) - 3\lambda(1+\alpha)(2+\alpha)z^{\alpha}\big)\cdot\lambda\alpha(1+\alpha)(2+\alpha)z^{\alpha-1} = 2c_0 - 11c_1 + 18c_2 + c_2^2 + o(1).$$

This gives

$$9+2c_1-3c_2 = 0, \qquad 2\alpha-1=0, \qquad -3\lambda\alpha(1+\alpha)^2(2+\alpha)^2 = 2c_0-11c_1+18c_2+c_2^2. \tag{7}$$

According to [FI, p17], this "corresponds to" the behavior in the $g=0$ case of Conjecture 2; this can **at most** describe a suitable lim sup. The desired claim would follow from the following conjecture.

Conjecture 6 *If the numbers $n_{0,d}$ are given by (2), the numbers $\sqrt[d]{n_{0,d}}$ are eventually increasing, i.e. there exists $d^*\in\mathbb{Z}^+$ such that*

$$\sqrt[d]{n_{0,d}} \le \sqrt[d+1]{n_{0,d+1}} \qquad \forall d\ge d^*.$$

Some thoughts on this conjecture are in Section 6.

3 Proof of Proposition 3

Let

$$F_1(z) = \sum_{d=1}^{\infty} n_{1,d}e^{dz}.$$

By [P, (8)],

$$n_{1,d} = \frac{(d-1)(d-2)}{216}n_{0,d} + \frac{1}{27d}\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge1}}(3d_1^2-2d_1)d_2 n_{0,d_1}n_{1,d_2}. \tag{8}$$

This implies that

$$(9+2F_0'-3F_0'')F_1' = \frac{1}{8}\big(F_0'''-3F_0''+2F_0'\big). \tag{9}$$

By this identity and (5), the series for F_1 converges at z if $\operatorname{Re} z < x_0$ and so

$$\limsup_{d\longrightarrow\infty}\sqrt[d]{n_{1,d}} \le \limsup_{d\longrightarrow\infty}\sqrt[d]{n_{0,d}}.$$

The opposite inequality follows directly from (8); it also holds for lim inf. This establishes Proposition 3.

[1] This is proved using the Frobenius method, as P. Sarnak suggested.

If the $g=0$ case of Conjecture 2 is true, (8) implies that

$$\liminf_{d\longrightarrow\infty}\frac{\ln n_{0,d}-d\ln b}{\ln d}\ge -\frac{3}{2}\,.$$

By (4) and Section 5, F_1 admits an expansion around $z=x_0$ of the form

$$F_1(x_0+z)=z^{-1}\sum_{d=0}^{\infty}b_d z^{d/2}\qquad\text{with}\quad b_0=-\frac{a_5}{48}\neq 0.$$

Thus, F_1 does not converge at $z=x_0$ and so

$$\limsup_{d\longrightarrow\infty}\frac{\ln n_{0,d}-d\ln b}{\ln d}\ge -1\,.$$

4 Comments on counts in $\mathbb{P}^3$

For each $d\in\mathbb{Z}^+$ and $p\in\mathbb{Z}^{\ge 0}$, let

$$n_{0,d}(p)=\frac{N_{0,d}(p)}{(2d+p)!}\,.$$

By the recursion of [RT, Theorem 10.4],

$$\begin{aligned}
n_{0,d}(0)&=\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\frac{(d_2-d_1)d_1^2d_2(2d_2+1)}{d(d-1)(2d-1)}n_{0,d_1}(0)n_{0,d_2}(1),\\
n_{0,d}(1)&=\frac{d}{2d+1}n_{0,d}(0)+\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\frac{d_1^3d_2(2d_2+1)}{d(2d-1)(2d+1)}n_{0,d_1}(0)n_{0,d_2}(1)\,,\\
n_{0,d}(p)&=\frac{d}{2d+p}n_{0,d}(p-1)+\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\sum_{\substack{p_1+p_2=p\\ 0\le p_i\le 2d_i}}f(d_1,d_2,p_1,p_2)\,n_{0,d_1}(p_1)n_{0,d_2}(p_2),\\
n_{0,d}(2d)&=\frac{1}{2}n_{0,d}(2d-1)+\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}}\frac{d_1d_2(d(4d_1-1)(4d_2-1)+8d_1d_2-d)}{8d(2d-1)(4d-1)}n_{0,d_1}(2d_1)n_{0,d_2}(2d_2),
\end{aligned}$$

