Math53: Ordinary Differential Equations
Winter 2004

Problem Set 5 Solutions

Section 6.1: 2,18 (16pts)

6.1:2; 8pts: *For the initial value problem*

$$y' = y, \qquad y(0) = 1,$$

compute the first five iterations of Euler's method with step size $h=0.1$. Then solve the initial value problem exactly and compare the obtained estimate for $y(0.5)$ with its exact value.
We start with $t_0=0$, $y_0=1$ and $f(t,y)=y$.
In the first iteration, we get that $t_1=t_0+h=0.1$, $y_1=y_0+y_0h=1.1$.
In the second iteration we get that $y_2=y_1+y_1h=1.21$ and $t_2=t_1+h=0.2$ and so on.
The first five iterations are given in the following table:

k	t_k	y_k	$f(t_k,y_k)=y_k$	$f(t_k,y_k)h$
0	0.0	1.0000	1.0000	0.1000
1	0.1	1.1000	1.1000	0.1100
2	0.2	1.2100	1.2100	0.1210
3	0.3	1.3310	1.3310	0.1331
4	0.4	1.4641	1.4641	0.1464
5	0.5	1.6105	–	–

The exact value of the solution $y(t)=e^t$ at .5 is $e^{1/2}\approx 1.6487$.

6.1:18; 8pts: *For the initial value problem*

$$x' = y, \quad y' = -x, \qquad x(0) = 1, \quad y(0) = -1,$$

compute the first five iterations of Euler's method with step size $h=0.1$.
We start with $t_0=0$, $x_0=1$, and $y_0=-1$. We also have that $f(t,x,y)=y$ and $g(t,x,y)=-x$, so from here, the iteration proceeds with

$$y_{k+1} = x_k + y_k h \qquad \text{and} \qquad x_{k+1} = y_k - x_k h.$$

The first five iterations are arranged in the following table:

t_k	x_k	y_k	$f(t_k,x_k,y_k)h = y_k h$	$g(t_k,x_k,y_k)h = -x_k h$
0.0	1.0000	−1.0000	−0.1000	−0.1000
0.1	0.9000	−1.1000	−0.1100	−0.0900
0.2	0.7900	−1.1900	−0.1190	−0.0790
0.3	0.6710	−1.2690	−0.1269	−0.0671
0.4	0.5441	−1.3361	−0.1336	−0.0544
0.5	0.4105	−1.3905	–	–

Unfortunately, I made the mistake of not asking you to solve the IVP exactly. This problem is another example of how useful complex numbers can be. So here is the exact solution.
We re-write this IVP as

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The characteristic polynomial for this equation is $\lambda^2+1=0$. Its roots are $\lambda_1, \lambda_2 = \pm i$. We first find an eigenvector for λ_1:

$$\begin{pmatrix} 0-i & 1 \\ -1 & 0-i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -ic_1 + c_2 = 0 \\ -c_1 - ic_2 = 0 \end{cases} \iff c_2 = ic_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

The complex conjugate of $\mathbf{v}_1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$, must then be an eigenvector with eigenvalue $\lambda_2 = \bar{\lambda}_1$. Thus, the general solution to the system of ODEs is

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 = C_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

Plugging in the initial condition, we obtain

$$\mathbf{y}(0) = C_1 \begin{pmatrix} 1 \\ i \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \iff \begin{cases} C_1 + C_2 = 1 \\ iC_1 - iC_2 = -1 \end{cases} \iff \begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = i \end{cases} \iff \begin{cases} C_1 = \frac{1+i}{2} \\ C_2 = \frac{1-i}{2} \end{cases}$$

$$\begin{aligned} \implies \mathbf{y}(t) &= \frac{1+i}{2} e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1-i}{2} e^{-it} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{e^{it}+e^{-it}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{e^{it}-e^{-it}}{2} i \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \cos t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (i \sin t) i \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos t - \sin t \\ -\cos t - \sin t \end{pmatrix}. \end{aligned}$$

The value of the last expression at .5 Radians is $\mathbf{y}(.5) \approx \begin{pmatrix} .398 \\ -1.357 \end{pmatrix}$.

