Math53: Ordinary Differential Equations
Winter 2004

Corrections to Lecture on 1/15

This handout concerns the behavior of solutions of ODE

$$y' = (y-1)(y-2)e^{ty}, \qquad y = y(t). \tag{1}$$

I am sorry for causing a lot of confusion with it last Thursday, but I hope you'll find at least part of this note helpful. In particular, you are encouraged to read at least up to Lemma 2, as the first two thirds of this handout hopefully clarify some things directly relevant to this course.

First of all, let's call a real number y^* a *limiting value for the ODE*

$$y' = Q(t,y), \qquad y = y(t), \tag{2}$$

if there exists a solution $y = y(t)$ of (2) such that $y(t) \longrightarrow \infty$ and $t \longrightarrow \infty$. If $y = y^*$ is a constant solution of (2), y^* is a limiting value of (2), but the converse need not be true.

Theorem 1 *(a) There are two constant, or equilibrium, solutions of (1): $y=1$ and $y=2$.*
(b) The set S of limiting values for (1) consists of 1, 2, and the interval $[0,\infty)$.
(c) If y is a solution of (1) and $y(t_0)>2$ for some t_0,
(c-i) $y(t)>2$ and $y'(t)>0$ for all t;
(c-ii) $y(t)\longrightarrow\infty$ as $t\longrightarrow\infty$.
(d) If y is a solution of (1) and $1<y(t_0)<2$ for some t_0,
(d-i) $1<y(t)<2$ and $y'(t)<0$ for all t;
(d-ii) $y(t)\longrightarrow 1$ as $t\longrightarrow\infty$.
(e) If y is a solution of (1) and $y(t_0)<1$ for some t_0,
(e-i) $y(t)<1$ and $y'(t)>0$ for all t;
(e-ii) $y(t)\longrightarrow y^*$ as $t\longrightarrow\infty$ for some $y^*\le 0$ or $y^*=1$.
(f) There exists a "critical" solution y_{cr} of (1) such that
(f-i) $y_{cr}(t)<0$ for all t;
(f-ii) if y is a solution of (1) and $y(t_0)\le y_{cr}(t_0)$ for some t_0, $\lim_{t\longrightarrow\infty} y(t)\le 0$;
(f-iii) if y is a solution of (1) and $y_{cr}(t_0)<y(t_0)<1$ for some t_0, $y(t)\longrightarrow 1$ as $t\longrightarrow\infty$.

If y^* is a real number, the constant function $y=y^*$ is a solution of (2) if and only if $Q(t,y^*)=0$ for all t. In our case,

$$Q(t,y) = (y-1)(y-2)e^{ty}.$$

Thus, $y=y^*$ is a solution of (1) if and only if $y^*=1,2$, as claimed in part (a) of the theorem. The corresponding solution curves, in the ty-plane, are the horizontal lines $y=1$ and $y=2$; see the first plot in Figure 1.

Using the uniqueness-of-solutions theorem for first-order ODEs, i.e. Theorem 7.16 in the textbook, in addition to part (a) of Theorem 1, we are able to prove (c-i), (d-i), and (e-i). Theorem 7.16 implies that no two solution curves of the ODE (1) can cross. Since the horizontal line $y=2$ is a

Figure 1: Sketching Solution Curves for ODE (1)

solution curve, any other solution curve that starts above this line must stay above it. In other words, if y is a solution of (1) and $y(t_0)>2$ for some t_0, then $y(t)>2$ for all t. Since

$$y'(t) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} > 0 \qquad \text{if} \quad y(t) > 2,$$

we conclude that $y'(t)>0$ for all t if $y(t_0)>2$ for some t_0, as claimed in part (c-i) of Theorem 1. We show the conclusion of this argument in the second plot of Figure 1 by drawing a solution curve of (1) above the horizontal solution line $y=2$. Such a solution curve must be heading up, by the second conclusion of (c-i), though we do not know *yet* whether it approaches ∞ or some finite value y^* as $t\longrightarrow\infty$.

