A Comparison Theorem for Gromov-Witten Invariants in the Symplectic Category

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May 19, 2009

Abstract

We exploit the geometric approach to the virtual fundamental class, due to Fukaya-Ono and Li-Tian, to compare the virtual fundamental classes of stable maps to a symplectic manifold and a symplectic submanifold whenever all constrained stable maps to the former are contained in the latter to first order. Various special cases of the comparison theorem in this paper have long been used in the algebraic category; some of them have also appeared in the symplectic setting. Combined with the inherent flexibility of the symplectic category, the main theorem leads to a confirmation of Pandharipande's Gopakumar-Vafa prediction for GW-invariants of Fano classes in 6-dimensional symplectic manifolds. The proof of the main theorem uses deformations of the Cauchy-Riemann equation that respect the submanifold and Carleman's Similarity Principle for solutions of perturbed Cauchy-Riemann equations. In a forthcoming paper, we apply a similar approach to relative Gromov-Witten invariants and the absolute/relative correspondence in genus 0.

Contents

*Partially supported by a Sloan fellowship and DMS Grant 0604874

1 Introduction

Gromov-Witten invariants are certain counts of pseudo-holomorphic curves in symplectic manifolds that play prominent roles in symplectic topology, algebraic geometry, and string theory. These are usually rational numbers, and their precise relations with some sort of integer enumerative counts of curves are rarely clear. However, it is well-known that genus 0 GW-invariants of Fano manifolds are precisely counts of rational curves; this observation is key to enumerating rational curves in projective space in [14, Section 5] and [27, Section 10]. String theory predicts an amazing integral structure for GW-invariants of Calabi-Yau threefolds. These predictions originate in [2], [6], and [7] and are extended to all threefolds in [25].

GW-invariants of a symplectic manifold X are determined by the virtual fundamental class (VFC) of the space of stable J-holomorphic maps to X. The main statement of this paper, Theorem 1.2, compares the VFC for stable maps meeting specified constraints in the ambient manifold with the VFC for stable maps to a submanifold containing the images of all such constrained maps to first order. It leads immediately to Corollary 1.3, which in a way is a succinct re-formulation of the main conclusion of [17], and with a bit more work to Theorem 1.4, which confirms the "Fano case" of the Gopakumar-Vafa prediction of [25, Section 0.2]. Theorem 1.2 is obtained by deforming the Cauchy-Riemann equation in two stages so that the first stage respects the submanifold. Carleman's Similarity Principle is used to take advantage of properties of solutions of Cauchy-Riemann equations that are preserved by a large class of perturbations of the equations. In a forthcoming paper [33], we will apply similar geometric principle to study relative GW-invariants and the absolute/relative correspondence in genus 0 with applications to birational geometry in the spirit of Hu-Li-Ruan ([8], [9], [18]) and McDuff ([21]).

The author would like to thank R. Pandharipande for bringing the "Fano case" of the Gopakumar-Vafa prediction to the author's attention, D. McDuff for detailed comments and suggestions on an earlier version of this paper, and T. Graber, T.-J. Li, D. Maulik, and Y. Ruan for related discussions.

1.1 A comparison theorem for GW-invariants

We will denote by $\bar{\mathbb{Z}}^+$ the set of non-negative integers. Let (X,ω) be a compact symplectic manifold. If $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, and J is an ω-tame[1] almost complex structure on X, denote by $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ the moduli space of equivalence classes of stable S-marked genus g degree β J-holomorphic maps to X. For each $j\in S$, there is a well-defined evaluation map

$$\mathrm{ev}_j:\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\longrightarrow X. \tag{1.1}$$

As standard in GW-theory, we will denote by

$$\psi_j\in H^2\big(\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\big)$$

the first chern class of the universal cotangent line bundle for the j-th marked point. The space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ carries a natural VFC, which is independent of J and will be denoted by $[\overline{\mathfrak{M}}_{g,S}(X,\beta)]^{vir}$; see Section 3.1. If the (real) dimension of X is $2n$, then

$$\dim\big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir}=2\big(\langle c_1(TX),\beta\rangle+(n-3)(1-g)+|S|\big). \tag{1.2}$$

[1] an almost complex structure on (X,ω) is ω-tame if $\omega(v,Jv)>0$ for all $v\in TX$ with $v\neq0$

If J is regular[2], then $\overline{\mathfrak{M}}_{0,S}(X,\beta;J)$ is a topological manifold with a preferred choice of orientation and

$$\big[\overline{\mathfrak{M}}_{0,S}(X,\beta)\big]^{vir} = \big[\overline{\mathfrak{M}}_{0,S}(X,\beta;J)\big].$$

If $a_j\in\bar{\mathbb{Z}}^+$ and $\kappa_j\in H_*(X;\mathbb{Z})$ for each $j\in S$, let

$$\big((\tau_{a_j}\kappa_j)_{j\in S}\big)_{g,\beta}^X \equiv \Big\langle \prod_{j\in S}\big(\psi_j^{a_j}\mathrm{ev}_j^*(\mathrm{PD}_X\kappa_j)\big), \big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir}\Big\rangle, \tag{1.3}$$

where $\mathrm{PD}_X\kappa_j\in H^*(X;\mathbb{Z})$ is the Poincare dual of κ_j in X.[3] In order to avoid any sign ambiguities, we define the number in (1.3) to be 0 if the dimension of κ_j is odd for some j. By (1.2), this number is zero unless

$$\sum_{j\in S}(2a_j+2n-\dim\kappa_j) = 2\big(\langle c_1(TX),\beta\rangle+(n-3)(1-g)+|S|\big). \tag{1.4}$$

The number (1.3) can be expressed as an integral on a "smaller" moduli space as follows. Choose cobordism representatives $f_j\colon M_j\longrightarrow X$ for κ_j, with $j\in S$.[4] Let

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J) = \Big\{\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,S}\big(X,\beta;J\big)\times\prod_{j\in S}M_j\colon \mathrm{ev}_j([u])=f_j(w_j)\ \forall\, j\in S\Big\}. \tag{1.5}$$

The space $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ of constrained stable maps also carries a virtual fundamental class and

$$\big((\tau_{a_j}\kappa_j)_{j\in S}\big)_{g,\beta}^X = \Big\langle \prod_{j\in S}\psi_j^{a_j}, \big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)\big]^{vir}\Big\rangle.$$

The subject of this subsection is a reduction of this GW-invariant of X to a combination of GW-invariants for its submanifolds.

Definition 1.1 *Let Y be a submanifold of X. A smooth map $f\colon M\longrightarrow X$ intersects Y properly if $f^{-1}(Y)\subset M$ is a smooth orientable even-dimensional submanifold of M and*

$$d_wf\big(T_w\big(f^{-1}(Y)\big)\big) = d_w(TM)\cap T_{f(w)}Y$$

for every $w\in f^{-1}(Y)$.

If $f: M\longrightarrow X$ intersects $Y\subset X$ transversally and M, X, and Y are orientable of even total dimension, then f intersects Y properly. However, a proper intersection need not be transverse. For example, any two real lines in $\mathbb{R}^n$ intersect properly, but not transversally if $n\geq 3$. Two curves that are tangent to each other do not intersect properly.

[2]an almost complex structure J is genus 0 regular if for every J-holomorphic map $u\colon\Sigma\longrightarrow X$, where Σ is a tree of Riemann spheres, the linearization $D_{J,u}$ of the $\bar\partial_J$-operator at u is surjective

[3]In the description of Section 3.1, $\big[\overline{\mathfrak{M}}_{g,S}(X,\beta)\big]^{vir}$ is a homology class in an arbitrarily small neighborhood of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in the space of equivalence classes of L^p_1-maps to X; there are well-defined evaluation maps ev_j and cohomology classes ψ_j on this space as well.

[4]We can assume that this is possible, since each κ_j can be replaced by a multiple for our purposes.

If $f: M \longrightarrow X$ intersects $Y \subset X$ properly and $NY \longrightarrow Y$ is the normal bundle of Y in X, the homomorphisms

$$d_w^{NY} f : T_w M \longrightarrow N_{f(w)} Y, \quad v \longrightarrow d_w f(v) + T_{f(w)} Y, \qquad w \in f^{-1}(Y),$$

have constant rank; the kernel of $d_w^{NY} f$ is $T_w(f^{-1}(Y))$. If M, X, and Y are oriented, an orientation on $f^{-1}(Y)$ then induces an orientation on the vector bundle

$$N^f Y \equiv f^* NY \big/ (\mathrm{Im}\, d^{NY} f) \longrightarrow f^{-1}(Y).$$

Let Y be a compact symplectic submanifold of X and

$$\iota_{Y*} : H_*(Y;\mathbb{Z}) \longrightarrow H_*(X;\mathbb{Z})$$

the homomorphism induced by the inclusion $\iota_Y : Y \longrightarrow X$. If $\beta_Y \in H_2(Y;\mathbb{Z})$ and J is an ω-tame almost complex structure on X which preserves $TY \subset TX|_Y$, then ι_Y induces an embedding

$$\overline{\mathfrak{M}}_{g,S}(Y, \beta_Y; J) \hookrightarrow \overline{\mathfrak{M}}_{g,S}(X, \iota_{Y*}\beta_Y; J).$$

If $f_j : M_j \longrightarrow X$, $j \in S$, are smooth maps as above, let

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y, \beta_Y; J) = \big\{ \big([u], (w_j)_{j\in S}\big) \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(X, \iota_{Y*}\beta_Y; J) \colon\ [u] \in \overline{\mathfrak{M}}_{g,S}(Y, \beta_Y; J) \big\}.$$

If in addition $u : \Sigma_u \longrightarrow Y$ is a J-holomorphic map from a nodal Riemann surface (see Section 2.1), let $\mathcal{H}_u$ denote the space of deformations of the complex structure on Σ_u. The linearization of the $\bar\partial_J$-operator for maps to X,

$$D_{J;u}^X : \mathcal{H}_u \oplus L_1^p(\Sigma_u; u^*TX) \longrightarrow L^p(\Sigma_u; T^*\Sigma_u^{0,1} \otimes_{\mathbb{C}} u^*TX), \quad p > 2,$$

induces a generalized Cauchy-Riemann operator

$$D_{J;u}^{NY} : L_1^p(\Sigma_u; u^*NY) \longrightarrow L^p(\Sigma_u; T^*\Sigma_u^{0,1} \otimes_{\mathbb{C}} u^*NY).$$

For each $j \in S$, define

$$\widetilde{\mathrm{ev}}_j : \ker D_{J;u}^{NY} \longrightarrow N_{z_j(u)} Y \qquad \text{by} \quad \xi \longrightarrow \xi(z_j(u)) + T_{z_j(u)} Y,$$

where $z_j(u) \in \Sigma_u$ is the j-th marked point; this homomorphism is the composition of the differential of the evaluation map (1.1) with the projection to the normal bundle.

Theorem 1.2 *Suppose (X,ω) is a compact symplectic manifold, $g \in \bar{\mathbb{Z}}^+$, S is a finite set, $\beta \in H_2(X;\mathbb{Z})$, $a_j \in \bar{\mathbb{Z}}^+$ for each $j \in S$, and $f_j : M_j \longrightarrow X$ is a cobordism representative for $\kappa_j \in H_*(X;\mathbb{Z})$ for each $j \in S$. If J is an ω-tame almost complex structure on X, Y is a compact almost complex submanifold of (X,J), and $\beta_Y \in H_2(Y;\mathbb{Z})$ are such that*

(a) $\iota_{Y*}(\beta_Y) = \beta$ and f_j intersects Y properly for each $j \in S$;

(b) for every $([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$*, the homomorphism*

$$\ker(D^{NY}_{J;u})\longrightarrow\bigoplus_{j\in S}N^{f_j}_{f_j(w_j)}Y,\qquad \xi\longrightarrow\big(\widetilde{\mathrm{ev}}_j(\xi)+(\mathrm{Im}\,d_{w_j}f_j)\big)_{j\in S}\,, \tag{1.6}$$

is an isomorphism,

then

(1) the space $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ *carries a natural VFC (dependent on the orientations of* $f_j^{-1}(Y)$*);*

(2) the vector spaces $\mathrm{cok}(D^{NY}_{J;u})$ *form a natural oriented vector orbi-bundle*

$$\mathrm{cok}\big(D^{NY}_J\big)\longrightarrow\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J);$$

(3) $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ *is a union of connected components of* $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ *and its contribution to the number (1.3) is given by*

$$\mathbf{C}_{g,\mathbf{f}}(Y,\beta_Y)=\Big\langle e\big(\mathrm{cok}(D^{NY}_J)\big)\prod_{j\in S}\psi_j^{a_j},\,\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}\Big\rangle. \tag{1.7}$$

Example A Suppose (X,J) is a Calabi-Yau 3-fold and $Y\subset X$ is a smooth isolated rational curve with $NY\approx\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. We can then apply Theorem 1.2 with $S=\emptyset$, $g=0$, and $\beta=d\iota_{Y*}([Y])$ for any $d\in\mathbb{Z}^+$. The assumption on the normal bundle implies that $\ker(D^{NY}_{J;u})$ is trivial and thus Condition (b) is satisfied. The right-hand side of (1.7) is then the famous multiple-cover contribution of $1/d^3$ ([2],[24, Section 27.5]).