Note that in the above IVP we never needed to use the real form of the general solution. We found the two constants C_1 and C_2 for the complex form. With these constants, the corresponding complex expression automatically reduces to a real one. The key formulas to remember are

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i};$$

they follow from $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$.

Section 6.2:2 (8pts)

For the initial value problem

$$y' = y, \qquad y(0) = 1,$$

compute the first five iterations of the second-order Runge-Kutta method with step size $h = 0.1$ and compare the obtained estimate for $y(0.5)$ with its exact value.
We begin with $t_0 = 0$, $y_0 = 1$, and $f(t,y) = y$. Thus, the initial slopes are

$$s_{0,1} = f(0,1) = 1 \qquad \text{and} \qquad s_{0,2} = f\big(t_0 + h, y_0 + s_{0,1} h\big) = f(0.1, 1.1) = 1.1.$$

From here, we iterate using:

$$s_{k,1} = f(t_k, y_k) = y_k, \qquad s_{k,2} = f\big(t_k+h, y_k+s_{k,1}h\big) = y_k + s_{k,1}h,$$
$$y_{k+1} = y_k + \frac{s_{k,1}+s_{k,2}}{2}h, \qquad t_{k+1} = t_k + h.$$

The first five iterations are presented in the following table:

t_k	y_k	$s_{k,1}$	$s_{k,2}$	$\frac{s_{k,1}+s_{k,2}}{2}h$
0.0	1.0000	1.0000	1.1000	0.1050
0.1	1.1050	1.1050	1.2155	0.1160
0.2	1.2210	1.2210	1.3431	0.1282
0.3	1.3492	1.3492	1.4842	0.1417
0.4	1.4909	1.4909	1.6400	0.1565
0.5	1.6474	−	−	−

Just as in 6.1:2, the exact value of $y(.5)$ is $e^{1/2} \approx 1.6487$. So, the approximation obtained after just five iterations, 1.6474, is quite good. Compare this with Euler's method!

PS5-Problem 4 (20pts)

(a; **7pts**) *Suppose y and $\tilde{y}$ are smooth functions on the interval $[c,d]$ and M is a positive number such that*

$$\big|y''(t)\big|, \big|\tilde{y}''(t)\big| \le M \qquad \text{for all} \qquad t \in [c,d].$$

Show that

$$\big|y(d) - \tilde{y}(d)\big| \le \big|y(c) - \tilde{y}(c)\big| + \big|y'(c) - \tilde{y}'(c)\big||d-c| + M|d-c|^2.$$

We will apply FTC to the function

$$z(t) = y(t) - \tilde{y}(t)$$

and its derivative to estimate the change in $z(t)$ from $t=c$ to $t=d$. We first note

$$|z''(s)| = \big|y''(s) - \tilde{y}''(s)\big| \le |y''(s)| + |\tilde{y}''(s)| \le M + M = 2M \quad \text{for all} \quad s \in [c,d],$$

by our assumption on y and $\tilde{y}$. On the other hand, by FTC, for all $t \in [c,d]$.

$$z'(t) = z'(c) + \int_c^t z''(s)\,ds \quad \Longrightarrow \quad \begin{aligned} \big|z'(t)\big| &\le \big|z'(c)\big| + \Big|\int_c^t z''(s)\,ds\Big| \le \big|z'(c)\big| + \int_c^t |z''(s)|\,ds \\ &\le \big|z'(c)\big| + 2M|t-c| = \big|z'(c)\big| + 2M(t-c). \end{aligned} \tag{1}$$

Similarly, by FTC,

$$z(d) = z(c) + \int_c^d z'(t)\,dt \qquad \Longrightarrow$$
$$\begin{aligned} |z(d)| &\le |z(c)| + \Big|\int_c^d z'(t)\,dt\Big| \le |z(c)| + \int_c^d |z'(t)|\,dt \\ &\le |z(c)| + \int_c^d \big(|z'(c)| + 2M(t-c)\big)\,dt = |z(c)| + |z'(c)||d-c| + M|d-c|^2, \end{aligned}$$

by (1). Since $z(t)=y(t)-\tilde{y}(t)$, we conclude that

$$\big|y(d)-\tilde{y}(d)\big| \le \big|y(c)-\tilde{y}(c)\big| + \big|y'(c)-\tilde{y}'(c)\big||d-c| + M|d-c|^2.$$

