Math53: Ordinary Differential Equations
Winter 2004

Solution Guide to Practice Problems

1 Preface

Most of these notes review Chapter 2 topics, using some practice problems as examples. Here is where to find answers or partial to the practice problems:

Problem 1: Example 3.1 below and PS1-2.4:13, Examples 2.8, 2.7, 3.2.

Problem 2: PS1 2.2:14, 2.2:18, Examples 3.2, 2.7, 3.1, 2.8.

Problem 3: Examples 2.7, 2.8.

Problem 4: Examples 2.9, 2.10.

Problem 5: Examples 2.5, 2.6.

Problem 6: PS1 2.1:8; there is a typo in the statement: "1" $\longrightarrow$ "0"

Problem 7: PS6 Problem 1a, last part of Unit 2 Summary.

Problem 8: Examples 3.3 and 3.4.

Problem 9: PS1 2.3:4, 2.5:4.

Problem 10: PS2 4.3:4, 4.3:10, 4.3:14, 4.5:2, 4.5:6, 4.5:16, 4.5:18, 4.5:32, 4.5:42, MTI-3.

Problem 11: PS2 4.3:26, MTI Problem 3, PS3 5.4:18, 5.4:36, -, MTII-1.
For (v), use LT or undetermined coefficients to get: $y(t) = \frac{1}{6}t^3e^{-3t} - te^{-3t}$.

Problem 12: PS2 4.1:14, 4.6:13, -.
The answer in (iii) is $\frac{1}{2}t^{-1}\ln^2|t|$.

Problem 13: (a) PS4 9.2:1, 9.2:4, 9.2:24, 9.2:26, 9.2:38, 9.2:40;
(b) compute as $Y(t)Y(0)^{-1}$, for distinct eigenvalues, or by splitting off λI, for repeated ones;
(c) $y = y_h + y_p$; find by y_p via $Y(t)$, e^{tA}, undetermined coefficients, or LT;
(d) either compute from the general one or in the same way directly.

Problem 14: MTII-2

Problem 15: -,-,-, PS4 9.4:14
(i) $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -3$ can be read off from the diagonal, since matrix is upper-triangular; $\mathbf{v}_1 = (1\,0\,0)^t$ for the same reason; $\mathbf{v}_2$ and $\mathbf{v}_3$ can be found in the usual way, i.e. as in PS4 9.4:14.
(ii) $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 3$ can be read off from the diagonal, since matrix is lower-triangular; $\mathbf{v}_3 = (0\,0\,1)^t$ for the same reason; $\mathbf{v}_2$ and $\mathbf{v}_3$ can be found in the usual way, i.e. as in PS4 9.4:14.
(iii) $\lambda_1 = \lambda_2 = -2,\ \lambda_3 = -3$; $\mathbf{v}_3 = (0\,0\,1)^t$; even though $\lambda_1 = \lambda_2$, we can find two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ from $c_1 + c_2 - c_3 = 0$, e.g. $\mathbf{v}_1 = (1\,0\,1)^t$ and $\mathbf{v}_1 = (0\,1\,1)^t$.

Problem 16: PS5 10.1:2, 10.1:8, 10.1:20, PS6 10.3:16, Unit 6 Summary, PS6 10.4:2

Problem 16': PS6 10.6:10, PS6 10.5:6

Problem 17: (a) 533/420 (?) (b) 193/144 (c) 57/20 (d) 73/24
On the exam, the numbers should be simpler than in (a).

2 First-Order Equations

2.1 Linear Equations

Any *linear* first-order ODE

$$y' + a(t) \cdot y = f(t), \qquad y = y(t). \tag{1}$$

can be solved by multiplying both sides by an *integrating factor*

$$P_a = P_a(t) = e^{\int a(t)dt}.$$

We need only one such integrating factor. Its key property is that

$$P_a'(t) = a(t) \cdot P_a(t) \qquad \Longrightarrow \qquad (P_a y)' = P_a y' + a \cdot P_a y. \tag{2}$$

Multiplying both sides of (1) by P_a and using the second identity in (2), we obtain:

$$\boxed{P_a = P_a(t) = e^{\int a(t)dt}} \qquad \boxed{y' + a(t) \cdot y = f(t), \quad y = y(t) \qquad \Longrightarrow \qquad (P_a y)' = P_a(t) f(t)}$$

The last equation above is solved by integrating both sides with respect to t.