By the same reasoning as in the previous paragraph, any solution curve of (1) that starts below the horizontal solution curve $y=1$ of (1) must stay below the line $y=1$. Thus, if $y(t_0)<1$ for some t_0, $y(t)<1$ for all t. Since

$$y'(t) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} > 0 \qquad \text{if} \quad y(t) < 1,$$

we conclude that $y'(t)>0$ for all t if $y(t_0)<1$ for some t_0, as claimed in part (e-i) of Theorem 1. The graph of such a solution is shown in the first plot of Figure 2. The curve must be heading up, though we do not yet what happens to it as $t\longrightarrow\infty$. Finally, any solution curve of (1) that starts between the horizontal solution curves $y=1$ and $y=2$ of (1) must stay between them. Thus, if $1<y(t_0)<2$ for some t_0, $1<y(t)<2$ for all t. Since

$$y'(t) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} < 0 \qquad \text{if} \quad 1 < y(t) < 2,$$

we conclude that $y'(t)<0$ for all t if $1<y(t_0)<2$ for some t_0, as claimed in part (d-i) of Theorem 1. We sketch the graph of such a solution of (1) in the second plot of Figure 2. This curve must be heading down, though we do not yet what happens to it as $t\longrightarrow\infty$.

It is far harder to prove (b). By (a), 1 and 2 are indeed limiting values for (1). The following lemma places a significant restriction on the set of limiting values for an ODE.

Lemma 1 *If $y=y(t)$ is a solution of the ODE $y'=Q(t,y)$ and $y(t)\longrightarrow y^*$ as $t\longrightarrow\infty$, then*

$$\int_t^\infty Q(s,y(s))ds \longrightarrow 0 \qquad \textit{as} \qquad t\longrightarrow 0.$$

Figure 2: Sketching Solution Curves for ODE (1)

The reason for this lemma is the Fundamental Theorem of Calculus:

$$\int_t^\infty Q(s,y(s))ds = \int_t^\infty y'(s)ds = \lim_{s\longrightarrow\infty} y(s) - y(t) = y^* - y(t).$$

Since $y(t)\longrightarrow y^*$ as $t\longrightarrow 0$, it follows that the same is the case for the integral on LHS.

Note that if $Q=Q(y)$ is independent of Q, i.e.

$$y' = Q(y), \qquad y = y(t),$$

is an autonomous ODE, the conclusion of Lemma 1 can be replaced by the simple statement that $Q(y^*)=0$. In particular, for an autonomous ODE, the set of limiting values corresponds to the set of constant solutions. On the other hand, as indicated by parts (a) and (b) of Theorem 1, for a non-autonomous ODE, the set of limiting values can be far larger than the set of constant solutions.

From Lemma 1, we are able to conclude that every limiting value y^* of (1) must either equal 1 or 2 or be nonpositive. Indeed, suppose $y^*>2$ is a limiting value. Then, there exists a solution $y=y(t)$ of (1) such that $y(t)\longrightarrow y^*$ as $t\longrightarrow\infty$. Since $y(t)\longrightarrow y^*$ as $t\longrightarrow\infty$, there exists $T>0$ such that

$$\begin{gathered} y(t) > y_1 \equiv \frac{y^*+2}{2} \in (2,y^*) \quad \text{for all} \quad t>T \qquad \Longrightarrow \\ Q(t,y(t)) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} \ge A \equiv (y_1-1)(y_1-2)e^{Ty_1} > 0 \quad \text{for all} \quad t>T. \end{gathered}$$

It follows that for all $t>T$,

$$\int_t^\infty Q(s,y(s))ds \ge \int_t^\infty A\,ds = A\big(\infty - y(t)\big) = \infty,$$

contrary to Lemma 1. In other words, $y^*>2$ cannot be a limiting value for (1). We conclude that there are no limiting values $y^*>2$ and every solution curve that starts above the horizontal line $y=2$ must approach ∞ as $t\longrightarrow\infty$, as claimed in (c-ii) and indicated in the first plot of Figure 3.

Remark: Actually, every solution curve that starts above the line $y=2$ goes to infinity in a finite time. This can be seen by comparing the solutions to the initial value problems:

$$y' = (y-1)(y-2)e^{ty}, \quad y(t_0) = y_0 > 2, \qquad \text{and} \qquad z' = \frac{(y_0-2)t_0^2}{4} z^2, \quad z(t_0) = y_0.$$

Since $y(t) \geq y_0$ for all $t \geq t_0$ by (c-i) and

$$(y-1)(y-2)e^{ty} \geq \frac{(y_0-2)t_0^2}{4} z^2 \quad \text{for all} \quad t \geq t_0,\ y \geq y_0,$$

