# Math53: Ordinary Differential Equations
# Autumn 2004

## Unit 4 Summary
*Systems of Linear ODEs*

**Extremely Important:** linear independence; basis for a vector space; matrix invertability and determinants; characteristic polynomial of a matrix; eigenvalues and eigenvectors; generalized eigenvectors and eigenspaces; matrix exponential.

**Very Important:** solving systems of linear first-order ODEs with constant coefficients, homogeneous and inhomogeneous; phase-plane sketches for planar autonomous systems of linear first-order ODEs with constant coefficients; stability of the origin as an equilibrium point of such systems; fundamental matrix; structure of solutions of homogeneous and inhomogeneous systems of linear first-order ODEs.

**Important:** high-order ODEs and systems of first-order ODEs; real form of general solution for systems with complex eigenvalues; uniqueness and existence theorem for systems of linear ODEs.

### Linear Algebra

Throughout this section $A$ denotes an $n \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

(1) Matrix $A$ is *nonsingular* if for every $\mathbf{v} \in \mathbb{R}^n$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{v} \quad \text{or} \quad \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{if} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Matrix $A$ is *invertible* if it has an *inverse*, i.e. there exists a matrix $B$ such that $AB = I = BA$, where $I = I_n$ is the *identity matrix*. If $AB = I$, then $BA = I$, provided that $A$ and $B$ are square matrices. If $AB = I$ and $AC = I$, then $B = C$. Thus, if $A$ has an inverse, it is unique, and denoted by $A^{-1}$. Furthermore,

$$\boxed{A \text{ is nonsingular} \iff A \text{ is invertible} \iff \det A \neq 0}$$

If $\det A \neq 0$, in the $n = 2$ case $A^{-1}$ is given by

$$\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \tfrac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det A = ad - bc}$$

In general, there is a three-step procedure for computing $A^{-1}$. The last step of this procedure involves division by $\det A$. If $A$ and $B$ are square matrices,

$$\boxed{\det(AB) = (\det A)\cdot(\det B) = \det(BA), \qquad \text{but} \qquad \det(A{+}B) \neq (\det A) + (\det B)}$$

(2) The set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$, or in any *vector space*, is *linearly independent* if

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}, \quad c_1, \ldots, c_k \in \mathbb{R} \;\; (\text{or } \mathbb{C}) \qquad \Longrightarrow \qquad c_1, \ldots, c_k = 0.$$

In other words, no nontrivial *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the zero vector $\mathbf{0}$. The set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$, or in any *vector space* $V$, is a *basis* for $\mathbb{R}^n$, or for $V$, if for every $\mathbf{v}$ in $\mathbb{R}^n$, or in $V$, there exists a *unique* tuple $(c_1, \ldots, c_n)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n.$$

Equivalently, the set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a *basis* for $V$ if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and *span* $V$, i.e. for every $\mathbf{v}$ in $V$, there exists a tuple $(c_1, \ldots, c_n)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n.$$

*Can you show that these two definitions are equivalent*? In the case of $\mathbb{R}^n$:
(i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ if and only if
(ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if
(iii) $\det\begin{pmatrix} | & \ldots & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & \ldots & | \end{pmatrix} \neq 0.$

(3) An *eigenvector* $\mathbf{v}$ for $A$ with *eigenvalue* $\lambda{\in}\mathbb{R}$ is a nonzero column $n$-vector such that

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}} \quad \text{or} \quad \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix} \quad \text{if} \quad \mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

If $\mathbf{v}$ is an eigenvector for $A$ with eigenvalue $\lambda$, so is $c\mathbf{v}$ for any number $c$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for $A$ with the *same* eigenvalue $\lambda$, so is $\mathbf{v}_1{+}\mathbf{v}_2$. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for $A$ with *distinct eigenvalues* $\lambda_1, \ldots, \lambda_k$, i.e. $\lambda_i \neq \lambda_j$ whenever $i \neq j$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. If some of these eigenvalues are the same, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ may or may not be linearly independent.