Example B If the image of each map f_j in Theorem 1.2 lies in Y, the second part of Condition (a) is automatically satisfied. Condition (b) is equivalent to the homomorphisms

$$\bigoplus_{j\in S}\widetilde{\mathrm{ev}}_j\colon\ker(D^{NY}_{J;u})\longrightarrow\bigoplus_{j\in S}N_{z_j(u)}Y,\qquad (u,w)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J),$$

being isomorphisms. For example, this is the case if $X=\mathbb{P}^n$, $Y=\mathbb{P}^1\subset X$, $S=\{1,2\}$, $g=0$, $\beta=\iota_{Y*}([Y])$ is the homology class of a line, $a_1,a_2=0$, and $f_1,f_2\colon pt\longrightarrow Y$ are maps to two distinct points. In this particular case,

$$\overline{\mathfrak{M}}_{0,\mathbf{f}}(X,\beta;J)=\overline{\mathfrak{M}}_{0,\mathbf{f}}(Y,\beta_Y;J),$$

where $\beta_Y=[Y]$, and $\mathrm{cok}(D^{NY}_J)$ is the zero vector bundle. Thus,

$$\big(pt,pt\big)^{\mathbb{P}^n}_{0,\beta}=\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{0,\beta}=\mathbf{C}_{0,\mathbf{f}}(Y,\beta_Y)={}^{\pm}\big|\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big|=\big(pt,pt\big)^{\mathbb{P}^1}_{0,\beta_Y}=1,$$

as expected.[5]

[5] This is the number of lines through 2 points in $\mathbb{P}^n$. In this particular case, each operator $D^{NY}_{J;u}$ is complex linear and its zero-dimensional kernel is positively oriented. In general, this need not be the case; see [17, Sections 9,10] for explicit sign computations.

Example C If each map f_j in Theorem 1.2 is transverse to Y, the second part of Condition (a) is again automatically satisfied. Condition (b) is equivalent to the injectivity of the operators $D^{NY}_{J;u}$ whenever $(u,w)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$. For example, this is the case if X is the blowup of $\mathbb{P}^n$, with $n\ge 2$, at a point, $Y\approx\mathbb{P}^{n-1}$ is the exceptional divisor, $S=\{1,2\}$, $g=0$, $\beta_Y\in H_2(Y;\mathbb{Z})$ is the homology class of a line in the exceptional divisor, $\beta=\iota_{Y*}(\beta_Y)$, $a_1,a_2=0$, and $f_1,f_2\colon\mathbb{P}^1\longrightarrow X$ are parametrizations of proper transforms of two distinct lines in $\mathbb{P}^n$ passing through the center of the blowup. In this particular case,

$$\overline{\mathfrak{M}}_{0,\mathbf{f}}(X,\beta;J)=\overline{\mathfrak{M}}_{0,\mathbf{f}}(Y,\beta_Y;J)$$

and $\mathrm{cok}(D^{NY}_J)$ is the zero vector bundle. Thus, if $\bar\ell$ denotes the homology class of f_1 and f_2,

$$\big(\bar\ell,\bar\ell\big)^X_{0,\beta}=\big((\tau_{a_j}\kappa_j)_{j\in S}\big)^X_{0,\beta}=\mathbf{C}_{0,\mathbf{f}}(Y,\beta_Y)={}^{\pm}\big|\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big|=\big(\bar\ell\cap Y,\bar\ell\cap Y\big)^Y_{0,\beta_Y}=1;$$

see Footnote 5.

Various special cases of Theorem 1.2, such as those in Examples A-C, are standard in the algebraic setting and are used in [3], [13], and [25], for example. Some special cases of Theorem 1.2 have appeared in the symplectic setting as well, including in [16], [23], and [29]. Examples B and C generalize Example A in two opposite directions. Corollary 1.3 below, which applies this theorem in the setting of [17], is yet another special case of Example C. The full statement of Theorem 1.2 mixes the two extreme cases of Examples B and C.

The striking conclusion of [17] is that all GW-invariants of a Kahler surface X of general type localize to a canonical divisor. The situation is particularly beautiful if X admits a smooth canonical divisor $\mathcal{K}_X$. If X is minimal, the GW-invariants of X in degrees other than multiples of $\mathcal{K}_X$ vanish. The GW-invariants of X in degrees $\mathcal{K}_X$ and $2\mathcal{K}_X$ are computed in [12] via an algebraic reformulation of [17] and shown to satisfy a conjecture of [19]. In the next paragraph we review the relevant statements from [17].

Let (X,J_0) be a minimal Kahler surface of general type and α the real part of a non-zero holomorphic $(2,0)$-form such that $D_\alpha\equiv\alpha^{-1}(0)$ is smooth (and reduced). Since X is minimal, D_α is connected. With $\langle\cdot,\cdot\rangle$ denoting the Riemannian metric on X, define

$$\begin{gathered}K_\alpha\in\Gamma\big(X;\mathrm{Hom}_{\mathbb{R}}(TX,TX)\big),\qquad R_\alpha\in\Gamma\big(D_\alpha;\mathrm{Hom}(TD_\alpha\otimes_{\mathbb{C}}ND_\alpha,ND_\alpha)\big),\qquad\text{by}\\ \langle v_1,K_\alpha v_2\rangle=\alpha(v_1,v_2)\quad\forall\,v_1,v_2\in T_xX,\ x\in X;\\ R_\alpha(v_1,v_2)=J_0\big\{\nabla_{v_2}K_\alpha\big\}(v_1)+T_xD_\alpha\quad\forall\,v_1\in T_xD_\alpha,\ v_2\in T_xX,\ x\in X.\end{gathered}\tag{1.8}$$

By [17, Lemmas 2.1 and 8.2], R_α is well-defined. The almost complex structure J_α on X described in [17, Section 2] agrees with J_0 along the smooth complex curve D_α. By [17, Lemma 2.3], every non-constant J_α-holomorphic map $u:\Sigma_u\longrightarrow X$ is in fact a J_0-holomorphic map to D_α and so lies in the homology class dD_α for some $d\in\mathbb{Z}^+$. By [17, Section 8], the operator on the normal bundle ND_α of D_α induced by the linearization of the $\bar\partial_{J_\alpha}$-operator for maps to X at such a map u is given by

$$D^{ND_\alpha}_{J_\alpha,u}=\bar\partial_{u^*ND_\alpha}+R_\alpha(df,\cdot)\colon L^p_1(\Sigma_u;u^*ND_\alpha)\longrightarrow L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*ND_\alpha\big),\tag{1.9}$$

where $\bar\partial_{u^*ND_\alpha}$ is the $\bar\partial$-operator in the holomorphic bundle $u^*(ND_\alpha,J_0)\longrightarrow\Sigma_u$. By [17, Proposition 8.6], $D^{ND_\alpha}_{J_\alpha,u}$ is injective. Corollary 1.3 below thus follows immediately from Theorem 1.2.

Corollary 1.3 *Suppose (X,J_0) is a minimal Kahler surface of general type, α is the real part of a non-zero holomorphic $(2,0)$-form such that $D_\alpha\equiv\alpha^{-1}(0)$ is smooth, $g\in\bar{\mathbb{Z}}^+$, $d\in\mathbb{Z}^+$, S is a finite set, $S_2\subset S$, $a_j\in\bar{\mathbb{Z}}^+$ for each $j\in S$, and $\kappa_j\in H_2(X;\mathbb{Z})$ for each $j\in S_2$. If R_α is defined by (1.8), then the cokernels of the operators (1.9) form a natural oriented vector orbi-bundle*

$$\mathrm{cok}\big(D_\alpha^N\big)\longrightarrow\overline{\mathfrak{M}}_{g,S}(D_\alpha,dD_\alpha)$$

and

$$\begin{aligned}\big((\tau_{a_j}\kappa_j)_{j\in S_2},(\tau_{a_j}1)_{j\in S-S_2}\big)_{g,d\mathcal{K}_X}^X\\ =\bigg(\prod_{j\in S_2}\langle c_1(T^*X),\kappa_j\rangle\bigg)\Big\langle e\big(\mathrm{cok}(D_\alpha^N)\big)\prod_{j\in S_2}\big(\mathrm{ev}_j^*PD_{D_\alpha}(pt)\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,S}(D_\alpha,dD_\alpha)\big]^{vir}\Big\rangle.\end{aligned}$$

1.2 The Fano case of the Gopakumar-Vafa prediction

GW-invariants are generally not integers. On the other hand, at least in the case of projective 3-folds (symplectic 6-manifolds), certain combinations of them are believed to be integers. Ideally these combinations would be precisely counts of curves of fixed genus and degree and passing through appropriate constraints. A projective 3-fold X is never ideal in this sense, but one might hope that X becomes ideal if its Kahler complex structure is replaced with a generic almost complex one. We show that this is indeed the case in the "Fano" case.

If (X,ω) is a compact symplectic manifold, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, $\beta\in H_2(X;\mathbb{Z})$, and J is an ω-tame almost complex structure on X, let

$$\mathfrak{M}_{g,S}^*(X,\beta;J)\subset\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$$

denote the subspace consisting of simple maps, i.e. J-holomorphic maps $u\colon\Sigma_u\longrightarrow X$ such that Σ_u is a smooth (connected) Riemann surface and $u^{-1}(u(z))=\{z\}$ for some $z\in\Sigma_u$. The last condition implies that u does not factor through a d-fold cover $\Sigma_u\longrightarrow\Sigma$, with $d>1$; see [22, Section 2.5]. If $f_j\colon M_j\longrightarrow X$, $j\in S$, are smooth maps from compact oriented manifolds of even dimensions, let

$$\mathfrak{M}_{g,\mathbf{f}}^*(X,\beta;J)=\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)\cap\bigg(\mathfrak{M}_{g,S}^*(X,\beta;J)\times\prod_{j\in S}M_j\bigg),$$

with $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)$ defined by (1.5). If $\mathfrak{M}_{g,\mathbf{f}}^*(X,\beta;J)$ is a finite set consisting of regular pairs $([u],(w_j)_{j\in S})$, we will denote its signed cardinality by $E_{g,\beta}^X(J,\mathbf{f})$.

If the (real) dimension of X is 6, the expected dimension of the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ is independent of the genus g; see (1.2). Thus, one can mix curve counts of different genera passing through the same constraints. Furthermore, if $\beta\in H_2(X;\mathbb{Z})$ and $\langle c_1(TX),\beta\rangle<0$, all degree β GW-invariants are zero, since the moduli space of unmarked maps has negative dimension. This leaves the "Calabi-Yau" case, $\langle c_1(TX),\beta\rangle=0$, and the "Fano" case, $\langle c_1(TX),\beta\rangle>0$. If $g,h\in\bar{\mathbb{Z}}^+$, define $C_{h,\beta}^X(g)\in\mathbb{Q}$ by

$$\sum_{g=0}^{\infty}C_{h,\beta}^X(g)t^{2g}=\bigg(\frac{\sin(t/2)}{t/2}\bigg)^{2h-2+\langle c_1(TX),\beta\rangle}. \qquad (1.10)$$

Theorem 1.4 *Suppose (X,ω) is a compact symplectic 6-fold, $\beta\in H_2(X;\mathbb{Z})$, $g\in\bar{\mathbb{Z}}^+$, S is a finite set, and $\kappa_j\in H_*(X;\mathbb{Z})$ for $j\in S$ are such that (1.4) is satisfied with $a_j=0$. If $\langle c_1(TX),\beta\rangle>0$,*

(1) there exists a dense open subset $\mathcal{J}_{\rm reg}(g,\beta)$ of the space of smooth ω-tame almost complex structures on X such that for all $h\le g$:

- *the moduli space $\mathfrak{M}^*_{h,S}(X,\beta;J)$ consists of regular maps;*
- *for a generic choice of pseudocycle representatives[6] $f_j: M_j\longrightarrow X$ for κ_j, $\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J)$ is a finite set of regular pairs $([u],(w_j)_{j\in S})$ such that u is an embedding;*

(2) the numbers $E^X_{h,\beta}(\mathbf{f},J)$, with $h\le g$, are independent of the choice of $J\in\mathcal{J}_{\rm reg}(g,\beta)$ and f_j and can thus be denoted $E^X_{h,\beta}((\kappa_j)_{j\in S})$;

(3) if $C^X_{g,\beta}(h)$ is defined by (1.10),

$$\big((\kappa_j)_{j\in S}\big)^X_{g,\beta}=\sum_{h=0}^{h=g}C^X_{h,\beta}(g-h)E^X_{h,\beta}\big((\kappa_j)_{j\in S}\big). \tag{1.11}$$

For $g=0,1$, (1.11) gives

$$\begin{aligned}\big((\kappa_j)_{j\in S}\big)^X_{0,\beta}&=E^X_{0,\beta}\big((\kappa_j)_{j\in S}\big),\\ \big((\kappa_j)_{j\in S}\big)^X_{1,\beta}&=E^X_{1,\beta}\big((\kappa_j)_{j\in S}\big)+\frac{2-\langle c_1(TX),\beta\rangle}{24}E^X_{0,\beta}\big((\kappa_j)_{j\in S}\big).\end{aligned} \tag{1.12}$$

The first identity expresses the well-known fact that the genus 0 GW-invariants of a Fano manifold are enumerative. The second identity in (1.12) is the $n=3$ case of the relation between the standard genus 1 GW-invariants and the reduced genus 1 GW-invariants constructed in [32] for all symplectic manifolds.