Suppose now that $f=f(t,y)$ is a smooth function and M_0, M_t, and M_y are positive numbers such that

$$\big|f(t,y)\big| \le M_0, \quad \big|f_t(t,y)\big| \le M_t, \quad \big|f_y(t,y)\big| \le M_y \qquad \text{for all} \qquad t\in[a,b],\ y\in(-\infty,\infty).$$

Let $y=y(t)$ be the solution to the initial value problem

$$y' = f(t,y), \qquad y(a)=y_0. \tag{2}$$

Given a positive integer N, let

$$h=\frac{b-a}{N}, \quad t_0=a, \quad t_{i+1}=t_i+h=h\cdot(i+1), \quad s_i=f(t_i,y_i), \quad y_{i+1}=y_i+s_ih;$$
$$\epsilon_i=\big|y(t_i)-y_i\big|, \qquad \tilde{y}_i(t)=y_i+s_i(t-t_i).$$

Note that

$$\epsilon_0=0, \quad \epsilon_N=y(b)-y_N, \quad \tilde{y}_i(t_i)=y_i, \quad \tilde{y}_i(t_{i+1})=y_{i+1}, \quad \tilde{y}_i'(t_i)=s_i, \quad \tilde{y}_i''(t)=0.$$

(b; **6pts**) *Use the ODE and the assumptions on f to show that*

$$|y''(t)| \le M_t+M_0M_y \qquad \text{and} \qquad \big|y'(t_i)-\tilde{y}_i'(t_i)\big| \le M_y\epsilon_i.$$

Since $y'(t)=f(t,y(t))$, by the chain rule

$$y''(t)=\frac{d}{dt}f\big(t,y(t)\big)=f_t\big(t,y(t)\big)+f_y\big(t,y(t)\big)\cdot y'(t)=f_t\big(t,y(t)\big)+f_y\big(t,y(t)\big)\cdot f\big(t,y(t)\big)$$
$$\Longrightarrow \qquad \big|y''(t)\big| \le \big|f_t(t,y(t))\big| + \big|f(t,y(t))\big|\big|f_y(t,y(t))\big| \le M_t+M_0M_y,$$

by our assumptions on f. On the other hand, by the same argument as in the first part of (a),

$$\big|y'(t_i)-\tilde{y}_i'(t_i)\big| = \big|f(t_i,y(t_i))-f(t_i,y_i)\big| \le M_y\big|y(t_i)-y_i\big| = M_y\epsilon_i.$$

(c; **3pts**) *Use part (a) to show that*

$$\epsilon_{i+1} \le \epsilon_i + M_y\epsilon_ih + (M_t+M_0M_y)h^2.$$

By parts (a) and (b),

$$\begin{aligned}\epsilon_{i+1}=\big|y(t_{i+1})-y_{i+1}\big| &= \big|y(t_{i+1})-\tilde{y}_i(t_{i+1})\big| \\ &\le \big|y(t_i)-\tilde{y}_i(t_i)\big| + \big|y'(t_i)-\tilde{y}_i'(t_i)\big||t_{i+1}-t_i| + (M_t+M_0M_y)|t_{i+1}-t_i|^2 \\ &\le \epsilon_i + M_y\epsilon_ih + (M_t+M_0M_y)h^2.\end{aligned}$$

(d; **4pts**) *Conclude that*

$$\epsilon_N \le \big(M_t + M_0 M_y\big)\frac{(1+M_y h)^N - 1}{M_y} h \le \frac{M_t + M_0 M_y}{M_y}\big(e^{M_y(b-a)} - 1\big)h.$$

By part (c),

$$\begin{aligned}\epsilon_N &\le (M_t + M_0 M_y)h^2 + (1+M_y h)\epsilon_{N-1}\\ &\le (M_t + M_0 M_y)h^2 + (1+M_y h)(M_t + M_0 M_y)h^2 + (1+M_y h)^2\epsilon_{N-2} \le \ldots\\ &\le (M_t + M_0 M_y)h^2 + (1+M_y h)(M_t + M_0 M_y)h^2 + \ldots + (1+M_y h)^{N-1}(M_t + M_0 M_y)h^2 + (1+M_y h)^N\epsilon_0.\end{aligned}$$