where

$$f(d_1,d_2,p_1,p_2)=\frac{(2d-1-p)!(2d_1+p_1)!(2d_2+p_2)!}{(2d+p)!(2d_1-p_1)!(2d_2-p_2)!}(p_2d_1-p_1d_2)\left(d_1^2\binom{2p-2}{2p_1}-d_2^2\binom{2p-2}{2p_2}\right).$$

The first recursion above holds for $d\ge 2$, while the remaining ones are valid for all $d\ge 1$; the third recursion is valid if $0<p<2d$.

These recursions involve negative coefficients, even in the case of the first pair (which is a closed pair of recursions). This makes obtaining a lower bound on the growth rather tricky.

5 On the existence of the expansion (6)

Suppose F_0 admits an expansion of the form

$$F_0(x_0+z) = \sum_{d=0}^{\infty} a_d z^{d/2}. \tag{10}$$

The differential equation (4) is equivalent to

$$a_1, a_3 = 0\,, \qquad (9+2a_2-6a_4)a_5 = 0\,,$$

$$\begin{aligned}(9+2a_2-6a_4)\frac{(d+2)(d+4)(d+6)}{8}a_{d+6} &= 2a_d - \frac{11(d+2)}{2}a_{d+2} + \frac{9(d+2)(d+4)}{2}a_{d+4}\\ &\quad - \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0}} \frac{(d_1+1)(d_1+3)(d_1+5)(d_2+3)}{32} a_{d_1+5}\big(4a_{d_2+3}-3(d_2+5)a_{d_2+5}\big)\\ &\quad + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0}} \frac{(d_1+2)(d_1+4)(d_2+2)(d_2+4)}{16} a_{d_1+4}a_{d_2+4}\,.\end{aligned}$$

The last equation holds for all $d\geq 0$.

If $F_0'''(z)$ blows up as $z\longrightarrow x_0$ from the left, $a_5\neq 0$. In this case, the last two conditions above are equivalent to

$$9+2a_2-6a_4 = 0\,, \qquad -\frac{675}{32}a_5^2 = 2a_0 - 11a_2 + 36a_4 + 4a_4^2\,, \tag{11}$$

$$\begin{aligned}\frac{45(d+2)(d+3)(d+5)}{32}a_5 a_{d+5} &= \frac{15(d+3)}{8}a_5 a_{d+3} - 2a_d + \frac{11(d+2)}{2}a_{d+2} - \frac{9(d+2)(d+4)}{2}a_{d+4}\\ &\quad + \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}} \frac{(d_1+1)(d_1+3)(d_1+5)(d_2+3)}{32} a_{d_1+5}\big(4a_{d_2+3}-3(d_2+5)a_{d_2+5}\big)\\ &\quad - \sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 0}} \frac{(d_1+2)(d_1+4)(d_2+2)(d_2+4)}{16} a_{d_1+4}a_{d_2+4}\,;\end{aligned} \tag{12}$$

the last equation is valid for $d\geq 1$. For any fixed a_0 and a_2 such that

$$4a_2^2 + 45a_2 + 18a_0 + 567 \neq 0, \tag{13}$$

these equations determine a_4, two possible values for $a_5\in\mathbb{C}^*$, and a_d for $d\geq 6$. In particular, $|a_d|\leq |n_{d+5}|$, where

$$n_d = C\big(1+|a_5|^{-1}\big)\sum_{\substack{d_1+d_2=d\\ d_1,d_2\ge 1}} \frac{d_1^3 d_2^3}{d^3} n_{d_1} n_{d_2} \qquad \forall d\geq 2$$

and n_1 and C are sufficiently large.[2] Thus, by Corollary 5 and (1), there exist $A, C\in\mathbb{R}^+$ such that

$$|a_d| \leq C' A^d \big(1+|a_5|^{-1}\big)^d \qquad \forall\, d\in\mathbb{Z}^+\,.$$