(4) The eigenvalues of $A$ are the roots of the *characteristic polynomial* for $A$:

$$\det\big(A{-}\lambda I\big) = \det\begin{pmatrix} a_{11}{-}\lambda & a_{12} & \ldots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22}-\lambda & & & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n,n-1} & a_{nn}{-}\lambda \end{pmatrix}$$

However, repeated roots of the characteristic polynomial may or may not correspond to different linearly independent eigenvectors. If the multiplicity of a root $\lambda$ of the characteristic polynomial is $q$, there exist $q$ linearly independent *generalized eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_q$ for $A$ with eigenvalue $\lambda$, i.e.

$$\boxed{(A-\lambda)^r \mathbf{v}_i = \mathbf{0} \quad \text{for some} \quad \mathrm{r}}$$

In fact, $r = q$ works in the given case. If $\mathbf{v}_i$ is an actual eigenvector, $r = 1$ suffices, by definition. Furthermore, $\mathbf{v}_1, \ldots, \mathbf{v}_q$ can be chosen in such a way that

$$A\mathbf{v}_1 = \lambda \mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_{i+1} = \mathbf{v}_i + \lambda \mathbf{v}_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, q-1.$$

Thus, it is always possible to find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of generalized eigenvectors for $A$ such that

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i + a_i \mathbf{v}_{i-1}, \qquad \text{where} \qquad a_i = 0 \text{ or } a_i = 1, \quad a_i = 0 \ \text{ if } i = 1 \text{ or } \lambda_{i-1} \neq \lambda_i,$$

where $\lambda_i$ is the eigenvalue corresponding to the generalized eigenvector $\mathbf{v}_i$. Then,

$$A = BDB^{-1}, \qquad \text{where} \qquad D = \begin{pmatrix} \lambda_1 & a_2 & 0 & \ldots \\ 0 & \lambda_2 & a_3 & \ldots \\ \vdots & \ldots & \ddots & \ddots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} | & \ldots & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & \ldots & | \end{pmatrix}. \tag{1}$$

*Can you check this?* The above basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and matrix $B$, however, may be complex. In such a case, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a $\mathbb{C}$-basis for $\mathbb{C}^n$, not an $\mathbb{R}$-basis for $\mathbb{R}^n$.

(5) If $A$ is an $n \times n$ matrix, the *exponential* of $A$ is the $n \times n$ matrix given by

$$\boxed{e^A = I_n + \tfrac{1}{1!}A + \tfrac{1}{2!}A^2 + \tfrac{1}{3!}A^3 + \ldots = \sum_{k=0}^{k=\infty} \tfrac{1}{k!}A^k} \qquad \text{where} \ \ A^0 = I_n, \ A^2 = AA, \ A^3 = AAA, \ldots$$

Note that this is the same power series as for $e^a$, if $a$ is a real or complex number. By definition, if $A$ is the zero matrix, $e^A = I_n$. Another property of the matrix exponential is

$$\boxed{\text{If } AB = BA, \quad \text{then} \quad e^{A+B} = e^A e^B = e^B e^A} \tag{2}$$

Using this property, we can conclude that

(i) $e^A$ is an invertible matrix and $(e^A)^{-1} = e^{-A}$;

(ii) if $H(t) = e^{tA}$, then $H'(t) = AH(t) = H(t)A$.

If $A$ is a diagonal matrix, then $e^A$ is also a diagonal matrix, and the diagonal entires of $e^A$ are the exponentials of the corresponding diagonal entries of $A$. For example,

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \Longrightarrow \qquad e^A = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix}$$

However, if $A$ is not a diagonal matrix, the entries of $e^A$ are *not* usually the exponentials of the entries of $A$, and it may be very hard to determine them directly from the power series definition of the exponential. On the other hand, it may be possible to find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$, or

$\mathbb{C}^n$, such that $e^A\mathbf{v}_i$ is easy to compute for each $i$. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, an arbitrary vector $\mathbf{v}$ has the form

$$\mathbf{v} = C_1\mathbf{v}_1 + \ldots + C_n\mathbf{v}_n, \quad C_1, \ldots, C_n \in \mathbb{C} \qquad \Longrightarrow \qquad e^A\mathbf{v} = C_1 e^A\mathbf{v}_1 + \ldots + C_n e^A\mathbf{v}_n.$$

This is usually sufficient for solving systems of linear ODEs with constant coefficients. The product $e^A\mathbf{v}_i$ can be computed for generalized eigenvectors of $A$. For example,

$$\boxed{A\mathbf{v}_1 = \lambda\mathbf{v}_1, \quad A\mathbf{v}_2 = a\mathbf{v}_1 + \lambda\mathbf{v}_2 \quad \Longrightarrow \quad e^A\mathbf{v}_1 = e^\lambda\mathbf{v}_1, \quad e^A\mathbf{v}_2 = ae^\lambda\mathbf{v}_1 + e^\lambda\mathbf{v}_2} \tag{3}$$

These two relations are sufficient for the $n=2$ case.

