

Math53: Ordinary Differential Equations Winter 2004

Solution Guide to Practice Problems

1 Preface

Most of these notes review Chapter 2 topics, using some practice problems as examples. Here is where to find answers or partial to the practice problems:

Problem 1: Example 3.1 below and PS1-2.4:13, Examples 2.8, 2.7, 3.2.

Problem 2: PS1 2.2:14, 2.2:18, Examples 3.2, 2.7, 3.1, 2.8.

Problem 3: Examples 2.7, 2.8.

Problem 4: Examples 2.9, 2.10.

Problem 5: Examples 2.5, 2.6.

Problem 6: PS1 2.1:8; there is a typo in the statement: "1" \longrightarrow "0"

Problem 7: PS6 Problem 1a, last part of Unit 2 Summary.

Problem 8: Examples 3.3 and 3.4.

Problem 9: PS1 2.3:4, 2.5:4.

Problem 10: PS2 4.3:4, 4.3:10, 4.3:14, 4.5:2, 4.5:6, 4.5:16, 4.5:18, 4.5:32, 4.5:42, MTI-3.

Problem 11: PS2 4.3:26, MTI Problem 3, PS3 5.4:18, 5.4:36, -, MTII-1.
For (v), use LT or undetermined coefficients to get: $y(t) = \frac{1}{6}t^3e^{-3t} - te^{-3t}$.

Problem 12: PS2 4.1:14, 4.6:13, -.
The answer in (iii) is $\frac{1}{2}t^{-1} \ln^2 |t|$.

Problem 13: (a) PS4 9.2:1, 9.2:4, 9.2:24, 9.2:26, 9.2:38, 9.2:40;
(b) compute as $Y(t)Y(0)^{-1}$, for distinct eigenvalues, or by splitting off λI , for repeated ones;
(c) $y = y_h + y_p$; find by y_p via $Y(t)$, e^{tA} , undetermined coefficients, or LT;
(d) either compute from the general one or in the same way directly.

Problem 14: MTII-2

Problem 15: -, -, -, PS4 9.4:14

(i) $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -3$ can be read off from the diagonal, since matrix is upper-triangular; $\mathbf{v}_1 = (100)^t$ for the same reason; \mathbf{v}_2 and \mathbf{v}_3 can be found in the usual way, i.e. as in PS4 9.4:14.

(ii) $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3$ can be read off from the diagonal, since matrix is lower-triangular; $\mathbf{v}_3 = (001)^t$ for the same reason; \mathbf{v}_2 and \mathbf{v}_1 can be found in the usual way, i.e. as in PS4 9.4:14.

(iii) $\lambda_1 = \lambda_2 = -2, \lambda_3 = -3$; $\mathbf{v}_3 = (001)^t$; even though $\lambda_1 = \lambda_2$, we can find two linearly independent eigenvectors \mathbf{v}_1 and \mathbf{v}_2 from $c_1 + c_2 - c_3 = 0$, e.g. $\mathbf{v}_1 = (101)^t$ and $\mathbf{v}_2 = (011)^t$.

Problem 16: PS5 10.1:2, 10.1:8, 10.1:20, PS6 10.3:16, Unit 6 Summary, PS6 10.4:2

Problem 16': PS6 10.6:10, PS6 10.5:6

Problem 17: (a) 533/420 (?) (b) 193/144 (c) 57/20 (d) 73/24

On the exam, the numbers should be simpler than in (a).

2 First-Order Equations

2.1 Linear Equations

Any *linear* first-order ODE

$$y' + a(t) \cdot y = f(t), \quad y = y(t). \quad (1)$$

can be solved by multiplying both sides by an *integrating factor*

$$P_a = P_a(t) = e^{\int a(t) dt}.$$

We need only one such integrating factor. Its key property is that

$$P_a'(t) = a(t) \cdot P_a(t) \quad \implies \quad (P_a y)' = P_a y' + a \cdot P_a y. \quad (2)$$

Multiplying both sides of (1) by P_a and using the second identity in (2), we obtain:

$$\boxed{P_a = P_a(t) = e^{\int a(t) dt}} \quad \boxed{y' + a(t) \cdot y = f(t), \quad y = y(t)} \quad \implies \quad \boxed{(P_a y)' = P_a(t) f(t)}$$

The last equation above is solved by integrating both sides with respect to t .

