Math53: Ordinary Differential Equations
Autumn 2004

Homework Assignment 7

Problem Set 7 is due by 2:15p.m. on Monday, 11/22, in MuddChem 101

Problem Set 7:

6.1: 2*,18*;   6.2:2*;   Problem F (see next page).

Note 1: In 6.1:2,18 and 6.2:2, also solve the initial-value problems exactly and compare your estimates with the exact values. Use a calculator for the book's part of 6.1:2,18 and 6.2:2.

Note 2: Most of Problem F is a summary of Euler's numerical method for solving ODEs. You do not need any material from Chapter 6 to do it, but it is motivated by Section 6.1.

Daily Assignments:

Date	*Read*	*Exercises*
11/18 R	6.1	6.1:2*,18*
11/19 F	6.2	6.2:2*, Problem F

**Note:* In 6.1:2,18 and 6.2:2, also solve the initial-value problem exactly and compare your estimates with the exact values. Use a calculator for the book's part of 6.1:2,18 and 6.2:2.

Hints for Problem F: For part (a), you will need to use the fundamental theorem of calculus, with definite integrals. Part (b) is independent of (a). Part (c) follows from (a) and (b). Part (d) follows from (c) and the fact that

$$\lim_{t\longrightarrow\infty}\left(1+\frac{1}{t}\right)^t = e$$

and the function $\left(1+t^{-1}\right)^t$ increases with t.

Problem F

The goal of this problem is to recover the error estimate (1.14) in Section 6.1.

(a) Suppose y and $\tilde{y}$ are smooth functions on the interval $[c,d]$ and M is a positive number such that

$$\big|y''(t)\big|, \big|\tilde{y}''(t)\big| \le M \qquad \text{for all} \qquad t\in[c,d].$$

Show that

$$\big|y(d)-\tilde{y}(d)\big| \le \big|y(c)-\tilde{y}(c)\big| + \big|y'(c)-\tilde{y}'(c)\big||d-c| + M|d-c|^2.$$

Suppose now that $f=f(t,y)$ is a smooth function and M_0, M_t, and M_y are positive numbers such that

$$\big|f(t,y)\big|\le M_0, \quad \big|f_t(t,y)\big|\le M_t, \quad \big|f_y(t,y)\big|\le M_y \qquad \text{for all} \qquad t\in[a,b],\ y\in(-\infty,\infty).$$

Let $y=y(t)$ be the solution to the initial value problem

$$y'=f(t,y), \qquad y(a)=y_0. \tag{1}$$

The first-order Euler's method is used to estimate the value of $y(b)$ as follows. We fix a positive integer N and put

$$h=\frac{b-a}{N}, \quad t_0=a, \quad t_{i+1}=t_i+h=h\cdot(i+1), \quad s_i=f(t_i,y_i), \quad y_{i+1}=y_i+s_ih.$$

This way, we start with the point (t_0,y_0), compute the slope s_0, and use it to compute y_1, which is an estimate for $y(t_1)$. We then repeat this procedure at (t_1,y_1), then at (t_2,y_2), and so on, as done in (1.5) and illustrated in Figures 1, 3, and 4 in Section 6.1. Our goal is to estimate the error

$$\epsilon_i=\big|y(t_i)-y_i\big|,$$

especially for $i=N$.

We know that $\epsilon_0=0$. We will estimate $\epsilon_{i+1}-\epsilon_i$. Let

$$\tilde{y}_i(t)=y_i+s_i(t-t_i).$$

Note that

$$\tilde{y}_i(t_i)=y_i, \quad \tilde{y}_i(t_{i+1})=y_{i+1}, \quad \tilde{y}_i'(t_i)=s_i, \quad \tilde{y}_i''(t)=0.$$

(b) Use the ODE and the assumptions on f to show that

$$|y''(t)|\le M_t+M_0M_y \qquad \text{and} \qquad \big|y'(t_i)-\tilde{y}_i'(t_i)\big|\le M_y\epsilon_i.$$

(c) Use part (a) to show that

$$\epsilon_{i+1}\le\epsilon_i+M_y\epsilon_ih+(M_t+M_0M_y)h^2.$$

(d) Conclude that

$$\epsilon_N\le\big(M_t+M_0M_y\big)\frac{(1+M_yh)^N-1}{M_y}h\le\frac{M_t+M_0M_y}{M_y}\big(e^{M_y(b-a)}-1\big)h.$$