(6) In order to compute $e^A$ for an arbitrary square matrix, one makes use of the relation

$$e^{BDB^{-1}} = Be^DB^{-1}$$

and (eq1). The exponential of the matrix $D$ as in (eq1) can be computed directly from the definition. This approach is analogous to the one described in Section 9.8: if $\{\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)\}$ is a fundamental set of solutions for the ODE, then

$$\boxed{Y(t) = \begin{pmatrix} | & \ldots & | \\ \mathbf{y}_1(t) & \ldots & \mathbf{y}_n(t) \\ | & \ldots & | \end{pmatrix} \qquad \Longrightarrow \qquad e^{tA} = Y(t)Y(0)^{-1}} \tag{4}$$

On the other hand, if $A$ has only one eigenvalue $\lambda$, $(A-\lambda I)^n$ is the zero matrix, and the power series for the exponential of $A-\lambda I$ quickly truncates. Since $\lambda I$ commutes with all matrices, one can compute $e^A$ by using (eq2) with $A=\lambda I$ and $B=A-\lambda I$.

## Systems of Linear ODEs with Constant Coefficients

(1) A system of first-order linear ODEs with constant coefficients can be written as

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \quad \mathbf{y}=\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix}, \quad \mathbf{f}=\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}.$$

This system is called *homogeneous* if $\mathbf{f} = 0$. A system of first-order linear ODEs with constant coefficients can be solved by the integrating factor method for first-order linear ODEs:

$$\boxed{\mathbf{y}' = A\mathbf{y} + \mathbf{f} \qquad \Longrightarrow \qquad \mathbf{y}(t) = e^{tA}\mathbf{v} + e^{tA}\int_{t_0}^t e^{-sA}\mathbf{f}(s)\,ds, \quad \mathbf{v} \in \mathbb{R}^n} \tag{5}$$

Note that the function $\mathbf{y}_h = \mathbf{y}_h(t)$ defined by (eq5) with $\mathbf{f}=\mathbf{0}$, i.e. the first term on the right-hand side, is the general solution of the corresponding homogeneous system of ODEs. Thus, the general solution to an inhomogeneous system of ODEs is given by

$$\boxed{\mathbf{y}' = A\mathbf{y} + \mathbf{f} \qquad \Longrightarrow \qquad \mathbf{y} = \mathbf{y}_p + \mathbf{y}_h} \tag{6}$$

where $\mathbf{y}_p$ is a solution to the inhomogeneous system, e.g. the function $\mathbf{y}$ corresponding to $\mathbf{v} = \mathbf{0}$ to (eq5). The relation (eq6) is valid for any system of linear ODEs, with constant or non-constant coefficients.

(2) The main difficulty in solving a system of linear ODEs with constant coefficients is dealing with the terms in (eq5) involving $e^{tA}$. This is not difficult to do if there is a basis for $\mathbb{R}^n$, or $\mathbb{C}^n$, of eigenvectors for $A$:

$$\boxed{\begin{array}{l} \mathbf{y}' = A\mathbf{y} \quad \Longrightarrow \quad \mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + C_n e^{\lambda_n t}\mathbf{v}_n, \quad C_1, \ldots, C_n \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \text{if} \quad \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \text{ is a basis for } \mathbb{R}^n \text{ (or } \mathbb{C}^n) \quad \text{and} \quad A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ldots, A\mathbf{v}_n = \lambda_n \mathbf{v}_n \end{array}} \qquad (7)$$

(3) If we are looking for real solutions, we will need to rewrite (eq7) in a different way if some of the eigenvalues $\lambda_i$ are complex, and not real. If $\mathbf{v}_i$ is an eigenvector for $A$ with eigenvalue $\lambda_i$ and $\lambda_i$ is complex, $\bar{\mathbf{v}}_i$ is an eigenvector for $A$ with eigenvalue $\bar{\lambda}_i$ and the vectors $\mathbf{v}_i$ and $\bar{\mathbf{v}}_i$ are linearly independent. Thus, if $n = 2$ and $A$ has an eigenvector $\mathbf{v}_1$ with a complex eigenvalue $\lambda_1$, then the two eigenvalues of $A$ are complex conjugates, $\lambda_1, \lambda_2 = a \pm ib$, and $\mathbb{C}^2$ has a basis of conjugate eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2 = \mathbf{w}_1 \pm i\mathbf{w}_2\}$. The general solution in this case can be written as

$$\boxed{\begin{array}{ll} \mathbf{y}' = A\mathbf{y} \Longrightarrow & \begin{array}{l} \mathbf{y}(t) = \big(A_1 \cos bt + A_2 \sin bt\big)e^{at}\mathbf{w}_1 + \big(A_2 \cos bt - A_1 \sin bt\big)e^{at}\mathbf{w}_2, \\ \quad = e^{at}(\mathbf{w}_1\ \mathbf{w}_2)\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad A_1, A_2 \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \text{if} \quad \lambda_1 = a + ib, \quad b \neq 0, \quad \mathbf{v}_1 = \mathbf{w}_1 + i\mathbf{w}_2 \neq 0, \text{ and } A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \end{array} \end{array}}$$