Note that before computing the integrating factor, you need to put the ODE into the form (1), which is *not* its normal form.

Example 2.1 (a) *Find the general solution to the ODE*

$$y' = \cos t - y \cos t.$$

First, we write this equation as $y' + y \cos t = \cos t$. The integrating factor $P(t)$ is the given by

$$\begin{aligned} P(t) = e^{\int \cos t dt} = e^{\sin t} &\implies y' + y \cos t = \cos t \iff e^{\sin t} (y' + y \cos t) = e^{\sin t} \cos t \\ &\iff (e^{\sin t} y)' = e^{\sin t} \cos t \iff e^{\sin t} y = \int e^{\sin t} \cos t dt \\ &\iff \boxed{y(t) = 1 + C e^{-\sin t}} \end{aligned}$$

(b) Find the solution to the initial value problem

$$y' = \cos t - y \cos t, \quad y(\pi) = 3.$$

We use the general solution found in (a) and solve for C :

$$y(0) = 1 + Ce^0 = 3 \implies C = 2 \implies \boxed{y(t) = 1 + 2e^{-\sin t}}$$

Example 2.2 (a) Find the general solution to the ODE

$$ty' = \sin t - 2y.$$

First, we write this equation as $y' + 2t^{-1}y = t^{-1} \sin t$. The integrating factor $P(t)$ is the given by

$$\begin{aligned} P(t) = e^{\int 2t^{-1} dt} = e^{2 \ln |t|} = t^2 \implies y' + 2t^{-1}y = t^{-1} \sin t &\iff t^2 y' + 2ty = t \sin t \iff (t^2 y)' = t \sin t \\ &\iff t^2 y(t) = \int t \sin t dt = -t \cos t + \int \cos t dt = -t \cos t + \sin t + C. \end{aligned}$$

Thus, the general solution is $\boxed{y(t) = Ct^{-2} - t^{-1} \cos t + t^{-2} \sin t}$

(b) Find the solution to the initial value problem

$$y' = \sin t - 2y, \quad y(\pi/2) = 0.$$

In this case we solve for the constant C using the expression preceding the boxed formula:

$$0 = 0 + 1 + C \implies C = -1 \implies \boxed{y(t) = -t^{-2} - t^{-1} \cos t + t^{-2} \sin t}$$

2.2 Separable Equations

Separable first-order ODEs are the equations of the form

$$y' = f(y) \cdot g(t), \quad y = y(t). \tag{3}$$

Equation (3) is solved by writing $y' = \frac{dy}{dt}$, moving all expressions involving y to LHS and all expressions involving t to RHS, and integrating both sides:

$$\boxed{\frac{dy}{dt} = f(y) \cdot g(t), \quad y = y(t) \implies \frac{dy}{f(y)} = g(t)dt \implies \int \frac{dy}{f(y)} = \int g(t)dt}$$

Once the two integrals are computed, one obtains a relation between y and t of the form

$$F(y) = G(t) + C \iff F(y) - G(t) = C. \tag{4}$$

These relations define solutions $y = y(t)$ of (3) *implicitly*. In some cases, it is possible to solve (4) for $y = y(t)$.

Note that this method involves division by $f = f(y)$ and may miss some of the constant solutions of (3). Such solutions are necessarily of the form $y = y^*$, where y^* is a real number such that $f(y^*) = 0$. If you are solving an IVP and it is possible to solve for $y = y(t)$ explicitly, make sure you take the correct branch, if there is more than one, of the appropriate level curve of $H = F - G$, e.g. the positive or negative square root, and not both. The correct branch is the one satisfying the initial condition $y(t_0) = y_0$.