By the proof of [22, Theorem 3.1.5], for a generic almost complex structure J on X all moduli spaces $\mathfrak{M}^*_{h,\emptyset}(X,\beta';J)$ are smooth and of the expected dimension, $\langle c_1(TX),\beta'\rangle$. In particular,

$$\langle c_1(TX),\beta'\rangle<0 \qquad\Longrightarrow\qquad \mathfrak{M}^*_{h,S}(X,\beta';J),\overline{\mathfrak{M}}_{h,S}(X,\beta';J)=\emptyset. \tag{1.13}$$

By a similar argument, for a generic J on X the evaluation maps

$$\mathrm{ev}_1,\mathrm{ev}_2\colon\mathfrak{M}^*_{g,\{1,2\}}(X,\beta;J)\longrightarrow X$$

are transverse, while the bundle section

$$\mathfrak{M}^*_{g,\{1\}}(X,\beta;J)\longrightarrow L_1^*\otimes\mathrm{ev}_1^*TX, \qquad [u]\longrightarrow du|_{z_1(u)}\,,$$

where $L_1\longrightarrow\mathfrak{M}^*_{g,\{1\}}(X,\beta;J)$ is the universal tangent line bundle at the marked point and $z_1(u)\in\Sigma_u$ is the marked point of u, is transverse to the zero set. Thus,

$$\mathfrak{M}^{sing}_{g,S}(X,\beta;J)\equiv\big\{[u]\in\mathfrak{M}^*_{g,S}(X,\beta;J)\colon\ u \text{ is not an embedding}\big\}$$

[6] After replacing κ_j by a multiple, M_j can be taken to be a smooth compact manifold.

is the image of a smooth map from a smooth manifold of (real) dimension two less than the dimension of $\mathfrak{M}^*_{g,S}(X,\beta;J)$. Thus, for a generic choice of pseudocycle representatives $f_j\colon M_j \longrightarrow X$ for κ_j, $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a 0-dimensional oriented sub-manifold of

$$\big(\mathfrak{M}^*_{g,S}(X,\beta;J)-\mathfrak{M}^{sing}_{g,S}(X,\beta;J)\big)\times\prod_{j\in S}M_j.$$

We next show that $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a finite set. If not, there is a sequence $([u_r],(w_{r,j})_{j\in S})$ in $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ converging to some

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J).$$

The image of u is a connected J-holomorphic curve in X of genus $h\le g$ with $k\ge 1$ irreducible components of degrees $\beta_1,\ldots,\beta_k\in H_2(X;\mathbb{Z})$ such that

$$d_1\beta_1+\ldots+d_k\beta_k=\beta\qquad\text{for some}\quad d_1,\ldots,d_k\in\mathbb{Z}^+.$$

By (1.13), $\langle c_1(TX),\beta_i\rangle\ge 0$ for all $i=1,\ldots,k$. Thus,

$$\sum_{i=1}^{i=k}\langle c_1(TX),\beta_i\rangle\le\langle c_1(TX),\beta\rangle.$$

The dimension-counting argument of [22, Section 6.6] then shows that $k=1$ and $d_1=1$. It then follows that the image of u is an irreducible J-holomorphic curve of degree β and genus $h<g$ that meets each of the maps f_j with $j\in S$.

While degree β genus $h<g$ J-holomorphic curves meeting the maps f_j can certainly exist for a generic J, they cannot be limits of other degree β curves meeting the maps f_j by the $\nu_r=0$ case of Proposition 3.2 for the following reason. If

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J),$$

the domain of u consists of two or more irreducible components. Furthermore, by the previous paragraph, the restriction of u to all components, except for one, is constant; let u_{eff} denote the effective part of u, i.e. the non-constant restriction. The domain $\Sigma_{u_{\text{eff}}}$ of u_{eff} is a smooth curve of genus $h<g$ with distinct points $(z_j(u_{\text{eff}}))_{j\in S}$ that are mapped to $(\text{ev}_j(u))_{j\in S}$ by u_{eff}. Thus,

$$\big([u_{\text{eff}}],(w_j)_{j\in S}\big)\in\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J);$$

by the previous paragraph, u_{eff} is an embedding onto a smooth J-holomorphic curve Y of genus h degree β meeting the maps f_j. Since the total evaluation map

$$\mathbf{ev}\equiv\prod_{j\in S}\text{ev}_j:\mathfrak{M}^*_{h,\mathbf{f}}(X,\beta;J)\longrightarrow X^S$$

is transverse to $\mathbf{f}$,

$$\ker(D^{NY}_{J;u_{\text{eff}}})\longrightarrow\bigoplus_{j\in S}N^{f_j}_{f_j(w_j)}Y,\qquad\xi\longrightarrow\big(\xi\big(z_j(u_{\text{eff}})\big)+T_{f_j(w_j)}Y+(\text{Im}\,d_{w_j}f_j)\big)_{j\in S},\qquad(1.14)$$

is surjective; see Section 1.1 for the notation. Since $u_{\rm eff}$ is a regular map,

$$\dim\ker(D^{NY}_{J;u_{\rm eff}})=\operatorname{ind}\big(D^{NY}_{J;u_{\rm eff}}\big)=2\big(\langle c_1(NY),Y\rangle+2(1-h)\big)=2\langle c_1(TX),\beta\rangle$$
$$=\sum_{j\in S}(4-\dim\,M_j)\le\sum_{j\in S}\dim\,N^{f_j}_{f_j(w)}Y;$$

the second-to-last equality holds by (1.4). Thus, the homomorphism in (1.14) is an isomorphism. On the other hand, $D^{NY}_{J;u}$ is the restriction of the operator $\bigoplus_i D^{NY}_{J;u_i}$ to

$$L^p_1(\Sigma_u;u^*NY)\subset\bigoplus_i L^p_1\big(\Sigma_{u;i};u_i^*NY\big),$$

where $\{\Sigma_i\}$ are the irreducible components of Σ and $u_i=u|_{\Sigma_{u;i}}$. If u_i is a constant map, then $D^{NY}_{J;u_i}$ is the usual $\bar\partial$-operator on the space of functions on Σ_{u_i} with values in $N_{u_i(\Sigma_i)}Y\approx\mathbb{C}^2$. Since Σ_u is a connected nodal Riemann containing $\Sigma_{u_{\rm eff}}$ as a component, $u|_{\Sigma_{\rm eff}}=u_{\rm eff}$, and u is constant on each of the irreducible components of $\Sigma_u-\Sigma_{u_{\rm eff}}$, it follows that the projection homomorphism

$$\ker D^{NY}_{J;u}\longrightarrow\ker D^{NY}_{J;u_{\rm eff}},\qquad \xi\longrightarrow\xi|_{\Sigma_{u_{\rm eff}}},$$

is an isomorphism. Thus, the homomorphism

$$\ker(D^{NY}_{J;u})\longrightarrow\bigoplus_{j\in S}N^{f_j}_{f_j(w_j)}Y,\qquad \xi\longrightarrow\big(\xi\big(z_j(u)\big)+T_{f_j(w_j)}Y+(\operatorname{Im}d_{w_j}f_j)\big)_{j\in S},$$

is an isomorphism, since the homomorphism (1.14) is. Therefore, by Proposition 3.2 there is no sequence in

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)-\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\supset\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$$

converging to $([u],(w_j)_{j\in S})$.

We have thus shown that $\mathfrak{M}^*_{g,\mathbf{f}}(X,\beta;J)$ is a compact oriented 0-dimensional manifold and its signed cardinality $E^X_{g,\beta}(\mathbf{f},J)$ is well-defined. The independence of $E^X_{g,\beta}(\mathbf{f},J)$ of the choices of J and f_j follows from (1.11), with $E^X_{h,\beta}((\kappa_j)_{j\in S})$ replaced by $E^X_{h,\beta}(\mathbf{f},J)$. In turn, this identity follows from Theorem 1.2 and the proof of [25, Theorem 3]. Let Y be a degree β J-holomorphic curve of genus $h\le g$ meeting each f_j. By the above, the assumptions of Theorem 1.2 are satisfied. By definition (see Section 2.4), the orbi-bundle $\operatorname{cok}(D^{NY}_J)$ is dual to the bundle $\ker((D^{NY}_J)^*)$ of kernels of the dual operators $(D^{NY}_J)^*$. For each

$$\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\subset\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J),$$

the operator $(D^{NY}_{J;u})^*$ is the natural extension of the operator $\bigoplus_i(D^{NY}_{J;u_i})^*$ to $(1,0)$-forms on Σ_u with poles at the nodes such that the residues at each node sum up to 0. Since $(D^{NY}_{J;u_{\rm eff}})^*$ is injective by the regularity of $u_{\rm eff}$, the projection

$$\eta\longrightarrow\bigoplus_{\Sigma_{u;i}\neq\Sigma_{u_{\rm eff}}}\eta|_{\Sigma_{u;i}}$$

to the contracted components is injective. Since $(D^{NY}_{J;u_i})^*=\bar\partial^*$ if u_i is constant, the image of this homomorphism is determined by Σ_u and is independent of $D^{NY}_{J;u_{\rm eff}}$ (as long as $D^{NY}_{J;u_{\rm eff}}$ is surjective).

Thus, $\mathrm{cok}(D_J^{NY})$ is isomorphic to the restriction to $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)$ of the obstruction bundle in [25, Section 3], i.e. the bundle of cokernels of the operators $D_{J;u}^{NY}$ as above, but for a holomorphic vector bundle NY. Thus,

$$\begin{aligned}\mathbf{C}_{g,\mathbf{f}}(Y,\beta_Y) &= \Big\langle e\big(\mathrm{cok}(D_J^{NY})\big)\prod_{j\in S}\psi_j^{a_j},\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,[Y];J)\big]^{vir}\Big\rangle \\ &= C_{h,\beta}^X(g-h)\,\mathrm{sgn}\big([u_{\mathrm{eff}}],(w_j)_{j\in S}\big)\end{aligned} \tag{1.15}$$

by (1.7) and [25, Theorem 3]. Since

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J) = \bigsqcup_{h=0}^{h=g}\ \bigsqcup_{([u],(w_j)_{j\in S})\in\mathfrak{M}_{h,\mathbf{f}}^*(X,\beta;J)} \overline{\mathfrak{M}}_{g,\mathbf{f}}(\mathrm{Im}\,u,[\mathrm{Im}\,u];J),$$

the identity (1.11) follows from (1.15).

Theorem 1.4 confirms (in strong form) the Fano case of [26, Conjecture 2(i)], i.e. that the numbers $E_{h,\beta}^X((\kappa_j)_{j\in S})$ *defined from GW-invariants by (1.11)* are integers. The Calabi-Yau case is fundamentally more difficult as it involves multiple covers of curves.[7] On the other hand, it might be possible to approach [26, Conjecture 2(ii)], i.e. that $E_{h,\beta}^X((\kappa_j)_{j\in S})=0$ for a fixed β and all sufficiently large g if X is projective, by studying possible limits of J_t-holomorphic curves with $J_t\in\mathcal{J}_{\mathrm{reg}}(g,\beta)$ as J_t approaches the standard complex structure on X and $\mathbb{P}^n$ and using the Castelnuovo bound [1, p116].

An algebro-geometric approach to Theorem 1.4 has recently been proposed in [11], at least in the usual, more narrow, meaning of *Fano* in algebraic geometry. The stable-map style invariants of smooth projective varieties defined in [11] are a priori integers in the case of Fano varieties, just like the numbers $E_{h,\beta}^X((\kappa_j)_{j\in S})$. In addition, in this Fano case, they are non-negative integers and satisfy the vanishing prediction of [26, Conjecture 2(ii)]. However, it remains to be shown that they are related to the GW-invariants in the required way, i.e. as in (1.11).

2 Analytic Preliminaries

In this section, we collect a number of background statements concerning solutions of perturbed Cauchy-Riemann equations. For the rest of the paper, fix a real number $p>2$. If Σ is a 2-dimensional manifold, this condition implies that any L_1^p-map $\Sigma\longrightarrow\mathbb{R}$ is continuous and in particular has a well-defined value at each point.