This expression is obtained by setting $C_1, C_2 = (A_1 \mp iA_2)/2$ in (eq7). Note that if $A_1$ and $A_2$ are arbitrary complex constants, so are $C_1$ and $C_2$. On the other hand, the solution corresponding to $A_1$ and $A_2$ is real if and only if $A_1$ and $A_2$ are real.

(4) Another potential problem with (eq7) is that $\mathbb{R}^n$, or $\mathbb{C}^n$, may not have a basis of eigenvectors for $A$. If so, we can use a basis of generalized eigenvectors. If $n = 2$ and $A$ has only one eigenvalue $\lambda$, by (eq3),

$$\boxed{\begin{array}{l} \mathbf{y}' = A\mathbf{y} \quad \Longrightarrow \quad \mathbf{y}(t) = \big(C_1 e^{\lambda t} + C_2 a t e^{\lambda t}\big)\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2, \quad C_1, C_2 \in \mathbb{R} \text{ (or } \mathbb{C}) \\ \text{if} \quad \mathbf{v}_1, \mathbf{v}_2 \text{ are lin. indep., } A\mathbf{v}_1 = \lambda \mathbf{v}_1, \text{ and } A\mathbf{v}_2 = a\mathbf{v}_1 + \lambda \mathbf{v}_2 \end{array}}$$

Once an eigenvector $\mathbf{v}_1$ for the eigenvalue $\lambda$ is found, $\mathbf{v}_2$ can be taken to be any vector in $\mathbb{R}^2$ which is not a multiple of $\mathbf{v}_1$, and the number $a$ is determined by computing $A\mathbf{v}_2$.

(5) The general solution to an inhomogeneous system of linear first-order ODEs with constant-coefficients is given by (eq5), or more generally by

$$\boxed{\mathbf{y}' = A\mathbf{y} + \mathbf{f} \quad \Longrightarrow \quad \mathbf{y}(t) = (e^{tA}B)\mathbf{v} + (e^{tA}B)\int_{t_0}^{t}(e^{sA}B)^{-1}\mathbf{f}(s)\,ds, \quad \mathbf{v} \in \mathbb{R}^n} \qquad (8)$$

for any invertible $n \times n$-matrix $B$. For a good choice of $B$, the product $e^{tA}B$ may be easier to compute than $e^{tA}$. For example,

$$\boxed{\begin{array}{l} \mathbf{y}' = A\mathbf{y} + \mathbf{f} \quad \Longrightarrow \quad \mathbf{y}(t) = Y(t)\mathbf{v} + Y(t)\int_{t_0}^{t} Y(s)^{-1}\mathbf{f}(s)\,ds, \quad \mathbf{v} \in \mathbb{R}^n \\ \text{if} \quad Y = Y(t) \text{ is a fundamental matrix for } \mathbf{y}' = A\mathbf{y} \text{ as in (eq4)} \end{array}}$$

(6) A solution to an initial value problem can be obtained directly by

$$\boxed{\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(t_0) = \mathbf{y}_0 \qquad \Longrightarrow \qquad \mathbf{y}(t) = e^{tA}\big(e^{-t_0 A}\mathbf{y}_0 + \textstyle\int_{t_0}^{t} e^{-sA}\mathbf{f}(s)\,ds\big)}$$

More generally, if $Y = Y(t)$ is any fundamental matrix for $\mathbf{y}' = A\mathbf{y}$,

$$\boxed{\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(t_0) = \mathbf{y}_0 \qquad \Longrightarrow \qquad \mathbf{y}(t) = Y(t)\big(Y(t_0)^{-1}\mathbf{y}_0 + \textstyle\int_{t_0}^{t} Y(s)^{-1}\mathbf{f}(s)\,ds\big)}$$

