Math53: Ordinary Differential Equations
Autumn 2004

Course Preview

In a single-variable calculus course, one learns how to find the derivative y' of a smooth function $y = y(t)$. One also learns how to find an antiderivative $y = y(t)$ of a function $f = f(t)$. For example,

$$
\begin{aligned}
& y'(t) = t^2 - \sin t, && (1)\\
\Longrightarrow \qquad & y(t) = \int (t^2 - \sin t)dt = \frac{1}{3}t^3 + \cos t + C, \quad t \in (-\infty, \infty), && (2)
\end{aligned}
$$

for some real constant C.

Relation (1) is a differential equation. More generally, a differential equation is an equation satisfied by the derivatives of a function. *Ordinary* differential equations (ODEs), such as (1), involve functions of a single *independent* variable, which in the case of (1) is t. In contrast, *partial* differential equation (PDEs), such as

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial s^2}, \qquad \text{where} \qquad y = y(s, t),$$

involve functions of two or more variables. Differential equations, both ordinary and partial, arise naturally in mathematics, in physical and social sciences, and in engineering.

This course describes a variety of approaches to studying the behavior of solutions of ODEs. The primary focus will be on finding *explicit solutions* to ODEs, whenever possible. For example, (2) is the *general* solution to (1). The differential equation (1) has a particularly simple form:

$$y'(t) = f(t). \tag{3}$$

Any equation of this form can of course be solved by integration.

Most ODEs that describe real-life phenomena are far more complicated than (3). In this course, we will study a number of techniques for solving many ODEs, often by essentially reducing them to (3). These techniques build up on the chain and product rules of one-variable differentiation and the change-of-variables and integration-by-parts formulas of one-variable integration.

We start by considering *first-order* ODEs, i.e. ODEs that involve only the first derivative of a function. *Linear* first-order ODEs, i.e. equations of the form

$$y'(t) = a(t) \cdot y(t) + b(t), \tag{4}$$

can be reduced to (3) by multiplying both sides of (4) by an appropriate function $\mu = \mu(t)$, called an *integrating factor*. Thus, it is possible to write down an explicit solution $y = y(t)$ for (4).

Most first-order ODEs that arise in applications can be put into the *normal form:*

$$y'(t) = Q(t, y(t)), \tag{5}$$

where Q is a function of two variables. For example, in the case of (4),

$$Q(t, y) = a(t) \cdot y + b(t).$$

If the function Q has a special form, it may be possible, using a technique from multivariable calculus, to find a function F of two variables such that

$$F(t, y(t)) = 0, \tag{6}$$

for all solutions of (5) and all admissible values of t. While it may not be possible to solve (6) explicitly for $y = y(t)$, (6) at least describes the graph of y in the (t, y)-plane.

We will continue on to *second-order* ODEs, i.e. equations that involve the second derivative of a function such as

$$y''(t) + p(t)y'(t) + q(t)y(t) = f(t). \tag{7}$$

If the functions $p(t)$, $q(t)$, and $f(t)$ are constant, the equation (7) can be solved explicitly for $y = y(t)$. There are a number of ways of trying to approach (7) in general. One of them is the *Laplace Transform*, an operator that replaces (7) with a seemingly unrelated, but equivalent, equation, which may be far easier to solve.

We will see that most high-order ODEs can be replaced by equivalent *systems* of first-order ODEs, i.e. equations like (5), but for vector-valued functions y. It is possible to find explicit solutions for a class of systems of ODEs. In other cases, some important qualitative information about solutions of a system can be obtained, even if it cannot be solved explicitly.

Many ODEs arising in applications can be solved numerically, for a fixed value of the parameter t, quite accurately. *Euler's Method* and *Runge-Kutta Methods*, that are motivated by the Taylor series expansion, lie behind many modern numerical ODE solvers. These methods will be studied in the course as well.