2.1 Nodal Riemann surfaces

Let $(E,\mathfrak{i})\longrightarrow\Sigma$ be an L_1^p-complex vector bundle over a smooth Riemann surface, i.e. a one-dimensional complex manifold. If $z\in\Sigma$ and

$$A_z\in\mathrm{Hom}_{\mathbb{R}}(E_z,T_z^*\Sigma^{0,1}\otimes_{\mathbb{C}}E_z),$$

[7] Theorem 1.4 and its proof also apply to the cases when $\langle c_1(TX),\beta\rangle=0$, but β is not a non-trivial integer multiple of another element of $H_2(X;\mathbb{Z})$.

we define

$$A_z^* \in \mathrm{Hom}_{\mathbb{R}}(T_z^*\Sigma^{1,0}\otimes_{\mathbb{C}}E_z^*, T_z^*\Sigma^{1,1}\otimes_{\mathbb{C}}E_z^*) \qquad \text{by}$$
$$\mathrm{Re}\big(v\wedge(A_z^*w)\big)=\mathrm{Re}\big((A_zv)\wedge w\big)\in\Lambda^2_{\mathbb{R}}(T_z^*\Sigma) \qquad \forall\, v\in E_z,\ w\in T_z^*\Sigma^{1,0}\otimes_{\mathbb{C}}E_z^*\,.$$

Since $\Lambda^2_{\mathbb{R}}(T_z^*\Sigma)$ is one-dimensional, A_z^* is well-defined. If

$$A\in L^p\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(E,T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\big),$$

this construction gives rise to an element

$$A^*\in L^p\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E^*,T^*\Sigma^{1,1}\otimes_{\mathbb{C}}E^*)\big) \qquad \text{s.t.}$$
$$\langle\!\langle\xi,A^*\eta\rangle\!\rangle\equiv\mathrm{Re}\Big(\int_\Sigma\xi\wedge(A^*\eta)\Big)=\mathrm{Re}\Big(\int_\Sigma(A\xi)\wedge\eta\Big)\equiv\langle\!\langle A\xi,\eta\rangle\!\rangle \tag{2.1}$$

for all $\xi\in L^p_1(\Sigma;E)$ and $\eta\in L^p_1(\Sigma;T^*\Sigma^{1,0}\otimes E^*)$.

Let $E\longrightarrow\Sigma$ be as above. If S is a finite subset of Σ, denote by

$$L^p_k\big(\Sigma;E(S)\big)\subset L^p_{k,loc}(\Sigma-S;E)$$

the subspace of sections η of E such that for every $z_0\in S$ there exist a neighborhood U of z_0 in Σ and a coordinate $w\colon U\longrightarrow\mathbb{C}$ such that

$$w(z_0)=0 \qquad \text{and} \qquad w\cdot\eta|_U\in L^p_k(U;E).$$

If $k\ge1$, an element η of $L^p_k(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E(S))$ has a well-defined residue at $z_0\in S$ given by

$$\mathrm{Res}_{z=z_0}\eta=\xi(z_0)\in E_{z_0} \qquad \text{if} \qquad \eta(z)=\frac{dw}{w(z)}\otimes\xi(z)\ \ \forall\ z\in U,\ \xi\in L^p_1(U;E).^8$$

If ϱ is a function assigning to each element $z_0\in S$ a real subspace $E'_{z_0}\subset E_{z_0}$, let

$$L^p_1\big(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E(\varrho)\big)=\big\{\eta\in L^p_1\big(\Sigma;E(S)\big)\colon\ \mathrm{Res}_{z=z_0}\eta\in E'_{z_0}\ \forall\,z_0\in S\big\}.$$

By a Riemann surface Σ we will mean a compact complex one-dimensional manifold with pairs of distinct points identified. In other words,

$$\Sigma=\widetilde{\Sigma}/\sim, \qquad \text{where} \quad x_i^{(1)}\sim x_i^{(2)} \quad i=1,\ldots,m, \tag{2.2}$$

for some smooth compact Riemann surface $\widetilde{\Sigma}$ and distinct points $x_i^{(1)},x_i^{(2)}\in\widetilde{\Sigma}$. The quotient map

$$\sigma\colon\widetilde{\Sigma}\longrightarrow\Sigma$$

[8] If $\eta\in L^p_k(\Sigma;T^*\Sigma^{1,0}\otimes_{\mathbb{C}}E(S-z_0))$, then $\mathrm{Res}_{z=z_0}\eta=0$. The converse is not true; for example, the residue of $\eta=\bar{z}\,dz/z$ is zero at $z=0$, but η is not even continuous at $z=0$. On the other hand, the converse is true if η lies in the kernel of a generalized CR-operator as in Subsection 2.2.

will be called the normalization of Σ; it is well-defined up to an isomorphism. We will denote by

$$\Sigma_{\rm sing}\equiv\big\{\sigma(x_i^{(1)}):\ i=1,\ldots,m\big\}\subset\Sigma \qquad\text{and}\qquad \widetilde\Sigma_{\rm sing}\equiv\big\{x_i^{(1)},x_i^{(2)}:\ i=1,\ldots,m\big\}\subset\widetilde\Sigma$$

the subset of singular points of Σ and its preimage under σ, respectively. Let $\Sigma^*\subset\Sigma$ be the subspace of smooth points, i.e. the complement of $\Sigma_{\rm sing}$.

If Y is a smooth manifold and Σ is a Riemann surface as above, an L^p_1-map $u:\Sigma\longrightarrow Y$ is an L^p_1-map

$$\widetilde u\colon\widetilde\Sigma\longrightarrow Y \qquad \text{s.t.}\quad \widetilde u\big(x_i^{(1)}\big)=\widetilde u\big(x_i^{(2)}\big)\ \ \forall\, i=1,\ldots,m.$$

By a vector bundle $E\longrightarrow\Sigma$, we will mean a topological complex vector bundle such that $\sigma^*E\longrightarrow\widetilde\Sigma$ is an L^p_1-complex vector bundle. Let

$$\begin{aligned} L^p_1(\Sigma;E)&=\big\{\xi\in L^p_1(\widetilde\Sigma;\sigma^*E):\ \xi(x_i^{(1)})=\xi(x_i^{(2)})\ \forall\, i=1,\ldots,m\big\};\\ L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}E\big)&=L^p\big(\widetilde\Sigma;T^*\widetilde\Sigma^{0,1}\otimes_{\mathbb C}\sigma^*E\big). \end{aligned}$$

If S is a finite subset of Σ^*, let $\widetilde S=\sigma^{-1}(S)$ and define

$$\begin{aligned} L^p_1\big(\Sigma;\mathcal K_\Sigma\otimes_{\mathbb C}E(S)\big)&=\Big\{\eta\in L^p_1\big(\widetilde\Sigma;T^*\widetilde\Sigma^{1,0}\otimes_{\mathbb C}\sigma^*E(\widetilde S\cup\widetilde\Sigma_{\rm sing})\big):\\ &\qquad\qquad \sum_{\widetilde z_0\in\sigma^{-1}(z_0)}\text{Res}_{z=\widetilde z_0}\eta(\widetilde z_0)=0\ \ \forall\, z_0\in\Sigma_{\rm sing}\Big\}, \qquad (2.3)\\ L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}\mathcal K_\Sigma\otimes_{\mathbb C}E(S)\big)&=L^p\big(\widetilde\Sigma;T^*\widetilde\Sigma^{0,1}\otimes_{\mathbb C}T^*\widetilde\Sigma^{1,0}\otimes_{\mathbb C}\sigma^*E(\widetilde S\cup\widetilde\Sigma_{\rm sing})\big). \end{aligned}$$

If ϱ is a function assigning to each element $z_0\in S$ a real subspace $E'_{z_0}\subset E_{z_0}$, let

$$L^p_1\big(\Sigma;\mathcal K_\Sigma\otimes_{\mathbb C}E(\varrho)\big)=\big\{\eta\in L^p_1\big(\Sigma;\mathcal K_\Sigma\otimes_{\mathbb C}E(S)\big):\ \text{Res}_{z=\sigma^{-1}(z_0)}\eta\in E'_{z_0}\ \forall\, z_0\in S\big\}. \qquad (2.4)$$

Similarly, we define

$$\begin{aligned} L^p_1\big(\Sigma;E(-S)\big)&=\big\{\xi\in L^p_1\big(\Sigma;E\big):\ \xi(z_0)=0\ \forall\, z_0\in S\big\},\\ L^p_1\big(\Sigma;E^*(-\varrho)\big)&=\big\{\xi\in L^p_1\big(\Sigma;E^*\big):\ \xi(z_0)\in\text{Ann}(E'_{z_0})\ \forall\, z_0\in S\big\}, \end{aligned}$$

where $\text{Ann}(E'_{z_0})\subset\text{Hom}_{\mathbb R}(E_{z_0},\mathbb R)$ is the annihilator of $E'_{z_0}\subset E_{z_0}$. The real pairings in (2.1) extend to pairings

$$\begin{aligned} &L^p_1\big(\Sigma;E\big)\otimes L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}\mathcal K_\Sigma\otimes_{\mathbb C}E^*(S)\big)\longrightarrow\mathbb R,\\ &L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb C}E\big)\otimes L^p_1\big(\Sigma;\mathcal K_\Sigma\otimes_{\mathbb C}E^*(S)\big)\longrightarrow\mathbb R. \end{aligned}$$

Furthermore, the equality in (2.1) holds for all $\eta\in L^p_1\big(\Sigma;\mathcal K_\Sigma\otimes_{\mathbb C}E^*(S)\big)$.

2.2 Generalized Cauchy-Riemann operators

Definition 2.1 *Let (Y,J) be an almost complex manifold and $(N,\mathfrak i)\longrightarrow(Y,J)$ a smooth vector bundle.*

(1) A $\bar\partial$-operator on $(N,\mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar\partial\colon \Gamma(Y;N)\longrightarrow \Gamma^{0,1}(Y;N)\equiv \Gamma\big(Y;T^*Y^{0,1}\otimes_{\mathbb{C}}N\big)$$

such that

$$\bar\partial\big(f\xi\big)=(\bar\partial f)\otimes\xi+f(\bar\partial\xi)\qquad \forall\ f\in C^\infty(Y),\ \xi\in\Gamma(Y;N).$$

(2) A smooth generalized Cauchy-Riemann operator (or smooth CR-operator) on $(N,\mathfrak{i})$ is a differential operator of the form

$$D=\bar\partial+A\colon \Gamma(Y;N)\longrightarrow \Gamma^{0,1}(Y;N), \tag{2.5}$$

where $\bar\partial$ is a $\bar\partial$-operator on $(N,\mathfrak{i})$ and

$$A\in\Gamma\big(Y;\mathrm{Hom}_{\mathbb{R}}(N,T^*Y^{0,1}\otimes_{\mathbb{C}}N)\big).$$

If ∇ is an affine connection in $(N,\mathfrak{i})$, the operator

$$\Gamma(Y;N)\longrightarrow\Gamma^{0,1}(Y;N),\qquad \xi\longrightarrow\frac{1}{2}\big(\nabla\xi+\mathfrak{i}\nabla\xi\circ J\big), \tag{2.6}$$

is a $\bar\partial$-operator on $(N,\mathfrak{i})$. Furthermore, any $\mathbb{C}$-linear CR-operator on $(N,\mathfrak{i})$ is a $\bar\partial$-operator, and any $\bar\partial$-operator on $(N,\mathfrak{i})$ is of the form (2.6) for some (not unique) connection ∇ in $(N,\mathfrak{i})$. In particular, A in the decomposition (2.5) can be assumed to be $\mathbb{C}$-anti-linear.

Let ∇^J be the J-linear connection in TY obtained from a Levi-Civita connection ∇ on Y and $A_Y(\cdot,\cdot)$ the Nijenhuis tensor of J:

$$\begin{aligned}\nabla^J_{\xi_1}\xi_2&=\frac{1}{2}\Big(\nabla_{\xi_1}\xi_2-J\nabla_{\xi_1}(J\xi_2)\Big)\\ A_Y(\xi_1,\xi_2)&=\frac{1}{4}\Big([\xi_1,\xi_2]+J[\xi_1,J\xi_2]+J[J\xi_1,\xi_2]-[J\xi_1,J\xi_2]\Big)\end{aligned}\qquad \forall\,\xi_1,\xi_2\in\Gamma(Y;TY).$$

We identify A_Y with the element

$$A_Y\in\Gamma\big(Y;\mathrm{Hom}_{\mathbb{R}}(TY,T^*Y^{0,1}\otimes_{\mathbb{C}}TY)\big),\qquad v\longrightarrow A_Y(\cdot,v).$$

Then,

$$\bar\partial_Y\equiv\frac{1}{2}\Big(\nabla^J\xi+J\nabla^J\circ j\Big),\ D_Y\equiv\bar\partial_Y+A_Y:\Gamma(Y;TY)\longrightarrow\Gamma^{0,1}(Y;TY)$$

are a $\bar\partial$-operator on TY and a smooth CR-operator on N, respectively.

Definition 2.2 *Let $(E,\mathfrak{i})$ be an L^p_1 complex vector bundle over a Riemann surface $(\Sigma,\mathfrak{j})$.*

(1) A $\bar\partial$-operator on $(E,\mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar\partial\colon L^p_1(\Sigma;E)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big)$$

such that

$$\bar\partial\big(f\xi\big)=(\bar\partial f)\otimes\xi+f(\bar\partial\xi)\qquad \forall\ f\in C^\infty(\Sigma),\ \xi\in\Gamma(\Sigma;E).$$

(2) *A generalized Cauchy-Riemann operator (or CR-operator) on* $(E,\mathfrak{i})$ *is a differential operator of the form*

$$D=\bar{\partial}+A\colon L^p_1(\Sigma;E)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E\big), \tag{2.7}$$

where $\bar{\partial}$ *is a* $\bar{\partial}$*-operator on* $(E,\mathfrak{i})$ *and*

$$A\in L^p\big(\Sigma;\mathrm{Hom}_{\mathbb{R}}(E,T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E)\big). \tag{2.8}$$

If ∇ is an affine connection in $(E,\mathfrak{i})$, the operator

$$L^p_1(\Sigma;E)\longrightarrow L^p(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}E),\qquad \xi\longrightarrow\frac{1}{2}\big(\nabla\xi+\mathfrak{i}\nabla\xi\circ\mathfrak{j}\big), \tag{2.9}$$

is the usual $\bar{\partial}$-operator for a unique holomorphic structure in $(E,\mathfrak{i})$. Furthermore, any $\mathbb{C}$-linear CR-operator is of the form (2.9).