*CAUTION:* What determines whether the general solution of $\mathbf{y}' = A\mathbf{y}$ involves a term with an extra $t$, e.g. $te^{\lambda t}$, is *NOT* whether $A$ has multiple eigenvalues, but whether it is possible, or not, to find enough eigenvectors of $A$ to form a basis for $\mathbb{R}^n$. For example, the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

both have a double (generalized) eigenvalue of $\lambda = 1$ and an eigenvector

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The first matrix also has

$$\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

as well any other nonzero vector, as an eigenvector. Thus, it is possible to find a basis of eigenvectors for $A_1$ and the general solution of $\mathbf{y}' = A_1\mathbf{y}$ will not involve an extra $t$. On the other hand, it is impossible to find an eigenvector for $A_2$ which is linearly independent of $\mathbf{v}_1$. Thus, the general solution of $\mathbf{y}' = A_2\mathbf{y}$ will involve an extra $t$. Since

$$A_2\mathbf{v}_2 = 1 \cdot \mathbf{v}_1 + \lambda \cdot \mathbf{v}_2,$$

from (2) and (4) above we conclude that

$$\begin{aligned} \mathbf{y}' = A_1\mathbf{y} \qquad &\Longrightarrow \qquad \mathbf{y}(t) = C_1 e^t \mathbf{v}_1 + C_2 e^t \mathbf{v}_2 \\ \mathbf{y}' = A_2\mathbf{y} \qquad &\Longrightarrow \qquad \mathbf{y}(t) = (C_1 + C_2 t)e^t \mathbf{v}_1 + C_2 e^t \mathbf{v}_2. \end{aligned}$$

Both of these expressions are in fact special cases of (4), with $a = 0$ in the first case and $a = 1$ in the second case.

## Qualitative Descriptions

(1) As is the case for linear ODEs, every initial-value problem

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad A = A(t), \quad \mathbf{f} = \mathbf{f}(t), \tag{9}$$

has a unique solution, provided the functions $A$ and $\mathbf{f}$ are continuous near $t_0$. Furthermore, the interval of the existence of the solution to (eq9) is the largest interval on which $A$ and $\mathbf{f}$ are defined. If $A$ is a constant matrix, it follows that the phase-space solution curves for the system $\mathbf{y}'=A\mathbf{y}$ do not intersect. *Can you explain why?*

(2) Every homogeneous system of linear ODEs $\mathbf{y}'=A\mathbf{y}$ has an equilibrium solution, $\mathbf{y}(t)=\mathbf{0}$. This solution can be *asymptotically stable*, *stable*, or *unstable*. If $A$ is a constant matrix and the real part of every eigenvalue of $A$ is negative, all solutions $\mathbf{y}=\mathbf{y}(t)$ approach $\mathbf{0}$ at $t\longrightarrow\infty$, and thus $\mathbf{0}$ is an *asymptotically stable* equilibrium point of the system. If the real parts of some eigenvalues of $A$ are negative and of the others are zero, some solutions $\mathbf{y}=\mathbf{y}(t)$ approach $\mathbf{0}$ at $t\longrightarrow\infty$, while others approach closed orbits. In this case, $\mathbf{0}$ is a *stable* equilibrium point of the system, as every solution starting near $\mathbf{0}$ stays near $\mathbf{0}$. Finally, if the real part of at least one eigenvalue of $A$ is positive, some solutions $\mathbf{y}=\mathbf{y}(t)$ move away from $\mathbf{0}$ and approach $\infty$ at $t\longrightarrow 0$, and thus $\mathbf{0}$ is an *unstable* equilibrium point of the system.

(3) If $A$ is a constant matrix, the system $\mathbf{y}'=A\mathbf{y}$ is *autonomous*, i.e. it does not involve $t$ explicitly. Thus, if $\mathbf{y}=\mathbf{y}(t)$ is a solution to this system, so is $\tilde{\mathbf{y}}(t)=\mathbf{y}(t-c)$. The latter solution traces the same curve $\mathbf{y}(t)$ in $\mathbb{R}^n$, but is delayed by time $c$. For this reason, the qualitative behavior of solutions of $\mathbf{y}'=A\mathbf{y}$ is well represented by the non-intersecting curves $\mathbf{y}(t)$ traced out in the *phase space*, i.e. $\mathbb{R}^n$.

(4) While systems of first-order ODEs arise in applications by themselves, they can also be used to replace high-order ODEs. For example, the initial value problem

$$y''' + y'y'' + ty = 0, \quad y(t_0) = y_0, \;\; y'(t_0) = y_1, \;\; y''(t_0) = y_2,$$

is equivalent to the initial value problem

$$\begin{pmatrix} y \\ u \\ v \end{pmatrix}' = \begin{pmatrix} u \\ v \\ -uv - ty \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} y(0) \\ u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}.$$

*Can you explain why?* Such replacements are often useful, because many numerical methods and methods of qualitative analysis apply only to first-order ODEs and systems of first-order ODEs.