Note that before computing the integrating factor, you need to put the ODE into the form (1), which is *not* its normal form.

Example 2.1 (a) *Find the general solution to the ODE*

$$y' = \cos t - y \cos t.$$

First, we write this equation as $y' + y\cos t = \cos t$. The integrating factor $P(t)$ is the given by

$$\begin{aligned} P(t) = e^{\int \cos t\, dt} = e^{\sin t} \quad &\Longrightarrow \quad y' + y\cos t = \cos t \iff e^{\sin t}(y' + y\cos t) = e^{\sin t}\cos t \\ &\iff (e^{\sin t} y)' = e^{\sin t}\cos t \iff e^{\sin t} y = \int e^{\sin t}\cos t\, dt \\ &\iff \boxed{y(t) = 1 + Ce^{-\sin t}} \end{aligned}$$

(b) *Find the solution to the initial value problem*

$$y' = \cos t - y\cos t, \qquad y(\pi) = 3.$$

We use the general solution found in (a) and solve for C:

$$y(0) = 1 + Ce^0 = 3 \implies C = 2 \implies \boxed{y(t) = 1 + 2e^{-\sin t}}$$

Example 2.2 (a) *Find he general solution to the ODE*

$$ty' = \sin t - 2y.$$

First, we write this equation as $y' + 2t^{-1}y = t^{-1}\sin t$. The integrating factor $P(t)$ is the given by

$$P(t) = e^{\int 2t^{-1}dt} = e^{2\ln|t|} = t^2 \implies y' + 2t^{-1}y = t^{-1}\sin t \iff t^2y' + 2ty = t\sin t \iff (t^2y)' = t\sin t$$
$$\iff t^2y(t) = \int t\sin t\,dt = -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C.$$

Thus, the general solution is $\boxed{y(t) = Ct^{-2} - t^{-1}\cos t + t^{-2}\sin t}$

(b) *Find the solution to the initial value problem*

$$y' = \sin t - 2y, \qquad y(\pi/2) = 0.$$

In this case we solve for the constant C using the expression preceding the boxed formula:

$$0 = 0 + 1 + C \implies C = -1 \implies \boxed{y(t) = -t^{-2} - t^{-1}\cos t + t^{-2}\sin t}$$

2.2 Separable Equations

Separable first-order ODEs are the equations of the form

$$y' = f(y)\cdot g(t), \qquad y = y(t). \tag{3}$$

Equation (3) is solved by writing $y' = \frac{dy}{dt}$, moving all expressions involving y to LHS and all expressions involving t to RHS, and integrating both sides:

$$\boxed{\frac{dy}{dt} = f(y)\cdot g(t), \quad y = y(t) \qquad \implies \qquad \frac{dy}{f(y)} = g(t)dt \qquad \implies \qquad \int\frac{dy}{f(y)} = \int g(t)dt}$$

Once the two integrals are computed, one obtains a relation between y and t of the form

$$F(y) = G(t) + C \qquad \iff \qquad F(y) - G(t) = C. \tag{4}$$

These relations define solutions $y = y(t)$ of (3) *implicitly*. In some cases, it is possible to solve (4) for $y = y(t)$.

Note that this method involves division by $f=f(y)$ and may miss some of the constant solutions of (3). Such solutions are necessarily of the form $y=y^*$, where y^* is a real number such that $f(y^*)=0$. If you are solving an IVP and it is possible to solve for $y=y(t)$ explicitly, make sure you take the correct branch, if there is more than one, of the appropriate level curve of $H=F-G$, e.g. the positive or negative square root, and not both. The correct branch is the one satisfying the initial condition $y(t_0)=y_0$.

Example 2.3 *Find the general solution to the ODE*

$$y' = (1+y^2)e^t.$$

Write $y'=\frac{dy}{dt}$ and split the variables:

$$\begin{aligned} y'=(1+y^2)e^t \quad &\Longleftrightarrow \quad \frac{dy}{1+y^2} = e^t dt \quad \Longleftrightarrow \quad \int \frac{dy}{1+y^2} = \int e^t dt \\ &\Longleftrightarrow \quad \tan^{-1} y = e^t + C \quad \Longleftrightarrow \quad \boxed{y = \tan(e^t+C)} \end{aligned}$$