If Σ and $N\longrightarrow Y$ are as above, an L^p_1-map $u\colon\Sigma\longrightarrow Y$ pulls back a smooth CR-operator D on N to a CR-operator D_u on $u^*N\longrightarrow\Sigma$ as follows. Suppose D is presented as in (2.5) with $\mathbb{C}$-anti-linear A and ∇ is a connection in $(N,\mathfrak{i})$ inducing the corresponding $\bar{\partial}$-operator. Let $\widetilde{u}\colon\widetilde{\Sigma}\longrightarrow Y$ be the map corresponding to u as in Subsection 2.1 and

$$\widetilde{\nabla}\colon\Gamma(\widetilde{\Sigma};\widetilde{u}^*N)\longrightarrow\Gamma(\widetilde{\Sigma};T^*\widetilde{\Sigma}\otimes_{\mathbb{R}}\widetilde{u}^*N)$$

the connection induced by ∇. Then,

$$D_{\widetilde{u}}=\frac{1}{2}\Big(\widetilde{\nabla}+\mathfrak{i}\circ\widetilde{\nabla}\circ\mathfrak{j}\Big)+A\circ\partial_J\widetilde{u},\qquad\text{where}\quad\partial_J\widetilde{u}=\frac{1}{2}\Big(du-J\circ d\widetilde{u}\circ\mathfrak{j}\Big),$$

is a generalized CR-operator on $\widetilde{u}^*(N,\mathfrak{i})$; $D_{\widetilde{u}}$ is independent of the choice of ∇ if u is $(J,\mathfrak{j})$-holomorphic.

Suppose (Y,J) is an almost complex manifold and D_Y is as above. If $(\Sigma,\mathfrak{j})$ is a Riemann surface and $u:\Sigma\longrightarrow Y$ is a $(J,\mathfrak{j})$-holomorphic L^p_1-map, then $D_u\equiv u^*D_Y$ is the linearization of the $\bar{\partial}_J$-operator on the space of L^p_1-maps from Σ, with complex structure fixed, to Y; see [22, Section 3.1]. If in addition, (Y,J) is an almost complex submanifold of an almost complex manifold (X,J), then

$$D_u\equiv D^Y_u\equiv u^*D_Y: L^p_1(\Sigma;u^*TY)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TY\big)$$

is the restriction of

$$D^X_u\equiv u^*D_X: L^p_1(\Sigma;u^*TX)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

Thus, D^X_u induces a CR-operator

$$D^{NY}_u: L^p_1(\Sigma;u^*NY)\longrightarrow L^p\big(\Sigma;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}u^*NY\big),$$

where $NY\equiv TX|_Y/TY$ is the complex normal bundle of Y in X. Moreover, there is a long exact sequence

$$\begin{aligned}0\longrightarrow\ker D^Y_u\longrightarrow\ker D^X_u\longrightarrow\ker D^{NY}_u&\\ \longrightarrow\mathrm{cok}\, D^Y_u\longrightarrow\mathrm{cok}\, D^X_u\longrightarrow\mathrm{cok}\, D^{NY}_u\longrightarrow 0\,.&\end{aligned}$$

The next lemma extends Serre duality from $\bar\partial$-operators to CR-operators. If D is as in (2.7), let

$$D^* = \bar\partial - A^* \colon L^p_1(\Sigma; \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E^*) \longrightarrow L^p(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E^*);$$

see (2.1) and (2.3) for notation. If $S \subset \Sigma$ is a finite subset of smooth points of Σ and ϱ is a function assigning to $z_0 \in S$ a complex subspace of $E^*_{z_0}$, D^* extends to an operator

$$D^*_\varrho \colon L^p_1\big(\Sigma; \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E^*(\varrho)\big) \longrightarrow L^p\big(\Sigma; T^*\Sigma^{0,1} \otimes_{\mathbb{C}} \mathcal{K}_\Sigma \otimes_{\mathbb{C}} E^*(S)\big);$$

see (2.4). Let D_ϱ be the restriction of D to the closed subspace $L^p_1(\Sigma; E(-\varrho))$ of $L^p_1(\Sigma; E)$.

Lemma 2.3 *Let D be a CR-operator on a complex vector bundle $(E, \mathfrak{i})$ over a Riemann surface $(\Sigma, \mathfrak{j})$. If S is a finite subset of smooth points of Σ and ϱ is a function assigning to $z_0 \in S$ a real subspace of $E^*_{z_0}$, the homomorphism*

$$\operatorname{cok} D_\varrho \longrightarrow \operatorname{Hom}_{\mathbb{R}}(\ker D^*_\varrho, \mathbb{R}), \qquad \eta \longrightarrow \langle\!\langle \eta, \cdot \rangle\!\rangle, \tag{2.10}$$

is an isomorphism.

Proof: If Σ is smooth and $S = \emptyset$, this is [10, Lemma 2.3.2]. Furthermore, by the twisting construction of [28, Lemma 2.4.1][9], the elements z_0 of S for which $\varrho(z_0) = E^*_{z_0}$ can be omitted from S. In the general case, the proof of [10, Lemma 2.3.2] shows that the homomorphisms

$$\ker D^*_\varrho \longrightarrow \operatorname{Hom}_{\mathbb{R}}(\operatorname{cok} D_\varrho, \mathbb{R}), \qquad \ker D_\varrho \longrightarrow \operatorname{Hom}_{\mathbb{R}}(\operatorname{cok} D^*_\varrho, \mathbb{R}), \tag{2.11}$$

induced by the pairing (2.1) are well-defined and injective. On the other hand, if $\widetilde{D}_\varrho$ and $\widetilde{D}^*_\varrho$ are the operators corresponding to D_ϱ and D^*_ϱ over the normalization $\sigma \colon \widetilde{\Sigma} \longrightarrow \Sigma$, dropping any matching conditions at the nodes and the other restricting conditions at the points of S, then

$$\begin{aligned} \operatorname{ind} D_\varrho &= \operatorname{ind} \widetilde{D}_\varrho - 2k \cdot m - \|\varrho\|, \\ \operatorname{ind} D^*_\varrho &= \operatorname{ind} \widetilde{D}^*_\varrho - 2k \cdot m - 2k \cdot |S| + \|\varrho\|, \end{aligned}$$

where k is the complex rank of E, m is the number of nodes in Σ, and

$$\|\varrho\| = \sum_{z_0 \in S} \dim_{\mathbb{R}} \varrho(z_0).$$

Since the kernel and cokernel of $\widetilde{D}^*_\varrho$ are isomorphic to the kernel and cokernel of a CR-operator on $T^*\widetilde{\Sigma} \otimes \sigma^* E^*$ twisted by the preimages of the nodes and the elements of S,

$$\operatorname{ind} \widetilde{D}^*_\varrho = -\operatorname{ind} \widetilde{D}_\varrho + 4km + 2k|S|.$$

It follows that $\operatorname{ind} D^*_\varrho = -\operatorname{ind} D_\varrho$ and thus the injective homomorphisms in (2.11) are in fact isomorphisms.

[9]This construction extends the usual procedure of twisting a holomorphic vector bundle by a divisor to generalized CR-operators; it can be seen as a manifestation of Carleman's Similarity Principle [4, Theorem 2.2].

2.3 Families of nodal Riemann surfaces

By a stratified space (of dimension k), we will mean a topological space $\overline{\mathfrak{M}}$ together with a partition

$$\overline{\mathfrak{M}} = \bigsqcup_{l=0}^{l=k} \mathfrak{M}^{(l)}$$

such that $\mathfrak{M}^{(l)}$ is a smooth manifold of (real) dimension $k-l$ and

$$\overline{\mathfrak{M}}^{(l)} - \mathfrak{M}^{(l)} \subset \bigsqcup_{l'=l+1}^{l=k} \mathfrak{M}^{(l')}.$$

If U is an open subspace of a stratified space $\overline{\mathfrak{M}}$ as above, then

$$U = \bigsqcup_{l=0}^{l=k} (\mathfrak{M}^{(l)} \cap U)$$

is also a stratified space. If $\overline{\mathfrak{M}}_1$ and $\overline{\mathfrak{M}}_2$ are stratified spaces, $\overline{\mathfrak{M}}_1 \times \overline{\mathfrak{M}}_2$ is a stratified space with the strata given by unions of the products of the strata of $\overline{\mathfrak{M}}_1$ and $\overline{\mathfrak{M}}_2$. A continuous map $\pi: \overline{\mathfrak{M}}_1 \longrightarrow \overline{\mathfrak{M}}_2$ between stratified spaces will be called a stratified map if the restriction of π to each stratum of $\overline{\mathfrak{M}}_1$ is a smooth map to a stratum of $\overline{\mathfrak{M}}_2$. A stratified map $\pi_V: V \longrightarrow \overline{\mathfrak{M}}$ will be called a stratified vector bundle if π_V is a topological vector bundle with fiber $\mathbb{C}^k$ and the transition maps from open subsets of $\overline{\mathfrak{M}}$ to $\mathrm{GL}_k\mathbb{C}$ are stratified.

For the purposes of Definition 2.4 below, we set

$$\pi_{\mathrm{std}} \equiv \pi_1 : \mathfrak{U}_{\mathrm{std}} \equiv \big\{(t,u,v) \in \mathbb{C}^3 :\ uv = t\big\} \longrightarrow \mathbb{C}$$

to be the projection to the first component. This is a stratified map with respect to the stratifications

$$\mathbb{C} = \mathbb{C}^* \sqcup \{0\}, \qquad \mathfrak{U}_{\mathrm{std}} = \pi_{\mathrm{std}}^{-1}(\mathbb{C}^*) \sqcup \big(\pi_{\mathrm{std}}^{-1}(0) - 0\big) \sqcup \{0\}.$$

For each $t \in \mathbb{C}^*$, define

$$\rho_t : \Sigma_t \equiv \pi_{\mathrm{std}}^{-1}(t) \longrightarrow \mathbb{R}^+ \qquad \text{by} \quad \rho_t(t,u,v) = u^2 + v^2 .$$

If in addition $\epsilon \in \mathbb{R}^+$, let

$$\Sigma_{t,\epsilon} = \big\{(t,u,v) \in \Sigma_t :\ |u|^2 + |v|^2 < \epsilon\big\}.$$

If $E \longrightarrow \Sigma_t$ is a normed vector bundle and $\eta \in L^p(\Sigma_t; E)$, let

$$\|\eta\|_{t,\epsilon} = \bigg(\int_{\Sigma_{t,\epsilon}} |\eta|^p\bigg)^{1/p} + \bigg(\int_{\Sigma_{t,\epsilon}} \rho_t^{-\frac{p-2}{p}} |\eta|^2\bigg)^{1/2} .$$

Definition 2.4 *A stratified map $\pi: \mathfrak{U} \longrightarrow \overline{\mathfrak{M}}$ is a flat stratified family of Riemann surfaces if*

- *each fiber $\Sigma_u \equiv \pi^{-1}(u)$ is a (possibly nodal) Riemann surface;*

- *if* $z_0\in\Sigma_{u_0}$ *is a smooth point, there are neighborhoods* U_{z_0} *of* u_0 *in* $\overline{\mathfrak{M}}$ *and* $\widetilde{U}_{z_0}$ *of* z_0 *in* $\mathfrak{U}$ *and a stratified isomorphism of fiber bundles*

$$\widetilde{\phi}_{z_0}\colon \widetilde{U}_{z_0} \longrightarrow U_{z_0}\times(\Sigma_{u_0}\cap\widetilde{U}_{z_0})$$

  over U_{z_0} *such that the restriction of* $\widetilde{\phi}_{z_0}$ *to each fiber of* π *is holomorphic and the restriction of* $\widetilde{\phi}_{z_0}$ *to* $\Sigma_{u_0}\cap\widetilde{U}_{z_0}$ *is the identity;*

- *if* $z_0\in\Sigma_{u_0}$ *is a node, there are neighborhoods* U_{z_0} *of* u_0 *in* $\overline{\mathfrak{M}}$ *and* $\widetilde{U}_{z_0}$ *of* z_0 *in* $\mathfrak{U}$, *a stratified space* U'_{z_0}, *and stratified embeddings*

$$\phi_{z_0}\colon U_{z_0}\longrightarrow U'_{z_0}\times\mathbb{C} \qquad \textit{and} \qquad \widetilde{\phi}_{z_0}\colon \widetilde{U}_{z_0}\longrightarrow U'_{z_0}\times\mathfrak{U}_{\mathrm{std}}$$

  such that the diagram

$$\begin{array}{ccc} \widetilde{U}_{z_0} & \xrightarrow{\ \widetilde{\phi}_{z_0}\ } & U'_{z_0}\times\mathfrak{U}_{\mathrm{std}} \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\pi_{\mathrm{std}}} \\ U_{z_0} & \xrightarrow{\ \phi_{z_0}\ } & U'_{z_0}\times\mathbb{C} \end{array}$$

  commutes and the restriction of $\widetilde{\phi}_{z_0}$ *to each fiber of* π *is holomorphic.*

Definition 2.5 *If* S *is a finite set, a stratified map* $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ *with stratified sections* $z_j\colon\overline{\mathfrak{M}}\longrightarrow\mathfrak{U}$, $j\in S$, *is a* flat stratified family of S-marked Riemann surfaces *if*

- $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ *is a flat stratified family of Riemann surfaces;*
- $z_j(u)\in\Sigma_u$ *is a smooth point for every* $u\in\overline{\mathfrak{M}}$ *and* $j\in S$;
- $z_{j_1}(u)\neq z_{j_2}(z)$ *for every* $u\in\overline{\mathfrak{M}}$, $j_1,j_2\in S$ *with* $j_1\neq j_2$.