[2]Replace the defining equation for a_{d+5} by the inequality for the absolute values; then replace this inequality, by a recursion with increasing terms; then use the increasing property to obtain a simpler recursion for some $\tilde{a}_{d+5}$ in terms of $\tilde{a}_{d_1+5}\tilde{a}_{d_2+5}$ with $d_1+d_2=d$; finally change the index $d+5$ to d.

It follows that (10) defines a singular solution of (4) on a neighborhood of $z\!=\!x_0$ for any choice of a_0 and a_2 such that (13) is satisfied.

If $4a_2^2+45a_2+18a_0+567>0$, $a_{2d}\in\mathbb{R}$ and $a_{2d+1}\in\mathrm{i}\mathbb{R}$ for all d, as can be seen by induction from the defining equation for a_{d+5} above. Thus, $F_0(x)\in\mathbb{R}$ for $x\in(x_0-\delta,x_0+\delta)$ for some $\delta\in\mathbb{R}^+$ and $F_0'''(x)\in\mathbb{R}^+$ for all $x\in(x_0-\delta,x_0)$ for some $\delta\in\mathbb{R}^+$ and one of the two possibilities for a_5. If in addition $0<a_0<a_2<a_4/2$, then

$$0<F_0(x)<F_0'(x)<F_0''(x)<F_0'''(x) \qquad \forall\, x\in(x_0-\delta,x_0) \tag{14}$$

for some $\delta\in(0,1)$. If $F_0\!=\!F_0(z)$ is any solution of (4) with $\mathrm{Re}\,z<x_0$ and $|z-x_0|<\delta$ which is real on $(x_0-\delta,x_0)$ and satisfies (14), then $9+2F_0'-3F_0''$ is a decreasing function on $(x_0-\delta,x_0)$. We show below that in fact there is at most one such solution $F_0\!=\!F_0(z)$ with

$$3F_0''(x_0)=9+2F_0'(x_0)<\infty\,;$$

this implies that the function (3) admits an expansion of the form (10) with $a_1,a_3\!=\!0$ and $a_5\!\neq\!0$.

Suppose F and G are solutions of (4) satisfying (14) with F_0 replaced by F and G such that

$$F(x^*)=G(x^*), \qquad F'(x^*)=G'(x^*), \qquad F''(x^*)<G''(x^*)$$

for some $x^*\in(x_0-1,x_0)$; this implies that $F'''(x^*)<G'''(x^*)$. If $F'''(x)<G'''(x)$ for all $x\in(x^*,x')$ and some $x'\in(x^*,x_0)$, then

$$0\le G'(x')-F'(x')=\int_{x^*}^{x'}\big(G''(x)-F''(x)\big)\mathrm{d}x\le\big(G''(x')-F''(x')\big)(x'-x^*)\le G''(x')-F''(x').$$

It follows that

$$F(x)<G(x), \quad F'(x)<G'(x), \quad F''(x)<G''(x), \quad F'''(x)<G'''(x) \qquad \forall\; x\in(x^*,x_0); \tag{15}$$

the first three inequalities also hold for $x\!=\!x_0$ if F,G,F',G',F'',G'' are continuous at $x\!=\!x_0$ from the left.