Example 2.4 (a) *Find the general solution to the ODE*

$$y' = y(y+2)t.$$

Write $y'=\frac{dy}{dt}$ and split the variables:

$$\begin{aligned} y'=y(y+2)t \quad &\Longleftrightarrow \quad \frac{dy}{y(y+2)} = tdt \Longleftrightarrow \frac{1}{2}\Big(\frac{1}{y} - \frac{1}{y+2}\Big)dy = t\,dt \Longleftrightarrow \int \Big(\frac{1}{y} - \frac{1}{y+2}\Big)dy = \int 2t\,dt \\ &\Longleftrightarrow \quad \ln|y| - \ln|y+2| = t^2 + C \quad \Longleftrightarrow \quad \ln\left|\frac{y}{y+2}\right| = t^2 + C \\ &\Longleftrightarrow \quad \frac{y}{y+2} = Ae^{t^2} \quad \Longleftrightarrow \quad y(t) = \frac{2}{Ce^{-t^2}-1} \end{aligned}$$

However, we also have two constant solutions: $y=0$ and $y=-2$. The latter corresponds to $A=0$. Thus, the general solution is $\boxed{y(t) = \frac{2}{Ce^{-t^2}-1} \quad \text{and} \quad y(t) = 0}$

Many ODEs are not separable, but some can be made separable through various manipulations. One class of such ODEs are the ODEs of the form

$$y' = f(t,y),$$

where f is a smooth function such that $f(st,sy) = f(t,y)$ for all s, t, and y. In this case, the substitution $y=tv$, where $v=v(t)$, reduces the ODE to

$$v + tv' = f(1,v).$$

This ODE is separable and can solved for $v=v(t)$ implicitly. We then replace v by y/t.

Example 2.5 *Find the general solution to the ODE*

$$y' = \frac{t-y}{t+y}$$

After making the substitution $y = tv$, we get

$$y' = \frac{t-y}{t+y} \iff v + tv' = \frac{1-v}{1+v} \iff tv' = -\frac{v^2+2v-1}{v+1} \iff \frac{v+1}{(v+1)^2-2}dv = -t^{-1}\,dt$$
$$\iff \frac{1}{2}\ln\left|(v+1)^2-2\right| = -\ln|t| + C \iff (v+1)^2-2 = At^{-2} \iff \boxed{(t+y)^2-2t^2 = A}$$

Example 2.6 *Find the general solution to the ODE*

$$(t^2+y^2)y' - ty = 0.$$

We first write this equation as $y' = ty/(t^2+y^2)$ and then substitute $y = tv$:

$$y' = \frac{ty}{t^2+y^2} \iff v + tv' = \frac{v}{1+v^2} \iff tv' = -\frac{v^3}{1+v^2} \iff \big(v^{-3}+v^{-1}\big)dv = -t^{-1}\,dt$$
$$\iff -\frac{1}{2}v^{-2} + \ln|v| = -\ln|t| + C \iff v^{-2}e^{v^{-2}} = At^2 \iff \boxed{y^2 = Ce^{t^2/y^2}}$$

2.3 Exact Equations

The first-order ODE

$$P(t,y) + Q(t,y)y' = 0 \quad \text{or} \quad P(t,y)dt + Q(t,y)dy = 0, \qquad y = y(t), \tag{5}$$

is *exact* $P_y = Q_t$ and the functions P and Q are smooth. If this is the case, there exists a smooth function $H = H(t,y)$ such that

$$H_t \equiv \frac{\partial H}{\partial t} = P \ \text{ and } \ H_y \equiv \frac{\partial H}{\partial y} = Q, \quad \text{or} \quad \vec{\nabla} H = P\hat{i} + Q\hat{j}, \quad \text{or} \quad dH \equiv H_t dt + H_y dy = Pdt + Qdy.$$

These three conditions are exactly the same. The function $H = H(t,y)$ can be found as follows. Using the condition $H_t = P$, we first find by integration that $H(t,y) = \tilde{H}(t,y) + \phi(y)$, where $\tilde{H}(t,y)$ is a fixed t-antiderivative of P and $\phi(y)$ determines an arbitrary function of y. We next take the y-derivative of $H(t,y)$, set it equal to Q, and thus obtain a condition $\phi(y)$. This condition should *not* involve t. If it does, either $P_y \neq Q_t$, and thus the function H does not exist, or there was a mistake made somewhere else. If we can find $H = H(t,y)$ such that $H_t = P$ and $H_y = Q$, then (5) is *implicitly* solved by

$$\boxed{P(t,y) + Q(t,y)y' = 0, \quad y = y(t) \quad \Longrightarrow \quad H(t,y) = C \quad \text{if} \quad H_t = P \text{ and } H_y = Q}$$

The solution curves for (5) will lie on the level curves of $H = H(t,y)$, i.e. on the curves described by the equations $H(t,y) = C$, for various constants C.