Definition 2.6 *If* $\pi\colon\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ *is a flat stratified family of* S*-marked Riemann surfaces and* Y *is a smooth manifold, a continuous map* $F\colon\mathfrak{U}\longrightarrow Y$ *is a* flat family of S-marked maps *if*

- *for every* $u\in\overline{\mathfrak{M}}$, *the restriction of* F *to* $\Sigma_u\equiv\pi^{-1}(u)$ *is an* L^p_1*-map;*
- *if* $z_0\in\Sigma_{u_0}$ *is a smooth point and* U_{z_0}, $\widetilde{U}_{z_0}$, *and* $\widetilde{\phi}_{z_0}$ *are as in Definition 2.4, there exists a compact neighborhood* $K_{z_0}(F)$ *of* z_0 *in* $\Sigma_{u_0}\cap\widetilde{U}_{z_0}$ *such that* $F\circ\widetilde{\phi}_{z_0}^{-1}|_{u\times K_{z_0}(F)}$ *converges to* $F|_{K_{z_0}(F)}$ *in the* L^p_1*-norm as* $u\in U_{z_0}$ *approaches* u_0;
- *if* $z_0\in\Sigma_{u_0}$ *is a node and* U_{z_0}, $\widetilde{U}_{z_0}$, ϕ_{z_0}, *and* $\widetilde{\phi}_{z_0}$ *are as in Definition 2.4,*

$$\lim_{\epsilon\longrightarrow 0}\ \lim_{\substack{(u',t)\longrightarrow\phi_{z_0}(u)\\ (u',t)\in\phi_{z_0}(U_{z_0})}} \big\|d(F\circ\widetilde{\phi}_{z_0}^{-1}|_{u'\times\Sigma_t})\big\|_{t,\epsilon}=0\,.$$

In the case of interest to us, $\overline{\mathfrak{M}}$ will be a family of S-marked stable maps to a smooth manifold Y. The fiber of $\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ over a point $u\colon\Sigma_u\longrightarrow Y$ will be the Riemann surface Σ_u.

2.4 Families of generalized CR-operators

Let D be a smooth CR-operator on a vector bundle $(N,\mathfrak{i})$ over an almost complex manifold (Y,J). Suppose $\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}$ is a flat stratified family of S-marked Riemann surfaces, $F:\mathfrak{U}\longrightarrow Y$ is a flat family of maps, $S_0\subset S$, and ϱ is a function assigning to each $z_0\in S_0$ a real subbundle of $\mathrm{ev}_j^*N^*$. For each $u\in\overline{\mathfrak{M}}$ and $z_0\in S$, let $\varrho_u(z_0)$ be the fiber of $\varrho(z_0)$ over u. Denote by $\ker^F_{\varrho;u}(D)$ and $\ker^F_{\varrho;u}(D^*)$ the kernels of the operators

$$\begin{aligned}&\big\{\big(F|_{\Sigma_u}\big)^*D\big\}_{\varrho_u}: L^p_1\big(\Sigma_u;\{F|^*_{\Sigma_u}N\}(-\varrho_u)\big)\longrightarrow L^p\big(\Sigma_u;T^*\Sigma^{0,1}\otimes_{\mathbb{C}}F|^*_{\Sigma_u}N\big),\\ &\big\{\big(F|_{\Sigma_u}\big)^*D\big\}^*_{\varrho_u}: L^p_1\big(\Sigma_u;\mathcal{K}_{\Sigma_u}\otimes_{\mathbb{C}}\{F|^*_{\Sigma_u}N\}(\varrho_u)\big)\\ &\qquad\qquad\longrightarrow L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}\mathcal{K}_{\Sigma_u}\otimes_{\mathbb{C}}\{F|^*_{\Sigma_u}N\}(\{z_j(u)\}_{j\in S_0})\big),\end{aligned}$$

respectively.

We topologize the sets

$$\ker^F_\varrho(D)\equiv\bigsqcup_{u\in\overline{\mathfrak{M}}}\ker^F_{\varrho;u}(D)\qquad\text{and}\qquad\ker^F_\varrho(D^*)\equiv\bigsqcup_{u\in\overline{\mathfrak{M}}}\ker^F_{\varrho;u}(D^*)$$

by point-wise convergence on compact subsets of the complement of the special (smooth and marked) points of the fiber. In other words, suppose $u_r\in\overline{\mathfrak{M}}$, $r\in\mathbb{Z}^+$, is a sequence converging to $u_0\in\overline{\mathfrak{M}}$ and $\xi_r\in\ker^F_{\varrho;u_r}(D')$ for $r\in\bar{\mathbb{Z}}^+$, where $D'=D,D^*$ and $\bar{\mathbb{Z}}^+=\{0\}\sqcup\mathbb{Z}$. The sequence $\{u_r\}$ converges to ξ_0 if for every smooth point $z_0\in\Sigma_{u_0}$, with $z_0\neq z_j(u)$ for $j\in S$, there exists a compact neighborhood $K_{z_0}(F)$ as in Definition 2.6 such that $\xi_r\circ\widetilde{\phi}^{-1}_{z_0}|_{u_r\times K_{z_0}(F)}$ converges pointwise to $\xi_0|_{K_{z_0}(F)}$.

By the Carleman Similarity Principle [4, Theorem 2.2], if the restriction of an element ξ of $\ker^F_{\varrho;u}(D')$ to an open subset of a component $\Sigma_{u;i}$ of Σ_u vanishes, then the restriction of ξ to $\Sigma_{u;i}$ is zero as well. This implies that the above convergence topology on $\ker^F_\varrho(D)$ is the topology inherited from the convergence topology on the bundle over $\overline{\mathfrak{M}}$ with fibers $L^p_1(\Sigma_u;u^*N)$ described in [15, Section 3].[10] Furthermore, if the dimension of $\ker^F_{\varrho;u}(D)$ is independent of u, then $\ker^F_\varrho(D)\longrightarrow\overline{\mathfrak{M}}$ is a vector bundle. By [27, Section 6], the analogous statement holds for $\ker^F_{\varrho;u}(D^*)$.[11] Lemma 2.3 then implies that $\ker^F_\varrho(D^*)\longrightarrow\overline{\mathfrak{M}}$ is a vector bundle if the dimension of $\ker^F_{\varrho;u}(D)$ is independent of $u\in\overline{\mathfrak{M}}$. If in addition, the vector bundles $\ker^F_\varrho(D)\longrightarrow\overline{\mathfrak{M}}$ and $\varrho(z_0)$, $z_0\in S$, are oriented (and S is ordered if any of the bundles $\varrho(z_0)$ is of odd rank), then the vector bundle

$$\ker^F_\varrho(D^*)\longrightarrow\overline{\mathfrak{M}}\tag{2.12}$$

has a canonical induced orientation, since $\ker^F_{\varrho;u}(D)$ and $(\ker^F_{\varrho;u}(D^*))^*$ are the kernel and cokernel of an operator obtained by a zeroth-order deformation from a first-order complex-linear Fredholm operator; the determinant line of such an operator has a canonical orientation defined via a homotopy of Fredholm operators (see the proof of [22, Theorem 3.1.5]).

[10]While [15, Section 3] concerns only the case $N=TY$, it applies to any vector bundle $N\longrightarrow Y$.

[11]While [27, Section 6] concerns only the case $N=TY$ and $S_0=\emptyset$, the argument applies to any vector bundle $N\longrightarrow Y$. Furthermore, the twisting construction of [28, Lemma 2.4.1] reduces the situation to the case $S_0=\emptyset$.

3 Configuration spaces and GW-invariants

We begin this section by reviewing the geometric approach to the virtual fundamental class in Gromov-Witten theory. Subsection 3.1 below combines the analytic portion of [15] with the topological portion of [5] in a way convenient for many geometric considerations. Subsection 3.2 applies this approach in the presence of a symplectic submanifold; Proposition 3.2 is a crucial technical observation needed for effective applications. Theorem 1.2 is then proved in Subsection 3.3.

3.1 Definition of GW-invariants

Let X be a compact manifold, $\beta\in H_2(X;\mathbb{Z})$, g a non-negative integer, and S a finite set. We denote by $\mathfrak{X}_{g,S}(X,\beta)$ the space of equivalence classes of stable L^p_1-maps $u\colon\Sigma_u\longrightarrow X$ from genus g Riemann surfaces with S-marked points, which may have simple nodes, to X of degree β, i.e.

$$u_*[\Sigma_u]=\beta\in H_2(X;\mathbb{Z}).$$

Let $\mathfrak{X}^0_{g,S}(X,\beta)$ be the subset of $\mathfrak{X}_{g,S}(X,\beta)$ consisting of the stable maps with smooth domains. The space $\mathfrak{X}_{g,S}(X,\beta)$ is topologized in [15, Section 3] using L^p_1-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The spaces $\mathfrak{X}_{g,S}(X,\beta)$ can be stratified by the smooth infinite-dimensional orbifolds $\mathfrak{X}_{\mathcal{T}}(X)$ of stable maps from domains of the same geometric type and with the same degree distribution between the components of the domain. The closure of the main stratum, $\mathfrak{X}^0_{g,S}(X,\beta)$, is $\mathfrak{X}_{g,S}(X,\beta)$.

If J is an almost complex structure on X, let

$$\Gamma^{0,1}_{g,S}(X,\beta;J)\longrightarrow\mathfrak{X}_{g,S}(X,\beta)$$

be the bundle of (TX,J)-valued $(0,1)$ L^p-forms. In other words, the fiber of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ over a point $[u]$ in $\mathfrak{X}_{g,S}(X,\beta)$ is the space

$$\Gamma^{0,1}_{g,S}(X,\beta;J)\big|_{[u]}=\Gamma^{0,1}(u;J)/\mathrm{Aut}(u),\quad\text{where}\quad\Gamma^{0,1}(u;J)=L^p\big(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

The total space of this bundle is topologized in [15, Section 3] using L^p-convergence on compact subsets of smooth points of the domain and certain convergence requirements near the nodes. The restriction of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ of $\mathfrak{X}_{g,S}(X,\beta)$ is a smooth vector orbibundle of infinite rank.

We define a continuous section of the bundle $\Gamma^{0,1}_{g,S}(X,\beta;J)\longrightarrow\mathfrak{X}_{g,S}(X,\beta)$ by

$$\bar\partial_J\big([u]\big)=\bar\partial_{J,\mathfrak{j}_u}u=\frac{1}{2}\big(du+J\circ du\circ\mathfrak{j}_u\big),$$

where $\mathfrak{j}_u$ is the complex structure on Σ_u. The zero set of this section is the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ of equivalence classes of stable J-holomorphic degree β maps from genus g curves with S-marked points into X. The restriction of $\bar\partial_J$ to each stratum of $\mathfrak{X}_{g,S}(X,\beta)$ is smooth. The section $\bar\partial_J$ of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ is Fredholm, i.e. the linearization of its restriction to every stratum $\mathfrak{X}_{\mathcal{T}}(X)$ has finite-dimensional kernel and cokernel at every point of $\bar\partial_J^{-1}(0)\cap\mathfrak{X}_{\mathcal{T}}(X)$. The index of the linearization of $\bar\partial_J$ at an element of

$$\mathfrak{M}^0_{g,S}(X,\beta;J)\equiv\overline{\mathfrak{M}}_{g,S}(X,\beta;J)\cap\mathfrak{X}^0_{g,S}(X,\beta)$$

is the expected dimension $\dim_{g,S}(X,\beta)$ of the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$. This is the dimension of the cycle

$$\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu) \equiv \{\bar{\partial}_J+\nu\}^{-1}(0)$$

for a small generic multi-valued perturbation

$$\nu \in \mathfrak{G}^{0,1}_{g,S}(X,\beta;J) \equiv \Gamma\big(\mathfrak{X}_{g,S}(X,\beta), \Gamma^{0,1}_{g,S}(X,\beta;J)\big)$$

of $\bar{\partial}_J$, where $\mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$ is the space of all continuous multisections[12] ν of $\Gamma^{0,1}_{g,S}(X,\beta;J)$ such that the restriction of ν to each stratum $\mathfrak{X}_{\mathcal{T}}(X)$ is smooth. Since the moduli space $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ is compact, so is $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)$ if ν is sufficiently small.