Example 2.3 Find the general solution to the ODE

$$y' = (1+y^2)e^t.$$

Write $y' = \frac{dy}{dt}$ and split the variables:

$$\begin{aligned} y' = (1+y^2)e^t &\iff \frac{dy}{1+y^2} = e^t dt \iff \int \frac{dy}{1+y^2} = \int e^t dt \\ &\iff \tan^{-1} y = e^t + C \iff \boxed{y = \tan(e^t + C)} \end{aligned}$$

Example 2.4 (a) Find the general solution to the ODE

$$y' = y(y+2)t.$$

Write $y' = \frac{dy}{dt}$ and split the variables:

$$\begin{aligned} y' = y(y+2)t &\iff \frac{dy}{y(y+2)} = t dt \iff \frac{1}{2} \left(\frac{1}{y} - \frac{1}{y+2} \right) dy = t dt \iff \int \left(\frac{1}{y} - \frac{1}{y+2} \right) dy = \int 2t dt \\ &\iff \ln|y| - \ln|y+2| = t^2 + C \iff \ln \left| \frac{y}{y+2} \right| = t^2 + C \\ &\iff \frac{y}{y+2} = Ae^{t^2} \iff y(t) = \frac{2}{Ce^{-t^2} - 1} \end{aligned}$$

However, we also have two constant solutions: $y = 0$ and $y = -2$. The latter corresponds to $A = 0$. Thus, the general solution is $\boxed{y(t) = \frac{2}{Ce^{-t^2} - 1} \text{ and } y(t) = 0}$

Many ODEs are not separable, but some can be made separable through various manipulations. One class of such ODEs are the ODEs of the form

$$y' = f(t, y),$$

where f is a smooth function such that $f(st, sy) = f(t, y)$ for all s, t , and y . In this case, the substitution $y = tv$, where $v = v(t)$, reduces the ODE to

$$v + tv' = f(1, v).$$

This ODE is separable and can be solved for $v = v(t)$ implicitly. We then replace v by y/t .

Example 2.5 Find the general solution to the ODE

$$y' = \frac{t-y}{t+y}$$

After making the substitution $y = tv$, we get

$$\begin{aligned} y' = \frac{t-y}{t+y} &\iff v + tv' = \frac{1-v}{1+v} \iff tv' = -\frac{v^2+2v-1}{v+1} \iff \frac{v+1}{(v+1)^2-2} dv = -t^{-1} dt \\ &\iff \frac{1}{2} \ln |(v+1)^2-2| = -\ln |t| + C \iff (v+1)^2-2 = At^{-2} \iff \boxed{(t+y)^2-2t^2 = A} \end{aligned}$$

Example 2.6 Find the general solution to the ODE

$$(t^2+y^2)y' - ty = 0.$$

We first write this equation as $y' = ty/(t^2+y^2)$ and then substitute $y = tv$:

$$\begin{aligned} y' = \frac{ty}{t^2+y^2} &\iff v + tv' = \frac{v}{1+v^2} \iff tv' = -\frac{v^3}{1+v^2} \iff (v^{-3}+v^{-1})dv = -t^{-1} dt \\ &\iff -\frac{1}{2}v^{-2} + \ln |v| = -\ln |t| + C \iff v^{-2}e^{v^{-2}} = At^2 \iff \boxed{y^2 = Ce^{t^2/y^2}} \end{aligned}$$

2.3 Exact Equations

The first-order ODE

$$P(t, y) + Q(t, y)y' = 0 \quad \text{or} \quad P(t, y)dt + Q(t, y)dy = 0, \quad y = y(t), \quad (5)$$

is *exact* $P_y = Q_t$ and the functions P and Q are smooth. If this is the case, there exists a smooth function $H = H(t, y)$ such that

$$H_t \equiv \frac{\partial H}{\partial t} = P \quad \text{and} \quad H_y \equiv \frac{\partial H}{\partial y} = Q, \quad \text{or} \quad \vec{\nabla} H = P\hat{i} + Q\hat{j}, \quad \text{or} \quad dH \equiv H_t dt + H_y dy = Pdt + Qdy.$$

These three conditions are exactly the same. The function $H = H(t, y)$ can be found as follows. Using the condition $H_t = P$, we first find by integration that $H(t, y) = \tilde{H}(t, y) + \phi(y)$, where $\tilde{H}(t, y)$ is a fixed t -antiderivative of P and $\phi(y)$ determines an arbitrary function of y . We next take the y -derivative of $H(t, y)$, set it equal to Q , and thus obtain a condition $\phi(y)$. This condition should *not* involve t . If it does, either $P_y \neq Q_t$, and thus the function H does not exist, or there was a mistake made somewhere else. If we can find $H = H(t, y)$ such that $H_t = P$ and $H_y = Q$, then (5) is *implicitly* solved by

$$\boxed{P(t, y) + Q(t, y)y' = 0, \quad y = y(t) \implies H(t, y) = C \quad \text{if} \quad H_t = P \quad \text{and} \quad H_y = Q}$$

The solution curves for (5) will lie on the level curves of $H = H(t, y)$, i.e. on the curves described by the equations $H(t, y) = C$, for various constants C .