Example 2.7 *Show that the equation*

$$(1 - y\sin t)dt + (\cos t)dy = 0$$

is exact, and solve it.
With $P(t,y)=1-y\sin t$ and $Q(t,y)=\cos t$, we get:

$$\frac{\partial P}{\partial y}=-\sin t=\frac{\partial Q}{\partial t}.$$

Thus, the equation is exact. We solve it by setting

$$\begin{aligned} H(t,y) &= \int P(t)\,dt = \int (1-y\sin t)dt = t + y\cos t + \phi(y) \\ Q(t,y) = \cos t &= \frac{\partial H}{\partial y} = \cos t + \phi'(y) \implies \phi'(y)=0. \end{aligned}$$

Thus, we can take $\phi(y)=0$, and the solution to the ODE is $\boxed{H(t,y)=t+y\cos t=C}$

Example 2.8 *Show that the ODE*

$$2t-y^2+(y^3-2ty)y'=0$$

is exact and solve it.
This ODE is equivalent to $(2t-y^2)dt+(y^3-2ty)dy=0$. Since

$$(2t-y^2)_y=-2y=(y^3-2ty)_t,$$

the mixed partials are equal. Since $2t-y^2$ and y^3-2ty are defined for all t and y, it follows that the ODE is exact. In order to solve it, we need to find $H=H(t,y)$ such that $H_t=(2t-y^2)$ and $H_y=(y^3-2ty)$:

$$\begin{aligned} H_t(t,y)=2t-y^2 &\implies H(t,y)=\int(2t-y^2)\,dt=t^2-ty^2+\phi(y) \\ H_y(t,y)=y^3-2ty &\implies 0-2ty+\phi'(y)=y^3-2ty \implies \phi'(y)=y^3 \\ &\implies \phi(y)=\int y^3dy=\frac{1}{4}y^4 \implies H(t,y)=\frac{1}{4}y^4-ty^2+t^2. \end{aligned}$$

Thus, the general solution $y=y(t)$ of the above ODE is implicitly defined by

$$\boxed{\tfrac{1}{4}y^4-ty^2+t^2=C} \qquad \text{or} \qquad \boxed{y^4-4ty^2+4t^2=C}$$

While most ODEs are not exact, many can be made exact by multiplying both sides by a nonzero *integrating factor* $\mu=\mu(t,y)$:

$$P(t,y)+Q(t,y)y'=0 \quad\iff\quad \mu(t,y)P(t,y)+\mu(t,y)Q(t,y)y'=0.$$

The latter equation is exact if $(\mu P)_y=(\mu Q)_t$. Expanding this relation, we obtain a condition involving μ, μ_t, μ_y, and the known functions P and Q. In general, this condition is very complicated. However, if we are trying to find an integrating factor μ which is a function of t only or y only, which is not always possible, the relation simplifies significantly.

Example 2.9 *Show that the equation*

$$y\,dt + (t^2y - t)\,dy = 0$$

is not exact. Suppose it has an integrating factor which is a function of t alone. Find the integrating factor and use it to solve the equation.
Since $(y)_y = 1$ and $(t^2y-t)_t = (2ty-1)$, the mixed partials are not equal and the equation is not exact. If $\mu = \mu(t)$ is an integrating factor for this equation,

$$\big(\mu(t)y\big)_y = \big(\mu(t)(t^2y-t)\big)_t \iff \mu(t) = \mu'(t)(t^2y-t) + \mu(t)(2ty-1)$$
$$\iff 2\mu(t)(1-ty) = t\mu'(t)(ty-1) \iff \mu'(t) = -2t^{-1}\mu(t).$$