For a generic choice of ν, $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)$ admits a stratification by orbifolds of (the expected) even dimensions; see the first remark below. The main stratum is

$$\mathfrak{M}^0_{g,S}(X,\beta;J,\nu) \equiv \overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu) \cap \mathfrak{X}^0_{g,S}(X,\beta).$$

Since $\mathfrak{X}_{g,S}(X,\beta)$ is locally a Banach space, there exist arbitrary small neighborhoods $\mathcal{U}$ of

$$\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu) - \mathfrak{M}^0_{g,S}(X,\beta;J,\nu)$$

in $\mathfrak{X}_{g,S}(X,\beta)$ such that

$$H_l\big(\mathcal{U};\mathbb{Q}\big) = \{0\} \qquad \forall\, l \geq \dim_{g,S}(X,\beta)-1.$$

Since $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu) - \mathcal{U}$ is compact, via the pseudocycle construction of [22, Chapter 7], [27, Section 1], and [31, Section 3.2], $\mathfrak{M}^0_{g,S}(X,\beta;J,\nu)$ determines a homology class

$$\begin{aligned}\big[\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)\big] \in H_{\dim_{g,S}(X,\beta)}(\mathcal{W},\mathcal{U};\mathbb{Q}) \\ \approx H_{\dim_{g,S}(X,\beta)}(\mathcal{W};\mathbb{Q}),\end{aligned}$$

for any small neighborhood $\mathcal{W}$ of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)$ in $\mathfrak{X}_{g,S}(X,\beta)$. The isomorphism between the two homology groups is induced by inclusion. Since ν can be chosen to be arbitrarily small, this procedure defines a rational homology class in an arbitrary small neighborhood of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in $\mathfrak{X}_{g,k}(X,\beta)$.

Remark 1: The strata of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)$ locally are unions of finitely many smooth suborbifolds of a smooth orbifold. The branches of the strata correspond to the branches of ν; see [20]. We will call such objects orbifolds, nevertheless, as these generalized orbifolds are just as suitable for the topological purposes of [5], [15], and this paper; see [5, Sections 3,4] for details.

Remark 2: The above construction defines a homology class

$$\Omega_{\mathcal{W}} \in H_{\dim_{g,S}(X,\beta)}(\mathcal{W};\mathbb{Q})$$

for every neighborhood $\mathcal{W}$ of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in $\mathfrak{X}_{g,S}(X,\beta)$. Furthermore, if

$$\iota_{\mathcal{W}',\mathcal{W}}\colon \mathcal{W} \longrightarrow \mathcal{W}'$$

[12] Our term *multisection* corresponds to *locally liftable multisection* described by [5, Definition 3.5].

is the inclusion map of a neighborhood $\mathcal{W}$ into a larger neighborhood $\mathcal{W}'$, then

$$\iota_{\mathcal{W}',\mathcal{W}*}\Omega_{\mathcal{W}} = \Omega_{\mathcal{W}'}.$$

Thus, the above construction defines VFC for $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ as an element of the inverse limit of the homology groups $H_*(\mathcal{W};\mathbb{Q})$ under inclusion, taken over all neighborhoods of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ in $\mathfrak{X}_{g,S}(X,\beta)$. If (X,J) is algebraic, $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ is a deformation retract of a neighborhood $\mathcal{W}$, and one can then define VFC for $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$ as a homology class in $\overline{\mathfrak{M}}_{g,S}(X,\beta;J)$. However, these formalities are not essential for defining GW-invariants as intersection numbers of $\overline{\mathfrak{M}}_{g,S}(X,\beta;J,\nu)$ with certain natural classes on $\mathfrak{X}_{g,S}(X,\beta)$. In particular, (1.7) is viewed in this paper as a relation between the cardinalities of two zero-dimensional oriented orbifolds.

If $\nu\in\mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$ and $f_j\colon M_j\longrightarrow X$, $j\in S$, are smooth maps, let

$$\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu) = \Big\{\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,S}\big(X,\beta;J,\nu\big)\times\prod_{j\in S}M_j\colon \mathrm{ev}_j([u])=f_j(w_j)\ \forall\, j\in S\Big\}.$$

3.2 Symplectic submanifolds and GW-invariants

This subsection describes a general analytic setup for comparing GW-invariants of a symplectic submanifold Y with those of the ambient manifold X. We regularize spaces of maps to Y without destroying the relative structure of spaces of maps to Y versus maps to X.

Definition 3.1 *If (X,J) is an almost complex manifold and $Y\subset X$ is an almost complex submanifold, a tuple $\big(\pi_Y\colon U_Y\longrightarrow Y, TU_Y^h\big)$ is a J-regularized tubular neighborhood of Y in X if*

- *U_Y is a tubular neighborhood of Y in X;*
- *$\pi_Y\colon U_Y\longrightarrow Y$ is a bundle projection map such that $\pi_Y|_Y=\mathrm{id}_Y$ and $\ker d_y\pi_Y$ is a complex subspace of (T_yX,J) for every $y\in Y$;*
- *$TU_Y^h\longrightarrow U_Y$ is a complex subbundle of (TU_Y,J) such that $d_x\pi_Y\colon TU_Y^h\longrightarrow T_{\pi_Y(x)}Y$ is an isomorphism of real vector spaces for every $x\in U_Y$ and is the identity for every $x\in Y$.*

We note that every embedded almost complex submanifold Y of an almost complex manifold (X,J) admits a J-regularized tubular neighborhood. Let g be a J-invariant Riemannian metric on X and $\exp^g\colon TX\longrightarrow X$ the exponential map with respect to the Levi-Civita connection of the metric g. Identifying NY with the g-orthogonal complement of TY in $TX|_Y$, we obtain a smooth map

$$\exp^Y\colon NY\longrightarrow X$$

by restricting $\exp^g$. Since Y is an embedded submanifold of X, there exist tubular neighborhoods U_Y' and U_Y of Y in NY and in Y, respectively, such that the map

$$\exp\equiv\exp^Y\big|_{U_Y'}\colon U_Y'\longrightarrow U_Y$$

is a diffeomorphism. Furthermore, $\exp|_Y=\mathrm{id}_Y$ and $d_y\exp\colon T_yNY\longrightarrow T_yX$ is $\mathbb{C}$-linear for every $y\in Y$. Thus,

$$\pi_Y=\pi_{NY}\circ\exp|_{U_Y'}^{-1}\colon U_Y\longrightarrow Y,$$

where $\pi_{NY}: NY \longrightarrow Y$ is the bundle projection map, satisfies the middle condition in Definition 3.1. Furthermore, if $(\ker d\pi_Y)^\perp$ is the g-orthogonal complement of $\ker d\pi_Y$ in TU_Y,

$$d_x\pi_Y : (\ker d_x\pi_Y)^\perp \longrightarrow T_{\pi_Y(x)}Y$$

is an isomorphism and induces a complex structure J_Y in the vector bundle $(\ker d\pi_Y)^\perp \longrightarrow U_Y$ (which may differ from J). Let

$$TU_Y^h = \big\{ v - JJ_Y v : v \in (\ker d_x\pi_Y)^\perp \big\}.$$

Note that $T_xU_Y^h$ is a complex linear subspace of (T_xU_Y, J_x) for each $x \in U_Y$. Since $(\ker d_y\pi_Y)^\perp = T_yY$ and $J_Y|_y = J|_{T_yY}$ for every $y \in Y$,

$$d_y\pi_Y = \mathrm{id}: T_yU_Y^h \longrightarrow T_{\pi_Y(y)}Y$$

for every $y \in Y$. Thus,

$$d_x\pi_Y : T_xU_Y^h \longrightarrow T_{\pi_Y(x)}Y$$

is an isomorphism for every $x \in U_Y$ if U_Y is sufficiently small. We conclude that TU_Y^h satisfies the final condition in Definition 3.1.

Suppose Y is a compact almost complex submanifold of (X, J), S is a finite set, and $\beta \in H_2(X;\mathbb{Z})$. We will call $\nu \in \mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$ Y-horizontal if there exists a J-regularized tubular neighborhood $(\pi_Y : U_Y \longrightarrow Y, TU_Y^h)$ of Y in X with the property that for every element $[u]$ of $\mathfrak{X}_{g,S}(X,\beta)$ there is an open neighborhood U_u of $u^{-1}(Y)$ in $u^{-1}(U_Y)$ such that

$$\nu([u])|_z \in T_z^*\Sigma \otimes T_{u(z)}U_Y^h \qquad \forall\, z \in U_u.$$[13]

Let $\mathfrak{G}^{0,1}_{g,S}(X,Y,\beta;J) \subset \mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$ denote the subspace of Y-horizontal elements. If $\beta \in H_2(Y;\mathbb{Z})$, any $\nu_Y \in \mathfrak{G}^{0,1}_{g,S}(Y,\beta;J)$ can be extended to an element

$$\widetilde{\nu}_Y \in \Gamma^{0,1}_{g,S}(X,Y,\iota_*\beta;J).$$

Such an extension can be constructed by

1. choosing a J-regularized tubular neighborhood $(\pi_Y : U_Y \longrightarrow Y, TU_Y^h)$;
2. extending $\widetilde{\nu}_Y$ to the neighborhood $\mathfrak{X}_{g,k}(U_Y,\iota_*\beta)$ of $\mathfrak{X}_{g,k}(Y,\beta)$ in $\mathfrak{X}_{g,k}(X,\beta)$ using $d\pi_Y|_{TU_Y^h}^{-1}$;
3. extending to $\mathfrak{X}_{g,k}(X,\beta)$ using a bump function around $\mathfrak{X}_{g,k}(Y,\beta)$.

Proposition 3.2 *Suppose (X,ω) is a compact symplectic manifold, $g \in \bar{\mathbb{Z}}^+$, S is a finite set, $\beta \in H_2(X;\mathbb{Z})$, and $f_j : M_j \longrightarrow X$ is a smooth map from a compact manifold for each $j \in S$. Let J be an ω-tame almost complex structure on X and Y a compact almost complex submanifold of (X,J). If*

$$\nu_r \in \mathfrak{G}^{0,1}_{g,S}(X,Y,\beta;J) \qquad \textit{and} \qquad \big([u_r],(w_{r,j})_{j\in S}\big) \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu_r) - \mathfrak{X}_{g,S}(Y,\beta_Y) \times \prod_{j\in S} M_j$$

[13] Since U_u is an open subspace of Σ_u, this condition is well-defined even though $\nu(u)$ is in L^p.

are sequences such that $\nu_r \longrightarrow 0$ *and* $[u_r] \longrightarrow [u] \in \mathfrak{X}_{g,S}(Y,\beta_Y)$ *for some* $\beta_Y \in H_2(Y;\mathbb{Z})$*, then* $\iota_{Y*}(\beta_Y)=\beta$ *and there exist*

$$\big([u],(w_j)_{j\in S}\big) \in \overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J), \quad \xi\in\ker D^{NY}_{J;u}, \quad v_j\in T_{w_j}M_j \quad \forall\, j\in S,$$

such that

$$\xi\neq 0, \qquad \xi\big(z_j(u)\big)=d_{w_j}f_j(v_j) \quad \forall\, j\in S.$$

The rest of this subsection is dedicated to the proof of this proposition by adopting a now-standard rescaling argument. It is sufficient to consider the case $X=NY$ as smooth manifolds and $\pi_Y: NY\longrightarrow Y$ is the bundle projection map. After passing to a subsequence, it can be assumed that $([u_r],(w_{r,j})_{j\in S})$ converges to an element $([u],(w_j)_{j\in S})$ of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ and the topological types of the domains Σ_{u_r} of u_r are the same (but not necessarily the same as the topological type of Σ_u). The desired vector field ξ and tangent vectors v_j will be constructed by re-scaling u_j in the normal direction to Y and then taking the limit.

For each $j\in S$, let $N_jY\subset T_{w_j}M$ be a complement of $T_{w_j}(f_j^{-1}(Y))$ and

$$\exp_j: T_{w_j}M_j \longrightarrow M_j$$

a diffeomorphism onto a neighborhood of w_j in M_j such that

$$\exp_j(0)=w_j, \qquad d_0\exp_j=\mathrm{Id}, \qquad \exp_j(v)\in f_j^{-1}(Y) \quad \forall\, v\in T_{w_j}(f_j^{-1}(Y)).$$

For each $r\in\mathbb{Z}^+$, define

$$v^h_{r,j}\oplus v^\perp_{r,j}\in T_{w_j}(f_j^{-1}(Y))\oplus N_jY=T_{w_j}M_j \qquad \text{by} \quad \exp_j\big(v^h_{r,j}+v^\perp_{r,j}\big)=w_{r,j}\,.$$

Choose metrics on NY and N_jY, $j\in S$. By our assumptions,

$$\epsilon_r\equiv\sup_{z\in\Sigma_{u_r}}\big|u_r(z)\big|\in\mathbb{R}^+, \quad \lim_{r\longrightarrow\infty}\epsilon_r=0, \quad \lim_{r\longrightarrow\infty}v^h_{r,j}=0 \quad \forall\, j\in S, \quad \big|v^\perp_{r,j}\big|\le C\epsilon_r \quad \forall\, r\in\mathbb{Z}^+,\, j\in S,$$

for some $C\in\mathbb{R}^+$ independent of r and j. By the last condition, for each $j\in S$ (a subsequence of) the sequence

$$\widetilde{v}^\perp_{r,j}=\epsilon_r^{-1}v^\perp_{r,j}, \quad r\in\mathbb{Z}^+,$$

converges to some $v_j\in N_jY\subset T_{w_j}M_j$.