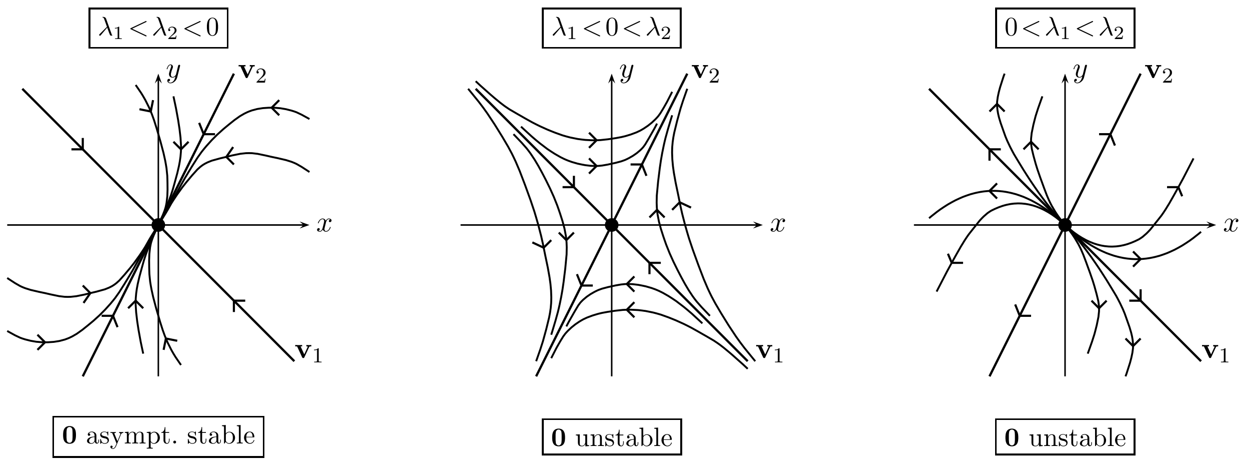
## Phase-Plane Portraits for Autonomous Systems

(1) In general, a solution $\mathbf{y}=\mathbf{y}(t)$ to a system of ODEs, such as $\mathbf{y}'=A\mathbf{y}+\mathbf{f}$, is a vector-valued function. In other words, $\mathbf{y}(t)$ is a point in $\mathbb{R}^n$ for each $t$. As the time $t$ changes, the point $\mathbf{y}(t)$ traces a curve in the phase space $\mathbb{R}^n$, i.e. the space of all possible states $\mathbf{y}$. Such curves are *phase-space solution curves* for the ODE. They do not specify what $\mathbf{y}(t)$ is at a given moment $t$, but they do show all values of $\mathbf{y}(t)$ taken as $t$ changes from $-\infty$ to $\infty$. If the system is autonomous, i.e. does not involve $t$ explicitly, such as $\mathbf{y}'=A\mathbf{y}$, the phase-space solution curves do not intersect. Below there is a discussion of what phase-plane portraits look like for $A=2{\times}2$ *const* in the most important cases.

(2) First, suppose $A$ has two distinct real nonzero eigenvalues, $\lambda_1$ and $\lambda_2$. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be corresponding eigenvectors. In this case, the general real solution is given by

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2, \qquad C_1, C_2 \in \mathbb{R}. \tag{10}$$

The phase-plane portraits in the three main cases are shown below.




In all three portraits, the origin $\mathbf{y} = \mathbf{0}$ is an equilibrium point, as is the case for all homogeneous linear equations $\mathbf{y}' = A\mathbf{y}$. This one-point solution curve corresponds to $C_1 = C_2 = 0$ in (eq10). All three portraits feature four distinguished rays, which correspond to the two eigenspaces of the matrix $A$, i.e. to the linear spans of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The two $\mathbf{v}_1$-rays are the solution curves for the ODE described by (eq10) with $C_1 \neq 0$ and $C_2 = 0$. Similarly, the two $\mathbf{v}_2$-rays are the solution curves described by (eq10) with $C_1 = 0$ and $C_2 \neq 0$.