The last equation is separable, and we can solve it. One nonzero solution is $\mu(t) = 1/t^2$. After multiplying the equation by $\mu = \mu(t)$, we get an exact equation:

$$y\,dt + (t^2y - t)\,dy = 0 \iff \frac{y}{t^2}\,dt + \frac{t^2y-t}{t^2}dy = 0$$
$$\Longrightarrow H(t,y) = \int \frac{y}{t^2}dt = -\frac{y}{t} + \phi(y)$$
$$H_y = \frac{t^2y-t}{t^2} \Longrightarrow -\frac{1}{t} + \phi'(y) = \frac{t^2y-t}{t^2} \Longrightarrow \phi'(y) = y \Longrightarrow \phi(y) = \frac{y^2}{2}.$$

Thus, $H(t,y) = -\frac{y}{t} + \frac{y^2}{2}$ and the general solution to the ODE is $\boxed{-\frac{y}{t} + \frac{y^2}{2} = C}$

Example 2.10 *Show that the equation*

$$y^2 + 2ty - t^2y' = 0$$

is not exact. Suppose it has an integrating factor that is a function of y alone. Find the integrating factor and use it to solve the equation.
Since $(y^2+2ty)_y = (2y+2t)$ and $(-t^2)_t = -2t$, the mixed partials are not equal and the equation is not exact. If $\mu = \mu(y)$ is an integrating factor for this equation,

$$\big(\mu(y)(y^2+2ty)\big)_y = \big(-\mu(y)t^2\big)_t \iff \mu'(y)(y^2+2ty) + \mu(y)(2y+2t) = -\mu(y)\cdot 2t$$
$$\iff (y+2t)y\mu'(y) = -2(y+2t)\mu(y) \iff \mu'(t) = -2y^{-1}\mu(t).$$

The last equation is separable, and we can solve it. One nonzero solution is $\mu(y) = 1/y^2$. After multiplying the equation by $\mu = \mu(y)$, we get an exact equation:

$$y^2 + 2ty - t^2y' = 0 \iff \frac{y^2+2ty}{y^2}\,dt - \frac{t^2}{y}dy = 0$$
$$\Longrightarrow H(t,y) = \int \frac{y^2+2ty}{y^2}dt = t + t^2y^{-1} + \phi(y)$$
$$H_y = -\frac{t^2}{y^2} \Longrightarrow -t^2y^{-2} + \phi'(y) = -\frac{t^2}{y^2} \Longrightarrow \phi'(y) = 0 \Longrightarrow \phi(y) = 0.$$

Thus, $H(t,y) = t + t^2y^{-1}$ and the general solution to the ODE is $\boxed{t + t^2y^{-1} = C}$

3 Qualitative Descriptions

3.1 Structure of Solutions of Linear ODEs and of Systems of Linear ODEs

A *homogeneous linear* first-order ODE is an ODE of the form

$$y' = a(t)y, \qquad y = y(t), \tag{6}$$

If $y_1 = y_1(t)$ is a nonzero solution of this equation, then $y(t) = C_1 y_1(t)$ is the general solution of (6). The general solution of any linear inhomogeneous equation

$$y' = a(t)y + f(t), \qquad y = y(t), \tag{7}$$

has the form $y = y_h + y_p$, where $y_p = y_p(t)$ is a fixed *particular* solution of (7) and $y_h = y_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (6) with the same $a = a(t)$ as in (7).

A *homogeneous linear* nth ODE is an equation of the form

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \ldots + a_{n-1}(t)y' + a_n(t)y, \qquad y = y(t). \tag{8}$$

If $y_1 = y_1(t), \ldots, y_n = y_n(t)$ is a set of n linearly independent solutions of (8), then

$$y(t) = C_1 y_1(t) + \ldots + C_n y_n(t)$$

is the general solution to (8). As in the first-order case, the general solution of any linear inhomogeneous equation

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \ldots + a_{n-1}(t)y' + a_n(t)y + f(t), \qquad y = y(t), \tag{9}$$

has the form $y = y_h + y_p$, where $y_p = y_p(t)$ is a fixed *particular* solution of (9) and $y_h = y_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (8) with the same $a = a(t)$ as in (9).