Since $\epsilon_0 = 0$, it follows that

$$\begin{aligned}\epsilon_N &\le (M_t + M_0 M_y)h^2\big(1 + (1+M_y h) + \ldots + (1+M_y h)^{N-1}\big)\\ &\le (M_t + M_0 M_y)h^2\frac{(1+M_y h)^N - 1}{(1+M_y h) - 1} = (M_t + M_0 M_y)\frac{(1+M_y h)^N - 1}{M_y}h.\end{aligned} \tag{3}$$

In order to obtain the final statement, recall that one definition of the number e is

$$e = \lim_{N\longrightarrow\infty}\Big(1 + \frac{1}{N}\Big)^N \qquad \Longrightarrow \qquad \lim_{N\longrightarrow\infty}\Big(1 + \frac{c}{N}\Big)^N = e^c \quad \text{for all} \quad c.$$

Furthermore, the sequence $(1 + c/N)^N$ is increasing with N, if $c > 0$. Since $h = (b-a)/N$, it follows from (3) that

$$\epsilon_N \le \frac{M_t + M_0 M_y}{M_y}\left(\Big(1 + \frac{M_y(b-a)}{N}\Big)^N - 1\right)h \le \frac{M_t + M_0 M_y}{M_y}\big(e^{M_y(b-a)} - 1\big)h.$$

Section 10.1: 2,8,19a,20 (38pts)

10.1:2; 10pts: *Sketch the nullclines for the system*

$$\begin{cases} x' = x(6 - 2x - 3y) = f(x,y) \\ y' = y(1 - x - y) = g(x,y) \end{cases}$$

Find the equilibrium points for the system and label them on your sketch with their coordinates. Use the Jacobian to classify each equilibrium point.
The x-nullcline is defined by the equation $x' = 0$ or $x(6-2x-3y) = 0$. It consists of the two lines $x = 0$ and $2x+3y = 6$. The y-nullcline is defined by the equation $y' = 0$ or $y(1-x-y) = 0$. It consists of the two lines $y = 0$ and $x+y = 1$. The equilibrium points are the intersections of the x-nullcline with the y-nullcline:

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff \begin{cases} x(6-2x-3y) = 0 \\ y(1-x-y) = 0 \end{cases} \iff \begin{cases} x=0 \text{ or } 2x+3y=6 \\ y=0 \text{ or } x+y=1 \end{cases}$$

$$\iff (x,y) = (0,0), (0,1), (3,0), \text{ or } \begin{cases} 2x+3y=6 \\ x+y=1 \end{cases}$$

Thus, the equilibrium points are $(0,0)$, $(0,1)$, $(3,0)$ and $(-3,4)$; see the first sketch in Figure 1. The Jacobian in this case is:

$$J(x,y) = \frac{\partial(f,g)}{\partial(x,y)} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 6-4x-3y & -3x \\ -y & 1-x-2y \end{pmatrix}.$$

The type of each equilibrium point (x_i, y_i) is determined by the eigenvalues of the matrix $J(x_i, y_i)$, *provided* the eigenvalues are distinct and have nonzero real parts:

$$J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \implies \lambda_1 = 6,\ \lambda_2 = 1 \implies \boxed{(0,0) \text{ is a nodal source}}$$

$$J(0,1) = \begin{pmatrix} 3 & 0 \\ -1 & -1 \end{pmatrix} \implies \lambda_1 = 3,\ \lambda_2 = -1 \implies \boxed{(0,1) \text{ is a saddle}}$$

$$J(3,0) = \begin{pmatrix} -6 & -9 \\ 0 & -2 \end{pmatrix} \implies \lambda_1 = -6,\ \lambda_2 = -2 \implies \boxed{(3,0) \text{ is a nodal sink}}$$

$$J(-3,4) = \begin{pmatrix} 6 & 9 \\ -4 & -4 \end{pmatrix} \implies \lambda^2 - 2\lambda + (-24+36) = 0 \implies \lambda_1, \lambda_2 = 1 \pm \sqrt{1-12} = 1 \pm i\sqrt{11}$$

$$\implies \boxed{(-3,4) \text{ is a spiral source}}$$

10.1:8; 10pts: *Sketch the nullclines for the system*

$$\begin{cases} x' = y, \\ y' = -\cos x - 0.5y \end{cases}$$

Find the equilibrium points for the system and label them on your sketch with their coordinates. Use the Jacobian to classify each equilibrium point.