If $C_1, C_2 \neq 0$, the second term in (eq10) dominates the first as $t \longrightarrow \infty$. Thus, the second term determines the slopes of the solution curves for $C_1, C_2 \neq 0$ as $t \longrightarrow \infty$. In other words, as $t \longrightarrow \infty$, such solution curves come closer and closer to being parallel to the $\mathbf{v}_2$-line. Similarly, as $t \longrightarrow -\infty$, these solution curves come closer and closer to being parallel to the $\mathbf{v}_1$-line. In the case of the middle sketch, these solution curves also approach the corresponding line. However, this is the case only in one of the two cases, $t \longrightarrow \infty$ or $t \longrightarrow -\infty$, in the first and the last sketch.

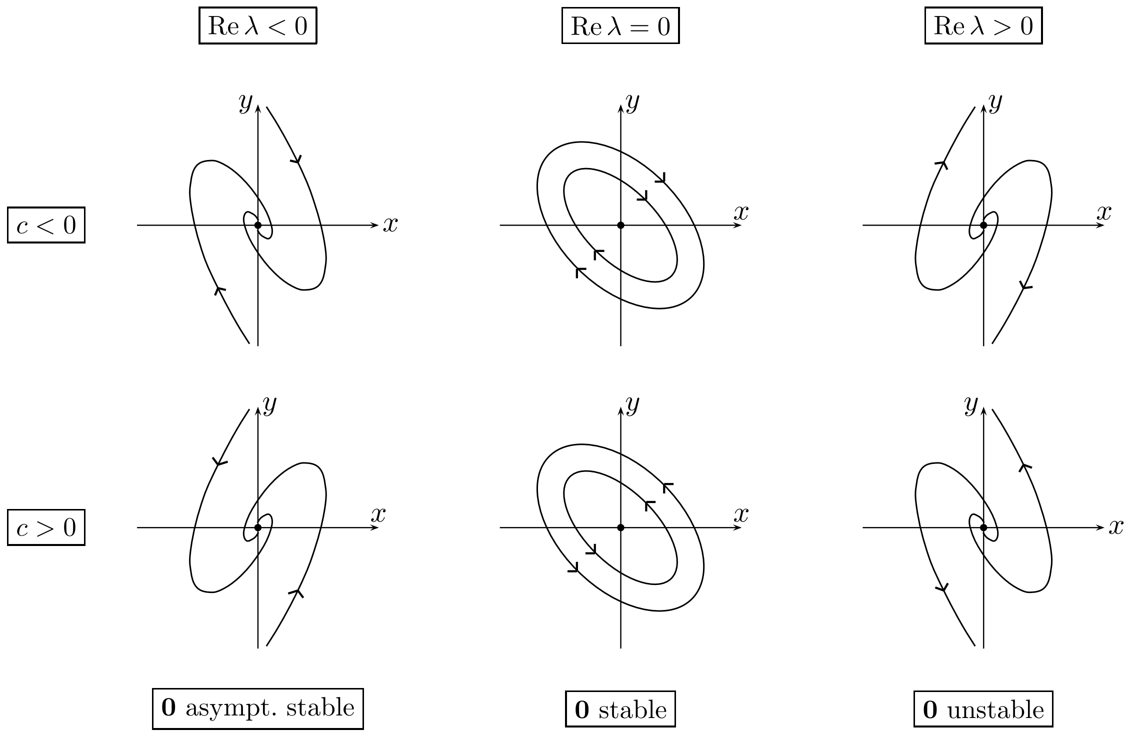
For example, in the last sketch, as $t \longrightarrow \infty$, the second term in (eq10) becomes much bigger than the first, but the first one is getting bigger and bigger nevertheless. Thus, the expression in (eq10) does not approach the second term as $t \longrightarrow 0$. On the other hand, as $t \longrightarrow -\infty$, the second term approaches zero much faster than the first term does. Thus, any solution curve with $C_1, C_2 \neq 0$, "leaves" the origin tangent to the $\mathbf{v}_1$-line.

*Can you sketch the phase-plane portrait if one of the eigenvalues is zero?*

(3) Next, suppose that

$$A = \begin{pmatrix} * & b \\ c & * \end{pmatrix}$$

has a complex eigenvalue $\lambda_1$. A corresponding eigenvector $\mathbf{v}_1$ must then also be complex. Furthermore, $\mathbf{v}_2 = \bar{\mathbf{v}}_1$ is an eigenvector of $A$ with eigenvalue $\lambda_2 = \bar{\lambda}_1$. The general *complex* solution of the ODE $\mathbf{y}' = A\mathbf{y}$ is given by (eq10), with $C_1, C_2 \in \mathbb{C}$. We can extract the general *real* solution as done above and in Section 9.2. However, in order to sketch the corresponding phase-plane portrait, the only information we need to know is the real part of the eigenvalues and the lower-left entry of the matrix $A$, i.e. $c$. All six possible phase-plane portraits are shown below.