Example 3.1 Since the characteristic polynomial for the second-equation equation

$$y'' + 5y' + 4y = 0 \tag{10}$$

is $\lambda^2 + 5\lambda + 4 = 0$, $y_1(t) = e^{-t}$ and $y_2(t) = e^{-4t}$ are solutions of (10). Since the ratio $y_1(t)/y_2(t) = e^{3t}$ is not a constant, $y_1 = y_1(t)$ and $y_2 = y_2(t)$ are linearly independent solutions. Thus,

$$y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{-t} + C_2 e^{-4t}$$

is the general solution of (10). Using the *Method of Undetermined Coefficients* or the *Laplace Transform*, we can find a particular solution to the inhomogeneous equation

$$y'' + 5y' + 4y = te^{-t}, \tag{11}$$

such as $y_p(t) = \frac{1}{6}t^2 e^{-t} - \frac{1}{9}te^{-t}$. Thus,

$$y(p) = C_1 y_1(t) + C_2 y_2(t) + y_p(t) = C_1 e^{-t} + C_2 e^{-4t} + \frac{1}{6}t^2 e^{-t} - \frac{1}{9}te^{-t}$$

is the general solution of (11).

Example 3.2 By direct substitution into the ODE, we can verify that $y_1(t)\!=\!t$ and $y_2(t)\!=\!t^{-3}$ are solutions to

$$t^2y'' + 3ty' - 3y = 0. \tag{12}$$

These solutions can be found by trying $y(t)\!=\!Ct^{\alpha}$ and finding that $\alpha\!=\!1$ or $\alpha\!=\!-3$. Since the ratio $y_1(t)/y_2(t)\!=\!t^4$ is not a constant, $y_1\!=\!y_1(t)$ and $y_2\!=\!y_2(t)$ are linearly independent solutions. Thus,

$$y(t) = C_1y_1(t) + C_2y_2(t) = C_1t + C_2t^{-3}$$

is the general solution of (12). Using *Variation of Parameters*, we can find a particular solution to the inhomogeneous equation

$$t^2y'' + 3ty' - 3y = t^{-1}, \tag{13}$$

such as $y_p(t)\!=\!-\frac{1}{4}t^{-1}$. Thus,

$$y(p) = C_1y_1(t) + C_2y_2(t) + y_p(t) = C_1t + C_2t^{-3} - \frac{1}{4}t^{-1}$$

is the general solution of (13).

Finally, a *homogeneous system* of linear first-order ODEs is an ODE of the form

$$\mathbf{y}' = A(t)\mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t), \tag{14}$$

where $A\!=\!A(t)$ is an $n\!\times\!n$-matrix, possibly dependent on t. If $\mathbf{y}_1\!=\!\mathbf{y}_1(t),\ldots,\mathbf{y}_n\!=\!\mathbf{y}_n(t)$ is a set of n linearly independent solutions of (8), then

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + \ldots + C_n\mathbf{y}_n(t)$$

is the general solution to (14). As in the previous two cases, the general solution to any inhomogeneous system of linear first-order ODEs

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \qquad \mathbf{y} = \mathbf{y}(t), \tag{15}$$

has the form $\mathbf{y}\!=\!\mathbf{y}_h\!+\!\mathbf{y}_p$, where $\mathbf{y}_p\!=\!\mathbf{y}_p(t)$ is a fixed *particular* solution of (15) and $\mathbf{y}_h\!=\!\mathbf{y}_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (14) with the same $A\!=\!A(t)$ as in (15).

3.2 Existence and Uniqueness Theorems

According to the Existence and Uniqueness Theorem for first-order ODEs, the initial value problem

$$y' = f(t,y), \qquad y(t_0) = y_0, \tag{16}$$

(a) has a solution $y\!=\!y(t)$ near t_0 if the function f is continuous near (t_0, y_0);
(b) has a unique solution $y\!=\!y(t)$ near t_0 if f and $\partial f/\partial y$ are continuous near (t_0, y_0).
Note that the applicability of this theorem to each given IVP depends on both the function y and

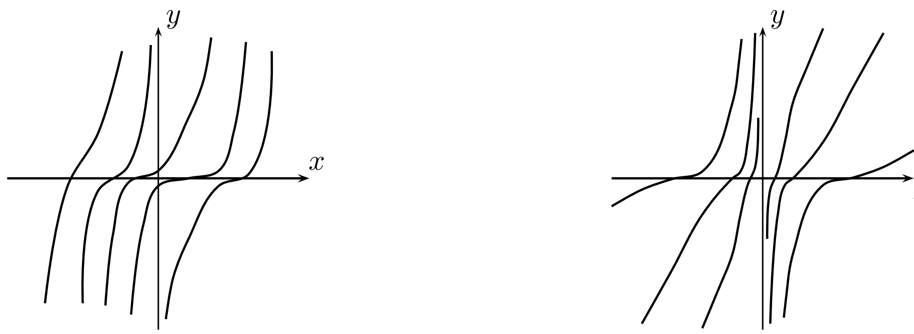

Figure 1: Sketches of Solution Curves for ODEs in (17) and in (18)

the initial condition (t_0, y_0). Whenever the assumptions of (a) or (b), the IVP is guaranteed to have a solution or a unique solution. However, even if they are not satisfied, the IVP may still have a solution or a unique solution.