Example 2.7 Show that the equation

$$(1 - y \sin t)dt + (\cos t)dy = 0$$

is exact, and solve it.

With $P(t, y) = 1 - y \sin t$ and $Q(t, y) = \cos t$, we get:

$$\frac{\partial P}{\partial y} = -\sin t = \frac{\partial Q}{\partial t}.$$

Thus, the equation is exact. We solve it by setting

$$\begin{aligned} H(t, y) &= \int P(t) dt = \int (1 - y \sin t) dt = t + y \cos t + \phi(y) \\ Q(t, y) &= \cos t = \frac{\partial H}{\partial y} = \cos t + \phi'(y) \implies \phi'(y) = 0. \end{aligned}$$

Thus, we can take $\phi(y) = 0$, and the solution to the ODE is $\boxed{H(t, y) = t + y \cos t = C}$

Example 2.8 Show that the ODE

$$2t - y^2 + (y^3 - 2ty)y' = 0$$

is exact and solve it.

This ODE is equivalent to $(2t - y^2)dt + (y^3 - 2ty)dy = 0$. Since

$$(2t - y^2)_y = -2y = (y^3 - 2ty)_t,$$

the mixed partials are equal. Since $2t - y^2$ and $y^3 - 2ty$ are defined for all t and y , it follows that the ODE is exact. In order to solve it, we need to find $H = H(t, y)$ such that $H_t = (2t - y^2)$ and $H_y = (y^3 - 2ty)$:

$$\begin{aligned} H_t(t, y) = 2t - y^2 &\implies H(t, y) = \int (2t - y^2) dt = t^2 - ty^2 + \phi(y) \\ H_y(t, y) = y^3 - 2ty &\implies 0 - 2ty + \phi'(y) = y^3 - 2ty \implies \phi'(y) = y^3 \\ &\implies \phi(y) = \int y^3 dy = \frac{1}{4}y^4 \implies H(t, y) = \frac{1}{4}y^4 - ty^2 + t^2. \end{aligned}$$

Thus, the general solution $y = y(t)$ of the above ODE is implicitly defined by

$$\boxed{\frac{1}{4}y^4 - ty^2 + t^2 = C} \quad \text{or} \quad \boxed{y^4 - 4ty^2 + 4t^2 = C}$$

While most ODEs are not exact, many can be made exact by multiplying both sides by a nonzero *integrating factor* $\mu = \mu(t, y)$:

$$P(t, y) + Q(t, y)y' = 0 \iff \mu(t, y)P(t, y) + \mu(t, y)Q(t, y)y' = 0.$$

The latter equation is exact if $(\mu P)_y = (\mu Q)_t$. Expanding this relation, we obtain a condition involving μ , μ_t , μ_y , and the known functions P and Q . In general, this condition is very complicated. However, if we are trying to find an integrating factor μ which is a function of t only or y only,

which is not always possible, the relation simplifies significantly.

Example 2.9 Show that the equation

$$y dt + (t^2 y - t) dy = 0$$

is not exact. Suppose it has an integrating factor which is a function of t alone. Find the integrating factor and use it to solve the equation.

Since $(y)_y = 1$ and $(t^2 y - t)_t = (2ty - 1)$, the mixed partials are not equal and the equation is not exact. If $\mu = \mu(t)$ is an integrating factor for this equation,

$$\begin{aligned} (\mu(t)y)_y = (\mu(t)(t^2 y - t))_t &\iff \mu(t) = \mu'(t)(t^2 y - t) + \mu(t)(2ty - 1) \\ &\iff 2\mu(t)(1 - ty) = t\mu'(t)(ty - 1) \iff \mu'(t) = -2t^{-1}\mu(t). \end{aligned}$$