All spirals make infinitely many loops around the origin, going away and toward the origin. The radii increase by a fixed factor with each full rotation. Note that the direction of rotation, i.e. positive (counterclockwise) or negative (clockwise), is the same as the sign of $c$.

*Can you explain why the phase-plane portraits look as depicted above? What can you say about the direction of rotation if* $c = 0$*?*

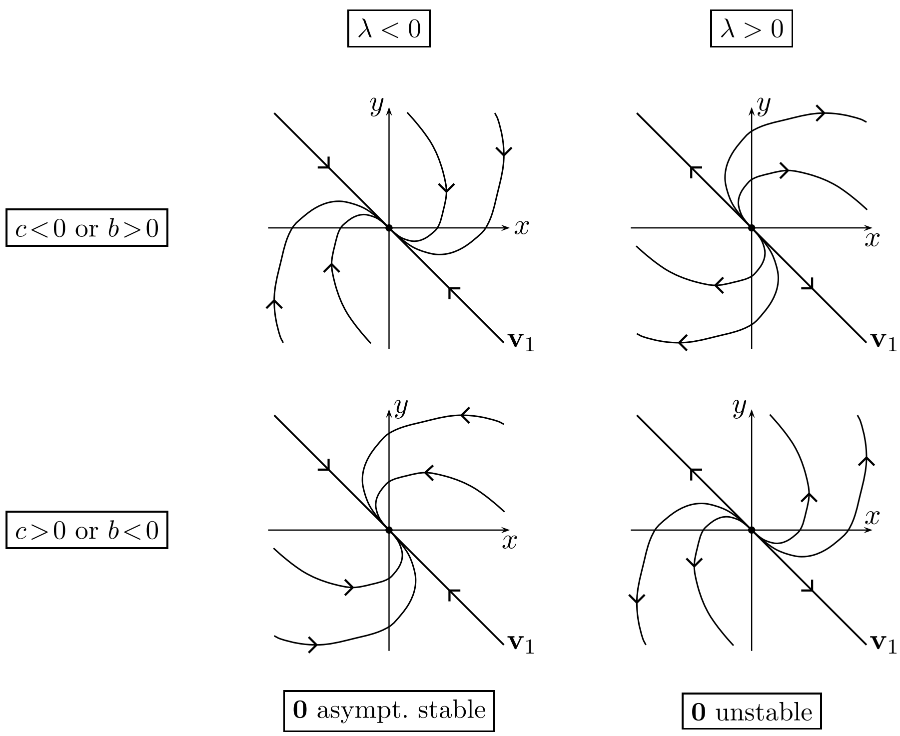
(4) Finally suppose that

$$A = \begin{pmatrix} * & b \\ c & * \end{pmatrix}$$

has a double eigenvalue $\lambda_1, \lambda_2 = \lambda$, and the $\lambda$-eigenspace is 1-dimensional. Let $\mathbf{v}_1$ be an eigenvector. If $\mathbf{v}_2$ is a vector linearly independent of $\mathbf{v}_1$, the general solution of $\mathbf{y}' = A\mathbf{y}$ is described by

$$\mathbf{v}_2 = a\mathbf{v}_1 + \lambda \mathbf{v}_2 \qquad \Longrightarrow \qquad \mathbf{y}(t) = \big(C_1 + C_2 at\big)e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2, \qquad C_1, C_2 \in \mathbb{R}. \tag{11}$$

The phase-plane portraits in the four main cases are shown below.




Note that in this case the phase-plane portraits are "half"-way between a *nodal* sink/source and a *spiral* sink/source. In particular, they all feature two opposite rays, but not four, directed according to the sign of the eigenvalue. There is a half-rotation, instead of infinitely many full ones, in each of them. The direction of rotation is determined by either of the off-diagonal entries of $A$, according to the same rule as in the complex-eigenvalues case. The difference is that the $c$-test for the direction of rotation always suffices in the complex-eigenvalues case, but not in this case. So, we may sometimes need to use the $b$-test for the direction of rotation, which is less natural, since negative/positive $b$ means positive/negative rotation.

*Can you sketch the phase-plane portrait if $\lambda = 0$? What can you say about the direction of rotation if $b$ and $c$ are both negative/zero/positive?*