The x-nullcline is defined by the equation $x'=0$ or $y=0$. The y-nullcline is defined by the equation $y'=0$ or $-\cos x - 0.5y = 0$. The equilibrium points are the points of intersection of the nullclines:

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff \begin{cases} y = 0 \\ y = -2\cos x \end{cases} \iff \begin{cases} y = 0 \\ \cos x = 0 \end{cases}$$

Thus, the equilibrium points are $(\frac{\pi}{2} + k\pi, 0)$, where k is any integer; see the second sketch in Figure 1. The Jacobian in this case is:

$$J(x,y) = \frac{\partial(f,g)}{\partial(x,y)} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sin x & -0.5 \end{pmatrix}.$$

Thus,

$$k \text{ even} \implies J(\tfrac{\pi}{2} + k\pi, 0) = \begin{pmatrix} 0 & 1 \\ 1 & -0.5 \end{pmatrix} \implies \lambda^2 + .5\lambda - 1 = 0 \implies \lambda_1, \lambda_2 = \frac{-.5 \pm \sqrt{.25+4}}{2}$$

$$\implies \boxed{(\tfrac{\pi}{2} + k\pi, 0) \text{ is a saddle}}$$

$$k \text{ odd} \implies J(\tfrac{\pi}{2} + k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix} \implies \lambda^2 + .5\lambda + 1 = 0 \implies \lambda_1, \lambda_2 = \frac{-.5 \pm \sqrt{.25-4}}{2}$$

$$\implies \boxed{(\tfrac{\pi}{2} + k\pi, 0) \text{ is a spiral sink}}$$

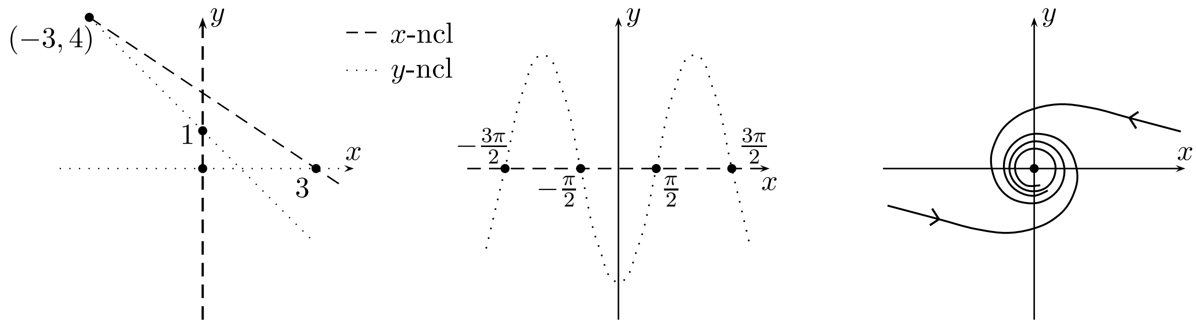

Figure 1: Phase-Plane Plot for Problems 10.1:2,8,20

10.1:19a; 4pts: *The polar coordinates of a point P with Cartesian coordinates (x, y) are (r, θ), where r is the radial length and θ is the angle with the positive x-axis. Using the relations:*

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad \textit{and} \qquad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

prove that:

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}, \qquad \frac{d\theta}{dt} = \frac{1}{r^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

Differentiating both sides with respect to t, we get:

$$\begin{aligned} r^2 = x^2 + y^2 \quad &\Longrightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \quad \Longrightarrow \quad r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \\ \tan\theta = \frac{y}{x} \quad &\Longrightarrow \quad \frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{x^2}(x\frac{dy}{dt} - y\frac{dx}{dt}) \quad \Longrightarrow \quad \frac{d\theta}{dt} = \frac{1}{r^2}\Big(x\frac{dy}{dt} - y\frac{dx}{dt}\Big). \end{aligned}$$