The last equation is separable, and we can solve it. One nonzero solution is $\mu(t) = 1/t^2$. After multiplying the equation by $\mu = \mu(t)$, we get an exact equation:

$$\begin{aligned} y dt + (t^2 y - t) dy = 0 &\iff \frac{y}{t^2} dt + \frac{t^2 y - t}{t^2} dy = 0 \\ &\implies H(t, y) = \int \frac{y}{t^2} dt = -\frac{y}{t} + \phi(y) \\ H_y = \frac{t^2 y - t}{t^2} &\implies -\frac{1}{t} + \phi'(y) = \frac{t^2 y - t}{t^2} \implies \phi'(y) = y \implies \phi(y) = \frac{y^2}{2}. \end{aligned}$$

Thus, $H(t, y) = -\frac{y}{t} + \frac{y^2}{2}$ and the general solution to the ODE is $\boxed{-\frac{y}{t} + \frac{y^2}{2} = C}$

Example 2.10 Show that the equation

$$y^2 + 2ty - t^2 y' = 0$$

is not exact. Suppose it has an integrating factor that is a function of y alone. Find the integrating factor and use it to solve the equation.

Since $(y^2 + 2ty)_y = (2y + 2t)$ and $(-t^2)_t = -2t$, the mixed partials are not equal and the equation is not exact. If $\mu = \mu(y)$ is an integrating factor for this equation,

$$\begin{aligned} (\mu(y)(y^2 + 2ty))_y = (-\mu(y)t^2)_t &\iff \mu'(y)(y^2 + 2ty) + \mu(y)(2y + 2t) = -\mu(y) \cdot 2t \\ &\iff (y + 2t)y\mu'(y) = -2(y + 2t)\mu(y) \iff \mu'(y) = -2y^{-1}\mu(y). \end{aligned}$$

The last equation is separable, and we can solve it. One nonzero solution is $\mu(y) = 1/y^2$. After multiplying the equation by $\mu = \mu(y)$, we get an exact equation:

$$\begin{aligned} y^2 + 2ty - t^2 y' = 0 &\iff \frac{y^2 + 2ty}{y^2} dt - \frac{t^2}{y} dy = 0 \\ &\implies H(t, y) = \int \frac{y^2 + 2ty}{y^2} dt = t + t^2 y^{-1} + \phi(y) \\ H_y = -\frac{t^2}{y^2} &\implies -t^2 y^{-2} + \phi'(y) = -\frac{t^2}{y^2} \implies \phi'(y) = 0 \implies \phi(y) = 0. \end{aligned}$$

Thus, $H(t, y) = t + t^2 y^{-1}$ and the general solution to the ODE is $\boxed{t + t^2 y^{-1} = C}$

3 Qualitative Descriptions

3.1 Structure of Solutions of Linear ODEs and of Systems of Linear ODEs

A *homogeneous linear* first-order ODE is an ODE of the form

$$y' = a(t)y, \quad y = y(t), \quad (6)$$

If $y_1 = y_1(t)$ is a nonzero solution of this equation, then $y(t) = C_1 y_1(t)$ is the general solution of (6). The general solution of any linear inhomogeneous equation

$$y' = a(t)y + f(t), \quad y = y(t), \quad (7)$$

has the form $y = y_h + y_p$, where $y_p = y_p(t)$ is a fixed *particular* solution of (7) and $y_h = y_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (6) with the same $a = a(t)$ as in (7).

A *homogeneous linear* n th ODE is an equation of the form

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \dots + a_{n-1}(t)y' + a_n(t)y, \quad y = y(t). \quad (8)$$

If $y_1 = y_1(t), \dots, y_n = y_n(t)$ is a set of n linearly independent solutions of (8), then

$$y(t) = C_1 y_1(t) + \dots + C_n y_n(t)$$

is the general solution to (8). As in the first-order case, the general solution of any linear inhomogeneous equation

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \dots + a_{n-1}(t)y' + a_n(t)y + f(t), \quad y = y(t), \quad (9)$$

has the form $y = y_h + y_p$, where $y_p = y_p(t)$ is a fixed *particular* solution of (9) and $y_h = y_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (8) with the same $a = a(t)$ as in (9).