For any $\delta\in\mathbb{R}^+$ and $y\!=\!(y_1,\ldots,y_n)\in\mathbb{C}^n$, let

$$B_\delta(y)=\big\{(y_1',\ldots,y_n')\in\mathbb{C}^n:\ |y_i'-y_i|<\delta\ \forall\,i\!=\!1,2,\ldots,n\big\}, \qquad B_\delta^{\mathbb{R}}(y)=B_\delta(y)\cap\mathbb{R}^n\,.$$

Suppose $a_0,a_2\in\mathbb{R}$ are such that

$$0<a_0<a_2<\frac{9}{10}, \qquad 4a_2^2+45a_2+18a_0+567>0.$$

Choose $\delta\in(0,1)$ such that for all $(a_0',a_2')\in B_\delta(a_0,a_2)$

$$0<\mathrm{Re}\,a_0'<\mathrm{Re}\,a_2'<\frac{9}{10}, \qquad \mathrm{Re}\,\big(4a_2'^2+45a_2'+18a_0'+567\big)>0,$$

the series F_0 in (10) converges whenever its coefficients are given by (11) and (12) with $a_1,a_3=0$ and (a_0,a_2) replaced by (a_0',a_2'), and the corresponding function $F_{a_0',a_2'}$ satisfies (14) if $(a_0',a_2')\in\mathbb{R}$. In particular,

$$\lim_{z\longrightarrow x_0} F_{a_0',a_2'}(z)=a_0', \quad \lim_{z\longrightarrow x_0} F_{a_0',a_2'}'(z)=a_2', \quad \lim_{z\longrightarrow x_0} F_{a_0',a_2'}''(z)=\frac{3}{2}+\frac{1}{3}a_2', \tag{16}$$

$$\Longrightarrow \qquad 9+2F_{a_0',a_2'}'(x)-3F_{a_0',a_2'}''(x)>0 \quad \text{if } (a_0',a_2')\in\mathbb{R}^2,\ x\in(x_0-\delta,x_0). \tag{17}$$

By (15) and (16), the restriction of the map

$$\Phi\colon B_\delta(x_0,a_0,a_2)\longrightarrow\mathbb{C}^3, \qquad (z,a_0',a_2')\longrightarrow\big(z,F_{a_0',a_2'}(z),F_{a_0',a_2'}'(z)\big),$$

to $B_\delta^{\mathbb{R}}(x_0,a_0,a_2)$ is an injective map to $\mathbb{R}^3$. Since Φ is a holomorphic map, the differential of Φ is nonsingular at (x_0,a_0,a_2) by the Weierestrass Preparation Theorem [GH, p8]. Thus, Φ is biholomorphic onto an open neighborhood V_{x_0} of (x_0,a_0,a_2) in $\mathbb{C}^3$ if $\delta\in\mathbb{R}^+$ is sufficiently small by the Inverse Function Theorem [GH, p18]. Its inverse is given by

$$V_{x_0}\longrightarrow B_\delta(x_0,a_0,a_2), \qquad (z,a_0',a_2')\longrightarrow\big(z,\Psi(z,a_0',a_2')\big),$$

for some holomorphic map $\Psi\colon V_{x_0}\longrightarrow\mathbb{C}^2$. Let $\delta'\in\mathbb{R}^+$ be such that $B_{\delta'}(x_0,a_0,a_2)\subset V_{x_0}$.

Suppose $F_0=F_0(z)$ is any solution of (4) defined for $z\in B_\delta(x_0)$ with $\operatorname{Re} z<x_0$ such that $F_0(x)\in\mathbb{R}$ for all $x\in(x_0-\delta,x_0)$, (14) is satisfied, and

$$\lim_{x\longrightarrow^- x_0} F_0(x)=a_0\,, \quad \lim_{x\longrightarrow^- x_0} F_0'(x)=a_2\,, \quad \lim_{x\longrightarrow^- x_0} F_0''(x)=\frac{3}{2}+\frac{1}{3}a_2\,, \quad \lim_{x\longrightarrow^- x_0} F_0'''(x)=\infty\,.$$

