Math53: Ordinary Differential Equations
Winter 2004

Homework Assignment 5

Problem Set 5 is due by 2:15p.m. on Friday, 3/5, in 380Y

Problem Set 5:

6.1: 2*,18; 6.2:2*; PS5-Problem 4 (see next page); 10.1: 2,8,19a,20*; 10.2: 4.

**Note 1:* In 6.1:2 and 6.2:2, also solve the initial-value problem exactly and compare the two estimates for $y(0.5)$ with the exact value of $y(0.5)$. Use a calculator for 6.1:2,18 and 6.2:2. In 10.1:2, skip (b); instead, sketch the phase-plane portrait near the origin.

**Note 2:* Most of PS5-Problem 4 is a summary of Euler's numerical method for solving ODEs. You do not need any material from Chapter 6 to do it, but it is motivated by Section 6.1

Daily Assignments:

Date	*Read*	*Exercises*
2/27 F	6.1	6.1:2*,18
3/1 M	6.2	6.2:2*
3/2 T		PS5-Problem 4
3/3 W	10.1	10.1:2,8,19a,20*
3/4 R	10.2	10.2:4

**Note:* In 6.1:2 and 6.2:2, also solve the initial-value problem exactly and compare the two estimates for $y(0.5)$ with the exact value of $y(0.5)$. Use a calculator for 6.1:2,18 and 6.2:2. In 10.1:2, skip (b); instead, sketch the phase-plane portrait near the origin.

About the Last Week of Class

According to the course schedule, Problem Set 6 is due on Friday, 3/12. However, no new material will be covered after the Monday or Tuesday of that week. The last few days of the quarter will be spent on review.

PS5-Problem 4

The goal of this problem is to recover the error estimate (1.14) in Section 6.1.

(a) Suppose y and $\tilde{y}$ are smooth functions on the interval $[c,d]$ and M is a positive number such that

$$\big|y''(t)\big|, \big|\tilde{y}''(t)\big| \le M \qquad \text{for all} \qquad t\in[c,d].$$

Show that

$$\big|y(d)-\tilde{y}(d)\big| \le \big|y(c)-\tilde{y}(c)\big| + \big|y'(c)-\tilde{y}'(c)\big|\big|d-c\big| + M|d-c|^2.$$

Suppose now that $f=f(t,y)$ is a smooth function and M_0, M_t, and M_y are positive numbers such that

$$\big|f(t,y)\big| \le M_0, \quad \big|f_t(t,y)\big| \le M_t, \quad \big|f_y(t,y)\big| \le M_y \qquad \text{for all} \qquad t\in[a,b],\ y\in(-\infty,\infty).$$

Let $y=y(t)$ be the solution to the initial value problem

$$y' = f(t,y), \qquad y(a) = y_0. \tag{1}$$

The first-order Euler's method is used to estimate the value of $y(b)$ as follows. We fix a positive integer N and put

$$h=\frac{b-a}{N}, \quad t_0=a, \quad t_{i+1}=t_i+h=h\cdot(i+1), \quad s_i=f(t_i,y_i), \quad y_{i+1}=y_i+s_ih.$$

This way, we start with the point (t_0,y_0), compute the slope s_0, and use it to compute y_1, which is an estimate for $y(t_1)$. We then repeat this procedure at (t_1,y_1), then at (t_2,y_2), and so on, as done in (1.5) and illustrated in Figures 1, 3, and 4 in Section 6.1. Our goal is to estimate the error

$$\epsilon_i = \big|y(t_i)-y_i\big|,$$

especially for $i=N$.

We know that $\epsilon_0=0$. We will estimate $\epsilon_{i+1}-\epsilon_i$. Let

$$\tilde{y}_i(t) = y_i + s_i(t-t_i).$$

Note that

$$\tilde{y}_i(t_i)=y_i, \quad \tilde{y}_i(t_{i+1})=y_{i+1}, \quad \tilde{y}_i'(t_i)=s_i, \quad \tilde{y}_i''(t)=0.$$

(b) Use the ODE and the assumptions on f to show that

$$|y''(t)| \le M_t + M_0M_y \qquad \text{and} \qquad \big|y'(t_i)-\tilde{y}_i'(t_i)\big| \le M_y\epsilon_i.$$

(c) Use part (a) to show that

$$\epsilon_{i+1} \le \epsilon_i + M_y\epsilon_ih + (M_t+M_0M_y)h^2.$$

(d) Conclude that

$$\epsilon_N \le \big(M_t+M_0M_y\big)\frac{(1+M_yh)^N-1}{M_y}h \le \frac{M_t+M_0M_y}{M_y}\big(e^{M_y(b-a)}-1\big)h.$$