10.1:20; 14pts: *The origin is an isolated equilibrium point of the system*

$$\begin{cases} x' = -y - x^3, \\ y' = x \end{cases}$$

(a; **6pts**) *Compute the linearization of the system near the origin. What kind of equilibrium point is predicted by this linearization?*

The linearization near the origin is $\mathbf{y}' = J(0,0)\mathbf{y}$. The Jacobian in this case is:

$$J(0,0) = \begin{pmatrix} -3x^2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial for the matrix is $\lambda^2+1=0$; its roots are $\lambda_1, \lambda_2 = \pm i$. Thus, the origin is a center for the *linearization* of the system at the origin; the direction of rotation is counterclockwise. This implies that the origin is *either* a center, a spiral source, or a spiral sink for the original system; the direction of rotation has to be counterclockwise.

(c; **2pts**) *Show that $r' = -x^4/r$.*
By 10.1:19a,

$$rr' = xx' + yy' = x(-y - x^3) + yx = -x^4 \qquad \Longrightarrow \qquad r' = -\frac{x^4}{r}$$

(d; **3pts**) *Show that $\theta' = 1 + x^3y/r^2$ and that $\frac{x^3y}{r^2} \longrightarrow 0$ as $r \longrightarrow 0$.*
By 10.1:19a,

$$r^2\theta' = xy' - yx' = x^2 - y(-y - x^3) = r^2 + x^3y \quad \Longrightarrow \quad \theta' = 1 + \frac{x^3y}{r^2}.$$

Since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$, by 10.1:19,

$$\left|\frac{x^3y}{r^2}\right| = \left|\frac{(r^3\cos^3\theta)(r\sin\theta)}{r^2}\right| \le r^2 \quad \Longrightarrow \quad \lim_{r\longrightarrow 0}\frac{x^3y}{r^2} = 0.$$

(e; **3pts**) *Use the above to explain the behavior of solution trajectories for the system near the origin.*
By (c), $r'(t) < 0$, unless $\theta = \frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$. Thus, r is nonincreasing. By (d),

$$1 - r^2 \le \theta'(t) \le 1 + r^2.$$

Thus, for $r < 1$, θ is strictly increasing. It follows that for $r < 1$, solution curves spiral toward the origin counterclockwise; see the last sketch in Figure 1. In particular, the origin is a spiral sink. However, the radius does not drop nearly as quickly with each period of rotation as it does for a planar system with complex eigenvalues with negative real part.

Section 10.2:4 (8pts)

Find the equilibrium points of the system

$$\begin{cases} x' = x + y, \\ y' = y(1 - x^2) \end{cases}$$

and analyze their stability.
The coordinates (x, y) of each equilibrium point satisfy the system

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} x + y = 0 \\ y(1 - x^2) = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y = 0, \text{ or } x = 1, \text{ or } x = -1 \\ y = -x. \end{cases}$$

Thus, there are three equilibrium points: $(0, 0)$, $(1, -1)$ and $(-1, 1)$. The Jacobian in this case is

$$J(x, y) = \frac{\partial(f, g)}{\partial x, y} = \begin{pmatrix} f_x & g_x \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2xy & 1-x^2 \end{pmatrix}.$$

The equilibrium point (x_i, y_i) is unstable if the real part of *an* eigenvalue of $J(x_i, y_i)$ is positive. It is asymptotically stable if the real part of *every* eigenvalue of $J(x_, y_i)$ is negative:

$$J(0, 0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = 1 > 0 \quad \Longrightarrow \quad \boxed{(0, 0) \text{ is unstable}}$$

$$J(1, -1) = J(-1, 1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \quad \Longrightarrow \lambda^2 - \lambda - 2 = 0 \Longrightarrow \lambda_1 = -1, \; \lambda_2 = 2$$

$$\Longrightarrow \quad \boxed{(1, -1) \text{ and } (-1, 1) \text{ are unstable/saddles}}$$