Example 3.3 (a) *For what values of t_0 and y_0, the IVP*

$$y' = \sqrt{|t-1|}y^{2/3}, \qquad y(t_0) = y_0, \tag{17}$$

is guaranteed by the Existence and Uniqueness Theorem to have a solution?
The function $f = f(t, y) = \sqrt{|t-1|}y^{2/3}$ is continuous everywhere. Thus, the existence part of the theorem guarantees that (17) has a solution for $\boxed{\text{all } (t_0, y_0)}$
(b) *For what values of t_0 and y_0, IVP (17) is guaranteed by the Existence and Uniqueness Theorem to have a unique solution?*
Since $\partial f/\partial y = (2/3)\sqrt{|t-1|}y^{-1/3}$, $\partial f/\partial y$ is continuous near $y \neq 0$ and is not even defined at $y = 0$. Thus, the uniqueness part of the theorem guarantees that IVP (17) has a unique solution if $\boxed{y_0 \neq 0}$ and does not apply if $y_0 = 0$.
(c) *For what values of t_0 and y_0, IVP (17) has a solution?*
By (a), (17) is guaranteed to have a solution for $\boxed{\text{all } (t_0, y_0)}$
(d) *For what values of t_0 and y_0, IVP (17) has a unique solution?*
By (b), (17) is guaranteed to have a unique solution if $y_0 \neq 0$. Thus, we need to somehow determine if (17) has a unique solution for $y_0 = 0$. Since the ODE is separable, we can solve it by separating variables:

$$y' = \sqrt{|t-1|}y^{2/3} \iff y^{-2/3}dy = \sqrt{|t-1|}dt \iff 3y^{1/3} = \frac{2}{3}\big(g(t) + C\big),$$

$$\text{where} \qquad g(t) = \begin{cases} |t-1|^{3/2}, & \text{if } t > 1; \\ -|t-1|^{3/2}, & \text{if } t < 1. \end{cases}$$

This separation of variables approach misses the solution $y(t) = 0$. The new solutions we found are defined for all $t = 0$. For any t_0, the above solution with $C = -g(t_0)$ is another solution to (17) with $(t_0, y_0) = (t_0, 0)$. We conclude that IVP (17) has a unique solution if $\boxed{y_0 \neq 0}$

Example 3.4 (a) *For what values of t_0 and y_0, the IVP*

$$|t|y' = \sqrt{|y|}, \qquad y(t_0) = y_0, \tag{18}$$

is guaranteed by the Existence and Uniqueness Theorem to have a solution?
Before applying the theorem, we need to rewrite the ODE in the *normal form*:

$$y' = |t|^{-1}\sqrt{|y|} = f(t,y), \qquad y(t_0) = y_0. \tag{19}$$

The function $f = f(t,y)$ is continuous near $t \neq 0$ and is not even defined at $t=0$. Thus, the existence part of the theorem guarantees that (18) has a solution if $\boxed{t_0 \neq 0}$ and does not apply if $t_0=0$.
(b) *For what values of* t_0 *and* y_0*, IVP (18) is guaranteed by the Existence and Uniqueness Theorem to have a unique solution?*
By part (a), we only need $t_0 \neq 0$. Since $\partial f/\partial y = \pm 1/2t\sqrt{|y|}$, $\partial f/\partial y$ is continuous near $y \neq 0$ and is not even defined at $y=0$. Thus, the uniqueness part of the theorem guarantees that IVP (18) has a unique solution if $\boxed{t_0 \neq 0 \text{ and } y_0 \neq 0}$ and does not apply if $t_0=0$ or $y_0=0$.
(c) *For what values of* t_0 *and* y_0*, IVP (18) has a solution?*
By (a), (18) is guaranteed to have a solution if $t_0 \neq 0$. So, we need to somehow determine if (18) has a solution for $t_0 = 0$. If $y = y(t)$ is a solution to the ODE (18) and is defined at $t = 0$, then $0 \cdot y'(0) = \sqrt{|y(0)|}$. Thus, IVP (18) has no solution if $t_0 = 0$ and $y_0 \neq 0$. On the other hand, $y(t) = 0$ is a solution to (18) with $t_0 = 0$ and $y_0 = 0$. We conclude that (18) has a solution if $\boxed{t_0 \neq 0 \text{ or } (t_0, y_0) = (0,0)}$
(d) *For what values of* t_0 *and* y_0*, IVP (18) has a unique solution?*
By (a), (18) is guaranteed to have a unique solution if $t_0 \neq 0$ and $y_0 \neq 0$. By (c), (18) has no solution if $t_0 = 0$ and $y_0 \neq 0$. So, we need to somehow determine if (18) has a unique solution for $y_0 = 0$. Since the ODE is separable, we can solve it by separating variables:

$$|t|y' = \sqrt{|y|} \iff \frac{dy}{\sqrt{|y|}} = \frac{dt}{|t|} \iff 2h(y) = g(t) + C, \qquad \text{where}$$

$$g(t) = \begin{cases} \ln t, & \text{if } t > 0; \\ -\ln|t|, & \text{if } t < 0; \end{cases} \qquad \text{and} \qquad h(y) = \begin{cases} \sqrt{y}, & \text{if } y > 0; \\ -\sqrt{|y|}, & \text{if } y < 0. \end{cases}$$

From this, we conclude that

$$y(t) = \begin{cases} -\frac{1}{4}\big(\ln|t| + C\big)\big|\ln|t| + C\big|, & t \in (-\infty, 0); \\ \frac{1}{4}\big(\ln|t| + C\big)\big|\ln|t| + C\big|, & t \in (0, \infty). \end{cases}$$

This separation of variables approach misses the solution $y(t)=0$. The new solutions we found are not defined for $t=0$ and thus $y(t)=0$ is the only solution to (18) with $(t_0,y_0)=(0,0)$. On the other hand, if $t_0 \neq 0$, the above solution with $C = -\ln|t_0|$ is another solution to (18) with $(t_0,y_0)=(t_0,0)$. We conclude that (18) has a unique solution if $\boxed{t_0 \neq 0 \text{ and } y_0 \neq 0 \quad \text{OR} \quad (t_0,y_0)=(0,0)}$

In Example 3.3, the Existence and Uniqueness Theorem predicts all cases when the IVP has a solution and when it has a unique solution. On the other hand, in Example 3.4, there is one case, $(t_0, y_0) = (0,0)$, in which the IVP has a unique solution, while the theorem cannot be used to predict even the existence of a solution. We plot solution curves for the ODEs in (17) and in (18) in Figure 1. There is a solution curve through every point (t_0, y_0) for which the corresponding IVP has a solution. Furthermore, solution curves intersect precisely at the points (t_0, y_0) for which only the uniqueness property for the corresponding IVP fails.

The Existence and Uniqueness Theorem for the first-order ODEs applies, word for word, to initial value problems involving systems of first-order equations. In other words, the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0,$$

(a) has a solution $\mathbf{y} = \mathbf{y}(t)$ near t_0 if the function $\mathbf{f}$ is continuous near $(t_0, \mathbf{y}_0)$;
(b) has a unique solution $\mathbf{y} = \mathbf{y}(t)$ near t_0 if $\mathbf{f}$ and $\partial \mathbf{f}/\partial \mathbf{y}$ are continuous near $(t_0, \mathbf{y}_0)$.
Since many high-order equations and systems of high-order equations can be re-written as systems of first-order equations, this theorem has implications for the existence and uniqueness of solutions to initial value problems involving high-order equations and systems of high-order equations.

The general Existence and Uniqueness Theorem makes it possible to approximately sketch solution curves just by looking at direction fields and to use numerical methods. If $\partial \mathbf{f}/\partial \mathbf{y}$ is not continuous near (t_0, y_0), we may not be able to obtain any error estimate for numerical methods that decay to zero as the step size decreases, as can be seen from PS5-Problem 4.

Sorry for the delay with this solution guide. Good luck with all your exams.