Example 3.1 Since the characteristic polynomial for the second-equation equation

$$y'' + 5y' + 4y = 0 \quad (10)$$

is $\lambda^2 + 5\lambda + 4 = 0$, $y_1(t) = e^{-t}$ and $y_2(t) = e^{-4t}$ are solutions of (10). Since the ratio $y_1(t)/y_2(t) = e^{3t}$ is not a constant, $y_1 = y_1(t)$ and $y_2 = y_2(t)$ are linearly independent solutions. Thus,

$$y(t) = C_1 y_1(t) + C_2 y_2(t) = C_1 e^{-t} + C_2 e^{-4t}$$

is the general solution of (10). Using the *Method of Undetermined Coefficients* or the *Laplace Transform*, we can find a particular solution to the inhomogeneous equation

$$y'' + 5y' + 4y = te^{-t}, \quad (11)$$

such as $y_p(t) = \frac{1}{6}t^2 e^{-t} - \frac{1}{9}te^{-t}$. Thus,

$$y(p) = C_1 y_1(t) + C_2 y_2(t) + y_p(t) = C_1 e^{-t} + C_2 e^{-4t} + \frac{1}{6}t^2 e^{-t} - \frac{1}{9}te^{-t}$$

is the general solution of (11).

Example 3.2 By direct substitution into the ODE, we can verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to

$$t^2y'' + 3ty' - 3y = 0. \quad (12)$$

These solutions can be found by trying $y(t) = Ct^\alpha$ and finding that $\alpha = 1$ or $\alpha = -3$. Since the ratio $y_1(t)/y_2(t) = t^4$ is not a constant, $y_1 = y_1(t)$ and $y_2 = y_2(t)$ are linearly independent solutions. Thus,

$$y(t) = C_1y_1(t) + C_2y_2(t) = C_1t + C_2t^{-3}$$

is the general solution of (12). Using *Variation of Parameters*, we can find a particular solution to the inhomogeneous equation

$$t^2y'' + 3ty' - 3y = t^{-1}, \quad (13)$$

such as $y_p(t) = -\frac{1}{4}t^{-1}$. Thus,

$$y(p) = C_1y_1(t) + C_2y_2(t) + y_p(t) = C_1t + C_2t^{-3} - \frac{1}{4}t^{-1}$$

is the general solution of (13).

Finally, a *homogeneous system* of linear first-order ODEs is an ODE of the form

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y} = \mathbf{y}(t), \quad (14)$$

where $A = A(t)$ is an $n \times n$ -matrix, possibly dependent on t . If $\mathbf{y}_1 = \mathbf{y}_1(t), \dots, \mathbf{y}_n = \mathbf{y}_n(t)$ is a set of n linearly independent solutions of (8), then

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + \dots + C_n\mathbf{y}_n(t)$$

is the general solution to (14). As in the previous two cases, the general solution to any inhomogeneous system of linear first-order ODEs

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y} = \mathbf{y}(t), \quad (15)$$

has the form $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p$, where $\mathbf{y}_p = \mathbf{y}_p(t)$ is a fixed *particular* solution of (15) and $\mathbf{y}_h = \mathbf{y}_h(t)$ is the general solution of the corresponding homogeneous equation, i.e. (14) with the same $A = A(t)$ as in (15).

3.2 Existence and Uniqueness Theorems

According to the Existence and Uniqueness Theorem for first-order ODEs, the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0, \quad (16)$$

- (a) has a solution $y = y(t)$ near t_0 if the function f is continuous near (t_0, y_0) ;
- (b) has a unique solution $y = y(t)$ near t_0 if f and $\partial f / \partial y$ are continuous near (t_0, y_0) .

Note that the applicability of this theorem to each given IVP depends on both the function y and

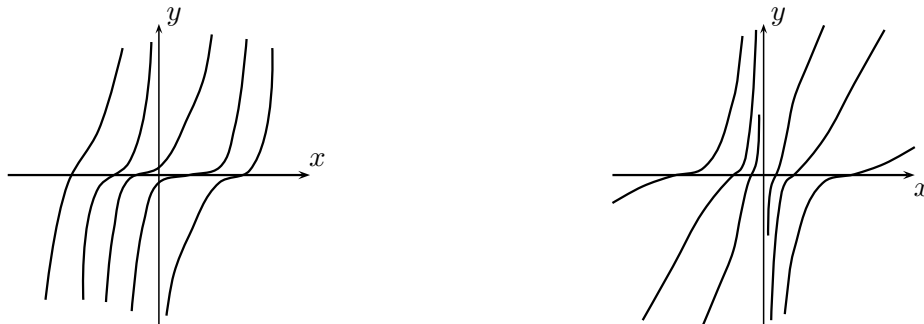


Figure 1: Sketches of Solution Curves for ODEs in (17) and in (18)

the initial condition (t_0, y_0) . Whenever the assumptions of (a) or (b), the IVP is guaranteed to have a solution or a unique solution. However, even if they are not satisfied, the IVP may still have a solution or a unique solution.