$y'(t) \geq z'(t)$ for all $t \geq t_0$. Since $y(t_0) = z(t_0)$, it follows that $y(t) \geq z(t)$ for all $t \geq t_0$. On the other hand,

$$z(t) = \frac{1}{A - \alpha t}, \qquad \text{where} \qquad \alpha = \frac{(y_0-2)t_0^2}{4}, \quad A = \alpha t_0 + \frac{1}{y_0}.$$

This function z blows up at the time $t_1 = t_0 + \frac{1}{\alpha y_0}$. Since $y(t) \geq z(t)$ for all $t \geq t_0$, the function $y(t)$ must blow up by the time t_1. *Please check all claims made in this paragraph.*

We next show that there are no limiting values y^* for (1) such that $1 < y^* < 2$. Suppose $y = y(t)$ is a solution of (1) and $y(t) \longrightarrow y^*$ as $t \longrightarrow \infty$. Then, by (d-i), $y(t) \in (y^*, 2)$ for all t. Furthermore, since $y(t) \longrightarrow y^*$ as $t \longrightarrow \infty$, there exists $T > 0$ such that

$$y(t) < y_1 \equiv \frac{y^*+2}{2} \in (y^*, 2) \quad \text{for all} \quad t > T \qquad \Longrightarrow$$
$$Q(t, y(t)) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} \leq A \equiv (y^*-1)(y_1-2)e^{Ty^*} < 0 \quad \text{for all} \quad t > T.$$

It follows that for all $t > T$,

$$\int_t^\infty Q(s, y(s))ds \leq \int_t^\infty A\, ds = A\big(\infty - y(t)\big) = -\infty,$$

contrary to Lemma 1. In other words, $y^* \in (1,2)$ cannot be a limiting value for (1) and every solution curve of (1) that starts between the horizontal solution curves $y = 1$ and $y = 2$ must approach the line $y = 1$, as claimed in (d-ii).

Finally, suppose $y = y(t)$ is a solution of (1) such that $\lim_{t \longrightarrow \infty} y(t) = y^*$ for some $y^* \in (0,1)$. Then, by (e-i), $y(t) < y^*$ for all t. Furthermore, since $y(t) \longrightarrow y^*$ as $t \longrightarrow \infty$, there exists $T > 0$ such that

$$y(t) \geq y_1 \equiv 0 \quad \text{for all} \quad t > T \qquad \Longrightarrow$$
$$Q(t, y(t)) = \big(y(t)-1\big)\big(y(t)-2\big)e^{ty(t)} \geq A = (y^*-1)(y^*-2) > 0 \quad \text{for all} \quad t > T.$$

It follows that for all $t > T$,

$$\int_t^\infty Q(s, y(s))ds \geq \int_t^\infty A\, ds = A\big(\infty - y(t)\big) = \infty,$$

contrary to Lemma 1. Thus, $y^* \in (0,1)$ cannot be a limiting value for (1) and every solution curve of (1) that has a point between the horizontal lines $y = 0$ and $y = 1$ must approach the line $y = 1$, as claimed in (e-ii).

So far, we have shown if y^* is a limiting value for (1), then $y^* = 1$, $y^* = 2$, or $y^* \leq 0$. Since $y = 1$ and $y = 2$ are solutions of (1), $y^* = 1$ and $y^* = 2$ are limiting values for (1). Thus, it remains to show that every nonpositive number y^* is a limiting value for (1). We also need to prove part (f) of Theorem 1.

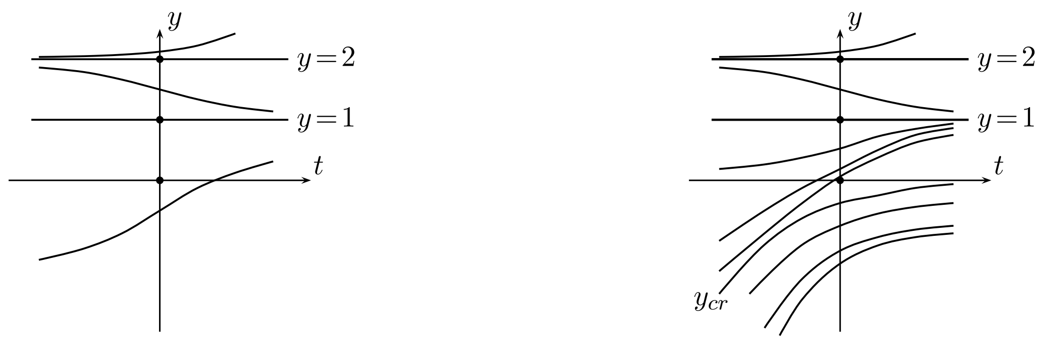

Figure 3: Sketching Solution Curves for ODE (1)