For each $r\in\mathbb{Z}^+$, we define

$$\begin{aligned}
&m_r: NY\longrightarrow NY && \text{by} \quad m_r(x)=\epsilon_r\cdot x;\\
&J_r\in\Gamma\big(NY;\mathrm{Hom}(T(NY),T(NY))\big) && \text{by} \quad J_r|_x=\big\{d_xm_r\big\}^{-1}\circ J_{\epsilon_rx}\circ d_xm_r;\\
&\widetilde{\nu}_r\in\mathfrak{G}^{0,1}_{g,S}(NY,\beta_Y;J_r) && \text{by} \quad \widetilde{\nu}_r(u)|_z=\big\{d_{u(z)}m_r\big\}^{-1}\circ\nu_r(m_r\circ u)|_z;\\
&\widetilde{u}_r:\Sigma_{u_r}\longrightarrow NY && \text{by} \quad \widetilde{u}_r(z)=\epsilon_r^{-1}\cdot u_r(z).
\end{aligned}$$

If in addition $j\in S$, define $\widetilde{f}_{r,j}\colon T_{w_j}M_j \longrightarrow NY$ by

$$\widetilde{f}_{r,j}\big(v^h+v^\perp\big)=\epsilon_r^{-1}\cdot f_j\big(\exp_j(v^h+\epsilon_r v^\perp)\big) \quad \forall\, v^h\in T_{w_j}(f_j^{-1}(Y)),\ v^\perp\in N_jY.$$

Then, for all $r\in\mathbb{Z}^+$,

$$\bar\partial_{J_r}\widetilde{u}_r+\widetilde{\nu}_r(\widetilde{u}_r)=0, \qquad \widetilde{u}_r(z_j(u_r))=\widetilde{f}_{r,j}\big(v_{r,j}^h+\widetilde{v}_{r,j}^\perp\big) \ \ \forall\, j\in S. \tag{3.1}$$

A straightforward computation in local coordinates shows that the sequence of almost complex structures J_r C^∞-converges on compact subsets of NY to an almost complex structure $\widetilde{J}$ such that $\widetilde{J}|_{TY}=J|_{TY}$. Since $\nu_r(u_r)$ converges to 0 in $\bigcup_r\Gamma^{0,1}_{g,S}(NY,\beta;J_r)$ and ν_r is an element of $\mathfrak{G}^{0,1}_{g,S}(NY,Y,\beta;J)$, $\widetilde{\nu}_r(\widetilde{u}_r)$ converges to 0 as well. Thus, by (3.1), $\widetilde{u}_r$ converges to some

$$\widetilde{u}\in\overline{\mathfrak{M}}_{g,S}(NY,\beta;\widetilde{J})\subset\mathfrak{X}_{g,S}(NY,\beta) \quad \text{s.t.} \quad \widetilde{u}(x_j(\Sigma_{\widetilde{u}}))=d_{w_j}f_j(v_j)\in N_{f_j(w_j)}Y \ \ \forall\, j\in S. \tag{3.2}$$

Since we must have $\pi_Y\circ\widetilde{u}=u$, $\widetilde{u}$ corresponds to an element

$$\xi\in\Gamma\big(\Sigma_u;u^*NY\big).$$

The first condition in (3.2) is equivalent to $D^{NY}_{J;u}\xi=0$.

3.3 Proof of Theorem 1.2

The first claim of Theorem 1.2 is immediate from the assumption that $f_j^{-1}(Y)$ is a smooth oriented manifold. Thus,

$$\big[\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)\big]^{vir}=\bigg(\prod_{j\in S}\mathrm{ev}_j^*\big(\mathrm{PD}_Y(f_{j*}[f_j^{-1}(Y)])\big)\bigg)\cap\big[\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y)\big]^{vir}.$$

The first part of the third claim holds by assumption (b) and the $\nu_r=0$ case of Proposition 3.2. The discussion in Subsections 2.2 and 2.4 implies the second claim of Theorem 1.2. Since the vector spaces

$$\ker\big((D^{NY}_{J;u})^*\big)\approx\mathrm{cok}\big(D^{NY}_{J;u}\big)^* \tag{3.3}$$

have constant rank and are oriented via the isomorphism (1.6), they form natural oriented bundles over the uniformizing charts for $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ described in [15, Section 3]. These bundles glue together to form an oriented vector orbi-bundle over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$.[14] In the notation of Subsections 2.2 and 2.4, this is also the bundle of the cokernels of the injective operators $D^{NY}_{J,\varrho;\mathbf{u}}\equiv(D^{NY}_{J;u})_\varrho$, where

$$[\mathbf{u}]\equiv\big([u],(w_j)_{j\in S}\big)\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J) \tag{3.4}$$

and ϱ is the function assigning to each element $j\in S$ the subbundle $\mathrm{Ann}(\mathrm{ev}_j^*(\mathrm{Im}\,d^{NY}f_j),\mathbb{R})$ of $\mathrm{ev}_j^*NY^*$.

It remains to verify (1.7), assuming (1.4). If φ is a section of the vector orbi-bundle

$$\bigoplus_{j\in S}a_jL_j^*\longrightarrow\mathfrak{X}_{g,S}(X,\beta),$$

[14]Note that neither the topologies of the bundles over the uniformizing charts nor the isomorphisms (3.3) depend on the Riemannian metrics over the uniformizing charts of [15, Section 3].

where $L_j \longrightarrow \mathfrak{X}_{g,S}(X,\beta)$ is the universal tangent line bundle for the j-th marked point, let

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J) = \big\{([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J)\colon \varphi([u])=0\big\};$$
$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu) = \big\{([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu)\colon \varphi([u])=0\big\}.$$

We will assume that φ is chosen generically so that

- φ is transverse to the zero set on every stratum of $\mathfrak{X}_{g,S}(X,\beta)$ and of $\mathfrak{X}_{g,S}(Y,\beta_Y)$;
- the restrictions of the total evaluation maps

  $$\mathbf{ev}\equiv\prod_{j\in S}\mathrm{ev}_j\colon \mathfrak{X}_{g,S}(X,\beta)\longrightarrow X^S,$$
  $$\mathbf{ev}_Y\equiv\prod_{j\in S}\mathrm{ev}_j\colon \mathfrak{X}_{g,S}(Y,\beta_Y)\longrightarrow Y^S,$$

  to each stratum of $\varphi^{-1}(0)$ and $\varphi^{-1}(0)\cap\mathfrak{X}_{g,S}(Y,\beta_Y)$ are transverse to

  $$\mathbf{f}\equiv\prod_{j\in S}f_j\colon\prod_{j\in S}M_j\longrightarrow X^S,$$
  $$\mathbf{f}|_Y\equiv\prod_{j\in S}f_j|_{f_j^{-1}(Y)}\colon\prod_{j\in S}f_j^{-1}(Y)\longrightarrow Y^S\,,$$

  respectively.

The left-hand side of (1.7) is the number of elements of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu)$, counted with appropriate multiplicities, that lie in a small neighborhood of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J)$, for a small generic element ν of $\mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$. We will take $\nu=\nu_Y+\nu_X$, where ν_Y is a small generic element of $\mathfrak{G}^{0,1}_{g,S}(X,Y,\beta;J)$ and ν_X is a small generic element of $\mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$ with respect to ν_Y.

Since $\nu_Y\in\mathfrak{G}^{0,1}_{g,S}(X,Y,\beta;J)$, for every element $[u]$ of $\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J,\nu_Y)$ the linearization

$$D^X_{J,\nu_Y;u}\colon \mathcal{H}_u\oplus L^p_1(\Sigma_u;u^*TX)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TX)$$

of the section $\bar\partial_J+\nu_Y$ for maps to X restricts to the linearization

$$D^Y_{J,\nu_Y;u}\colon \mathcal{H}_u\oplus L^p_1(\Sigma_u;u^*TY)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*TY)$$

of the section $\bar\partial_J+\nu_Y$ for maps to Y. Thus, $D^X_{J,\nu_Y;u}$ descends to a Fredholm operator

$$D^{NY}_{J,\nu_Y;u}\colon L^p_1(\Sigma_u;u^*NY)\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY).$$

If ν_Y is sufficiently small, by the last assumption in Theorem 1.2 the operator

$$D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}}\equiv\big(D^{NY}_{J,\nu_Y;u}\big)_{\varrho}\colon \big\{\xi\in L^p_1(\Sigma_u;u^*NY)\colon\ \xi(z_j(u))\in\mathrm{Im}\,d^{NY}_{w_j}f_j\ \forall\, j\in S\big\}$$
$$\longrightarrow L^p(\Sigma_u;T^*\Sigma_u^{0,1}\otimes_{\mathbb{C}}u^*NY)$$

is injective for every $[\mathbf{u}]\in\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ as in (3.4). Thus, the cokernels of these operators still form an oriented vector orbi-bundle over $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$, which will be denoted by $\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho})$.

Furthermore, $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is compact (because $\overline{\mathfrak{M}}_{g,\mathbf{f}}(Y,\beta_Y;J)$ is) and is a union of connected components of $\overline{\mathfrak{M}}_{g,\mathbf{f}}(X,\beta;J,\nu_Y)$ by Proposition 3.2. On the other hand, if ν_Y is generic,

$$\overline{\mathfrak{M}}_{g,S}(Y,\beta_Y;J,\nu_Y)\cap\varphi^{-1}(0)\subset\mathfrak{X}_{g,S}(Y,\beta_Y)$$

is stratified by branched orbifolds and the restriction of $\mathbf{ev}_Y$ to each stratum is transverse to $\mathbf{f}|_Y$. In particular, $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is also stratified by branched orbifolds.

The aim is then to determine the number of elements of

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(X,\beta;J,\nu_Y+\nu_X)\subset\mathfrak{X}_{g,S}(X,\beta)\times\prod_{j\in S}M_j$$

that lie in a small neighborhood of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ for any sufficiently small generic element ν_X of $\mathfrak{G}^{0,1}_{g,S}(X,\beta;J)$. The number of such elements near each stratum of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is determined via an obstruction analysis similar to [16, Section 3] and [30, Section 3]. Since the moduli $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ consists of regular elements viewed as maps to Y, the latter number is the number of solutions of an equation of the form

$$D^{NY}_{J,\nu_Y;u_\upsilon}\xi+\nu_X(u_\upsilon)+N_\upsilon(\xi)=0,\qquad \xi\in L^p_1(\Sigma_{u_\upsilon};u_\upsilon^*NY),\ \xi(z_j(u_\upsilon))\in\mathrm{Im}\,(d^{NY}_{w_j}f_j),\tag{3.5}$$

with small υ and ξ. In (3.5),

- $([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ is an element of a fixed stratum, i.e. the topological structure of Σ_u is fixed;
- υ is a small gluing parameter for Σ_u consisting of the smoothings of the nodes of Σ_u;
- $u_\upsilon\colon\Sigma_{u_\upsilon}\longrightarrow Y$ is the approximately (J,ν_Y)-map corresponding to υ;
- N_υ is a combination of a term quadratic in ξ and a term which is linear in ξ and ν_X.

Equation (3.5) has no solutions away from the subset of elements

$$\mathbf{u}\equiv([u],(w_j)_{j\in S})\in\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$$

for which $\nu_X(u)$ lies in the image of $D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}}$, i.e. the projection $\bar\nu_X(u)$ to $\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho;\mathbf{u}})$ is zero. For dimensional reasons, all zeros of $\bar\nu_X$ lie in the main stratum

$$\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)\cap\bigg(\mathfrak{X}^0_{g,S}(Y,\beta_Y)\times\prod_{j\in S}f_j^{-1}(Y)\bigg).$$

Thus, only the main stratum of $\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y)$ contributes to the left-hand side in (1.7). Furthermore, the solutions of the equation (3.5) for the main stratum, which no longer involves υ, correspond to the zeros of $\bar\nu_X$. As $\bar\nu_X$ extends to a continuous multi-section of the orbi-bundle

$$\mathrm{cok}(D^{NY}_{J,\nu_Y,\varrho})\longrightarrow\overline{\mathfrak{M}}^{\varphi}_{g,\mathbf{f}}(Y,\beta_Y;J,\nu_Y),\tag{3.6}$$

which is transverse to the zero set over every stratum, the left-hand side of (1.7) is the euler class of the bundle (3.6). On the other hand, the latter is the right-hand side of (1.7) by the last assumption in Theorem 1.2 and the definition of VFC in Section 3.1.

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