Example 3.3 (a) For what values of t_0 and y_0 , the IVP

$$y' = \sqrt{|t-1|}y^{2/3}, \quad y(t_0) = y_0, \quad (17)$$

is guaranteed by the Existence and Uniqueness Theorem to have a solution?

The function $f = f(t, y) = \sqrt{|t-1|}y^{2/3}$ is continuous everywhere. Thus, the existence part of the theorem guarantees that (17) has a solution for $\boxed{\text{all } (t_0, y_0)}$

(b) For what values of t_0 and y_0 , IVP (17) is guaranteed by the Existence and Uniqueness Theorem to have a unique solution?

Since $\partial f / \partial y = (2/3)\sqrt{|t-1|}y^{-1/3}$, $\partial f / \partial y$ is continuous near $y \neq 0$ and is not even defined at $y = 0$. Thus, the uniqueness part of the theorem guarantees that IVP (17) has a unique solution if $\boxed{y_0 \neq 0}$ and does not apply if $y_0 = 0$.

(c) For what values of t_0 and y_0 , IVP (17) has a solution?

By (a), (17) is guaranteed to have a solution for $\boxed{\text{all } (t_0, y_0)}$

(d) For what values of t_0 and y_0 , IVP (17) has a unique solution?

By (b), (17) is guaranteed to have a unique solution if $y_0 \neq 0$. Thus, we need to somehow determine if (17) has a unique solution for $y_0 = 0$. Since the ODE is separable, we can solve it by separating variables:

$$y' = \sqrt{|t-1|}y^{2/3} \iff y^{-2/3}dy = \sqrt{|t-1|}dt \iff 3y^{1/3} = \frac{2}{3}(g(t) + C),$$

$$\text{where } g(t) = \begin{cases} |t-1|^{3/2}, & \text{if } t > 1; \\ -|t-1|^{3/2}, & \text{if } t < 1. \end{cases}$$

This separation of variables approach misses the solution $y(t) = 0$. The new solutions we found are defined for all $t \neq 0$. For any t_0 , the above solution with $C = -g(t_0)$ is another solution to (17) with $(t_0, y_0) = (t_0, 0)$. We conclude that IVP (17) has a unique solution if $\boxed{y_0 \neq 0}$

Example 3.4 (a) For what values of t_0 and y_0 , the IVP

$$|t|y' = \sqrt{|y|}, \quad y(t_0) = y_0, \quad (18)$$

is guaranteed by the Existence and Uniqueness Theorem to have a solution?

Before applying the theorem, we need to rewrite the ODE in the *normal form*:

$$y' = |t|^{-1}\sqrt{|y|} = f(t, y), \quad y(t_0) = y_0. \quad (19)$$

The function $f = f(t, y)$ is continuous near $t \neq 0$ and is not even defined at $t = 0$. Thus, the existence part of the theorem guarantees that (18) has a solution if $t_0 \neq 0$ and does not apply if $t_0 = 0$.

(b) For what values of t_0 and y_0 , IVP (18) is guaranteed by the Existence and Uniqueness Theorem to have a unique solution?

By part (a), we only need $t_0 \neq 0$. Since $\partial f / \partial y = \pm 1 / 2t\sqrt{|y|}$, $\partial f / \partial y$ is continuous near $y \neq 0$ and is not even defined at $y = 0$. Thus, the uniqueness part of the theorem guarantees that IVP (18) has a unique solution if $t_0 \neq 0$ and $y_0 \neq 0$ and does not apply if $t_0 = 0$ or $y_0 = 0$.

(c) For what values of t_0 and y_0 , IVP (18) has a solution?