If $x^*\in(x_0-\delta',x_0)$ is sufficiently close to x_0, then

$$(a_0',a_2')\equiv\big(F_0(x^*),F_0'(x^*)\big)\in B_{\delta'}^{\mathbb{R}}(a_0,a_2)\,.$$

If $F''_{\Psi(x^*,a_0',a_2')}(x^*)<F_0''(x^*)$, choose $y\in\mathbb{R}^+\cap B_{x_0-x^*}(0)$ so that

$$F''_{\Psi(x^*+y,a_0',a_2')}(x^*+y)<F_0''(x^*).$$

Since (4) is a homogeneous differential equation, the function

$$\big\{z\in B_\delta(x_0-y)\colon\ \operatorname{Re} z<x_0-y\big\}\longrightarrow\mathbb{C}, \qquad G(z)=F_{\Psi(x^*+y,a_0',a_2')}(z+y),$$

is a solution of (4) satisfying

$$G(x^*)=a_0'=F_0(x^*), \qquad G'(x^*)=a_2'=F_0'(x^*)\,, \qquad G''(x^*)=F_{\Psi(x^*+y,a_0',a_2')}(x^*+y)<F_0''(x^*).$$

Furthermore, G is real on $(x_0-y-\delta,x_0-y)$ and satisfies (14) with F_0 replaced by G and x_0 by x_0-y, and $G'''(x)\longrightarrow\infty$ as $x\longrightarrow x_0-y$ from the left. However, this is impossible, since

$$G'''(x)<F_0'''(x) \qquad \forall x\in(x^*,x_0-y)$$

according to (15). If $F''_{\Psi(x^*,a_0',a_2')}(x^*)>F_0''(x^*)$, choose $y\in\mathbb{R}^+\cap B_{x^*-(x_0-\delta)}(0)$ so that

$$F''_{\Psi(x^*-y,a_0',a_2')}(x^*-y)>F_0''(x^*).$$

The function

$$\big\{z\in B_\delta(x_0+y):\ \mathrm{Re}\,z < x_0+y\big\} \longrightarrow \mathbb{C}, \qquad G(z) = F_{\Psi(x^*-y,a_0',a_2')}(z-y),$$

is a solution of (4) satisfying

$$G(x^*) = a_0' = F_0(x^*), \qquad G'(x^*) = a_2' = F_0'(x^*)\,, \qquad G''(x^*) = F_{\Psi(x^*-y,a_0',a_2')}(x^*-y) > F_0''(x^*).$$

Furthermore, G is real on $(x_0+y-\delta, x_0+y)$ and satisfies (14) with F_0 replaced by G and x_0 by x_0+y. However, this is impossible, since $F_0'''(x)\longrightarrow\infty$ as $x\longrightarrow x_0$, while

$$G'''(x) > F_0'''(x) \qquad \forall x\in(x^*,x_0)$$

according to (15). Finally, suppose $F_0''(x^*)=F''_{\Psi(x,a_0',a_2')}(x^*)$. Since

$$9 + 2F'_{\Psi(x^*,a_0',a_2')}(x^*) - 3F''_{\Psi(x^*,a_0',a_2')}(x^*) > 0$$

by (17), the differential equation (4) with F_0 replaced by G has a unique solution satisfying

$$G(x_0) = a_0'\,, \qquad G'(x_0) = a_2'\,, \qquad G''(x_0) = F''_{\Psi(x,a_0',a_2')}(x^*).$$

Thus, $F_0=F_{\Psi(x^*,a_0',a_2')}$. By the limiting conditions on F_0 and F_0' above, $\Psi(x^*,a_0',a_2')=(a_0,a_2)$, as required.

For the purposes of Section 3, we note that

$$9 + 2F_0'(x_0+z) - 3F_0''(x_0+z) = z^{1/2}\sum_{d=0}^{\infty}\frac{d+3}{4}\big(4a_{d+3} - 3(d+5)a_{d+5}\big)z^{d/2}.$$

In particular, the coefficient of the leading term above is $-45a_5/4 \neq 0$.