However, before continuing, we indicate the geometric meaning of (b) and (f) in the second plot of Figure 3. This plot shows the graph of the critical solution y_{cr}, which approaches the t-axis from below. Any solution curve that lies between this graph and the line $y=1$ approaches the latter as $t \longrightarrow \infty$. We have not said anything about what happens to the solution curves on the negative end, i.e. as $t \longrightarrow -\infty$. We can analyze the situation in a similar way to find that the limiting values on the negative end are 2 and any number in the interval $[0,1]$. In particular, there is a critical solution $\tilde{y}_{cr}$ such that $\tilde{y}_{cr} \longrightarrow 0$ as $t \longrightarrow -\infty$, while every solution curve below the graph of $\tilde{y}_{cr}$ blows up to $-\infty$. In fact, it does so in a finite time.

The next lemma shows that for every $y^* \leq 0$, there exists a limiting value $\tilde{y}^*$ arbitrary close to y^*.

Lemma 2 *For every $y^* \leq 0$ and $\epsilon > 0$, there exists $\tilde{y}^*$ and a solution of $y = y(t)$ of (1) such that $\lim_{t \longrightarrow \infty} y(t) = \tilde{y}^*$ and $|\tilde{y}^* - y^*| \leq \epsilon$.*

We can assume that $y^* < 0$. Let y be the solution of the initial value problem

$$y' = (y-1)(y-2)e^{ty}, \qquad y(t_0) = y^*,$$

where t_0 is a large number to be chosen latter. We will estimate the limiting value for y by comparing y with the solution $z = z(t)$ to the initial value problem

$$z' = (y^*-1)(y^*-2)e^{(y^*/2)t}, \qquad z(t_0) = y^*.$$

Since $y^* \leq y(t) \leq 1$ for all $t \geq t_0$,

$$\begin{aligned} \big(y(t)-1\big)\big(y(t)-2\big) \leq (y^*-1)(y^*-2) \quad \text{for all} \quad t \geq t_0 \qquad \Longrightarrow \\ y'(t) \leq z'(t) \quad \text{for all} \quad t \geq t_0 \quad \text{s.t.} \quad y(t) \leq y^*/2. \end{aligned}$$

Since $z(t_0) = y(t_0)$, it follows that for all $t \geq t_0$ such that $y(t) \leq y^*/2$,

$$\begin{aligned} y(t) \leq z(t) \leq z(t_0) + \int_{t_0}^{t} (y^*-1)(y^*-2)e^{(y^*/2)t} dt &= y^* + \frac{(y^*-1)(y^*-2)}{-y^*/2}\big(e^{t_0 y^*/2} - e^{ty^*/2}\big) \\ &\leq y^* + \frac{(y^*-1)(y^*-2)}{-y^*/2} e^{t_0 y^*/2}. \end{aligned}$$

Thus, if we choose $t_0 \geq \frac{2}{y^*}\big(\ln(\frac{y^{*2}}{4}) - \ln(y^*-1)(y^*-2)\big)$,

$$y(t) \leq z(t) \leq y^* + \frac{(y^*-1)(y^*-2)}{-y^*/2} e^{t_0 y^*/2} \leq \frac{1}{2} y^* \quad \text{for all} \quad t \geq 0.$$

Finally, if we choose t_0 that also satisfies $t_0 \geq \frac{2}{y^*}\big(\ln(\frac{|y^*|\epsilon}{2}) - \ln(y^*-1)(y^*-2)\big)$,

$$y(t) \leq z(t) \leq y^* + \frac{(y^*-1)(y^*-2)}{-y^*/2}e^{t_0 y^*/2} \leq y^* + \epsilon \quad \text{for all} \quad t \geq 0.$$

Thus, $\lim_{t\longrightarrow\infty} y(t) \leq y^* + \epsilon$, as needed.