By (a), (18) is guaranteed to have a solution if $t_0 \neq 0$. So, we need to somehow determine if (18) has a solution for $t_0 = 0$. If $y = y(t)$ is a solution to the ODE (18) and is defined at $t = 0$, then $0 \cdot y'(0) = \sqrt{|y(0)|}$. Thus, IVP (18) has no solution if $t_0 = 0$ and $y_0 \neq 0$. On the other hand, $y(t) = 0$ is a solution to (18) with $t_0 = 0$ and $y_0 = 0$. We conclude that (18) has a solution if $t_0 \neq 0$ or $(t_0, y_0) = (0, 0)$.

(d) For what values of t_0 and y_0 , IVP (18) has a unique solution?

By (a), (18) is guaranteed to have a unique solution if $t_0 \neq 0$ and $y_0 \neq 0$. By (c), (18) has no solution if $t_0 = 0$ and $y_0 \neq 0$. So, we need to somehow determine if (18) has a unique solution for $y_0 = 0$. Since the ODE is separable, we can solve it by separating variables:

$$|t|y' = \sqrt{|y|} \iff \frac{dy}{\sqrt{|y|}} = \frac{dt}{|t|} \iff 2h(y) = g(t) + C, \quad \text{where}$$

$$g(t) = \begin{cases} \ln t, & \text{if } t > 0; \\ -\ln |t|, & \text{if } t < 0; \end{cases} \quad \text{and} \quad h(y) = \begin{cases} \sqrt{y}, & \text{if } y > 0; \\ -\sqrt{|y|}, & \text{if } y < 0. \end{cases}$$

From this, we conclude that

$$y(t) = \begin{cases} -\frac{1}{4}(\ln |t| + C)|\ln |t| + C|, & t \in (-\infty, 0); \\ \frac{1}{4}(\ln |t| + C)|\ln |t| + C|, & t \in (0, \infty). \end{cases}$$

This separation of variables approach misses the solution $y(t) = 0$. The new solutions we found are not defined for $t = 0$ and thus $y(t) = 0$ is the only solution to (18) with $(t_0, y_0) = (0, 0)$. On the other hand, if $t_0 \neq 0$, the above solution with $C = -\ln |t_0|$ is another solution to (18) with $(t_0, y_0) = (t_0, 0)$. We conclude that (18) has a unique solution if $t_0 \neq 0$ and $y_0 \neq 0$ OR $(t_0, y_0) = (0, 0)$.

In Example 3.3, the Existence and Uniqueness Theorem predicts all cases when the IVP has a solution and when it has a unique solution. On the other hand, in Example 3.4, there is one case, $(t_0, y_0) = (0, 0)$, in which the IVP has a unique solution, while the theorem cannot be used to predict even the existence of a solution. We plot solution curves for the ODEs in (17) and in (18) in Figure 1. There is a solution curve through every point (t_0, y_0) for which the corresponding IVP has a solution. Furthermore, solution curves intersect precisely at the points (t_0, y_0) for which only the uniqueness property for the corresponding IVP fails.

The Existence and Uniqueness Theorem for the first-order ODEs applies, word for word, to initial value problems involving systems of first-order equations. In other words, the initial value problem

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \quad \mathbf{y}(t_0) = \mathbf{y}_0,$$

(a) has a solution $\mathbf{y} = \mathbf{y}(t)$ near t_0 if the function \mathbf{f} is continuous near (t_0, \mathbf{y}_0) ;

(b) has a unique solution $\mathbf{y} = \mathbf{y}(t)$ near t_0 if \mathbf{f} and $\partial\mathbf{f}/\partial\mathbf{y}$ are continuous near (t_0, \mathbf{y}_0) .

Since many high-order equations and systems of high-order equations can be re-written as systems of first-order equations, this theorem has implications for the existence and uniqueness of solutions to initial value problems involving high-order equations and systems of high-order equations.

The general Existence and Uniqueness Theorem makes it possible to approximately sketch solution curves just by looking at direction fields and to use numerical methods. If $\partial\mathbf{f}/\partial\mathbf{y}$ is not continuous near (t_0, \mathbf{y}_0) , we may not be able to obtain any error estimate for numerical methods that decay to zero as the step size decreases, as can be seen from PS5-Problem 4.

Sorry for the delay with this solution guide. Good luck with all your exams.