6 Some thoughts on Conjecture 6

The property of Conjecture 6 appears to be independent of the exact nature of $f(d_1,d_2)$. In the given case, it looks like $d_1^2d_2^2/d^2$. What seems to matter is that the degree of the polynomial on top in each variable separately (2 in this case) is at least as large as the degree of the polynomial in d at the bottom.

For example, if

$$f(d_1,d_2) = a\frac{d_1^k d_2^k}{d^k}\,,$$

for $a\in\mathbb{R}^+$ and $k\in\mathbb{R}^{\geq 0}$, then

$$n_d = \frac{(2d-2)!}{d!(d-1)!}\frac{1}{ad^k}\big(n_1a\big)^d\,.$$

Thus, the eventually increasing property in this case is equivalent to

$$\frac{\sum\limits_{r=d+1}^{2d-2}\ln r - \sum\limits_{r=1}^{d-1}\ln r - \ln a - k\ln d}{d} < \frac{\sum\limits_{r=d+2}^{2d}\ln r - \sum\limits_{r=1}^{d}\ln r - \ln a - k\ln(d+1)}{d+1} \qquad \forall\ d\geq d^*.$$

This is equivalent to

$$d\ln(d{+}1) + \sum_{r=d+1}^{2d-2} \ln r - \sum_{r=2}^{d-1} \ln r - \ln a + k\big(d\ln(1+\frac{1}{d}) - \ln d\big) < d\ln(2d-1) + d\ln 2. \qquad (18)$$

Since $\ln x$ is an increasing function,

$$\sum_{r=d+1}^{2d-2} \ln r < \int_{d+1}^{2d-1} \ln x = (x\ln x - x)\Big|_{d+1}^{2d-1} = (2d{-}1)\ln(2d{-}1) - (d{+}1)\ln(d{+}1) - (d{-}2),$$
$$\sum_{r=2}^{d-1} \ln r > \int_{1}^{d-1} \ln x = (x\ln x - x)\Big|_{1}^{d-1} = (d{-}1)\ln(d{-}1) - (d{-}2).$$

Thus, the left-hand side of (18) is bounded by

$$\begin{aligned}&(2d{-}1)\ln(2d{-}1) - (d{-}1)\ln(d{-}1) - \ln(d{+}1) - \ln a - k\big(\ln d - 1\big)\\ &\quad \le d\ln(2d{-}1) + (d{-}1)\ln 2 + (d{-}1)\ln\Big(1 + \frac{1}{2(d{-}1)}\Big) - \ln(d{+}1) - \ln a\\ &\quad \le d\ln(2d{-}1) + (d{-}1)\ln 2 + \frac{1}{2} - \ln(d{+}1) - \ln a \le d\ln(2d{-}1) + d\ln 2 - \ln(d{+}1) - \ln a.\end{aligned}$$

For d sufficiently large, the combination of the last two terms is negative, which establishes the claim in this case.

It seems that the dependence of the property of Conjecture 6 only on the asymptotic behavior of $f(d_1, d_2)$ may be related to the following statement. Let $p(q) \in q\mathbb{R}[q]$ be a polynomial with *positive* coefficients and vanishing constant term. Define the numbers n_d by

$$\sum_{d=1}^{\infty} n_d q^d = \sum_{d=1}^{\infty} \frac{(2d{-}2)!}{d!(d{-}1)!} \frac{1}{ad^k} q^d \big(1 + p(q)\big)^d .$$

It appears that the numbers $\sqrt[d]{n_d}$ are eventually increasing. In other words, this property is invariants under the change of variables,

$$q \longrightarrow \big(1 + p(q)\big) q$$

if $p(q)$ is a polynomial with *positive* coefficients and vanishing constant term. For the asymptotic behavior conclusion, $p(q)$ would perhaps need to be a power series with coefficients declining sufficiently quickly (perhaps a convergent one?)

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