Lemma 3 *If for each positive integer k, $y_k = y_k(t)$ is a solution of (1) such that $\lim_{t\longrightarrow\infty} y_k(t) = y_k^*$ for some $y_k^* < 0$ and $y_k^* < y_{k+1}^*$ for all k, then*
(a) $\lim_{k\longrightarrow\infty} y_k^* = y^*$ and $\lim_{k\longrightarrow\infty} y_k(1) = y_0$ for some $y^*, y_0 \leq 0$;
(b) if $y = y(t)$ is the solution of (1) such that $y(1) = y_0$, then $\lim_{t\longrightarrow\infty} y(t) = y^*$.

Since y_k^* is an increasing sequence of nonpositive integers, $\lim_{k\longrightarrow\infty} y_k^* = y^*$ for some $y^* \leq 0$. We next note that $y_k(1)$ is well-defined, i.e. any negative solution of (1) can be extended backwards past $t = 1$. The reason is that for some positive number A and for all k,

$$(y-1)(y-2)e^{ty} \leq (y-1)(y-2)e^{y} \leq A \ \text{ for all } t \geq 1,\ y \leq 0 \implies y_k(t) \geq y_k(t_0) - A(t_0 - 1) \ \text{ for all } t \geq 1,$$

for any number $t_0 \geq 1$. Thus, $y_k(t)$ is well-defined for all $t \geq 1$. We also observe that

$$y_k(1) < y_k^* \leq 0 \quad \text{and} \quad y_k(1) < y_{k+1}(1) \qquad \text{for all} \quad k.$$

The second inequality follows from the assumption $y_k^* < y_{k+1}^*$ and the fact that the solution curves corresponding to y_k and y_{k+1} do not intersect. Since $y_k(1)$ is an increasing sequence of nonpositive integers, $\lim_{k\longrightarrow\infty} y_k(1) = y_0$ for some $y_0 \leq 0$.

It remains to prove part (b) of Lemma 3. If y_0 and $y = y(t)$ are as in its statement, we first note that $y(t) \leq y^*$ for all $t \geq 1$. If not, let $t_1 \geq 1$ be such that $y(t_1) > y^*$ and let $\tilde{y}$ be the solution of (1) such that $\tilde{y}(t_1) = y^*$. Since the functions y_k are increasing and solution curves do not intersect

$$y_k(t_1) < y_k^* < y^* = \tilde{y}(t_1) < y(t_1) \ \text{ for all } \ k \quad \implies \quad y_k(1) < \tilde{y}(1) < y(1) = y_0 \ \text{ for all } \ k.$$

However, RHS contradicts the fact that $\lim_{k\longrightarrow\infty} y_k(1) = y_0$. Next, given $\epsilon > 0$, let m be a positive integer such that $|y_m^* - y^*| < \epsilon/2$ and let $T > 1$ be a large number such that $|y_m(T) - y_m^*| < \epsilon/4$. Such numbers m and T exist because $\lim_{k\longrightarrow\infty} y_k^* = y^*$ and $\lim_{t\longrightarrow\infty} y_m(t) \longrightarrow y_m^*$. Since $y_0 > y_m(1)$ and the solution curves corresponding to y and y_m do not intersect

$$y^* > y(t) \geq y(T) > y_m(T) > y_m^* - \frac{1}{4}\epsilon > y^* - \frac{3}{4}\epsilon \ \text{ for all } \ t \geq T.$$

Thus, $|y(t) - y^*| \leq \frac{3}{4}\epsilon$ for all $t \geq T$. It follows that $\lim_{t\longrightarrow\infty} y(t) = y^*$.

Lemmas 2 and 3 imply that every $y^* \leq 0$ is a limiting value for (1), and the proof of part (b) is complete. For part (f), the required critical solution y_{cr} is the solution of (1) such that

$$y_{cr}(1) = y_0 \equiv \sup\big\{y_0' < 0\colon \text{ if } y \text{ solves (1) and } y(1) = y_0',\text{ then } \lim_{t\longrightarrow\infty} y(t) \leq 0\big\}.$$

In other words, y_0 is either the largest number such that the solution y_{cr} to the initial value problem

$$y' = (y-1)(y-2)e^{ty}, \qquad y(1) = y_0, \tag{3}$$

tends to a nonpositive number as $t \longrightarrow \infty$ or the smallest number such that the solution y_{cr} to the above IVP approaches a positive number. By the same argument as in the first half of the previous paragraph, $y_{cr}(t) \leq 0$ for all $t \geq 1$. Thus, the former has to be the case, and y_{cr} satisfies all three properties of part (f) in Theorem 1.