MAT 645: Symplectic Topology
Spring 2014

Course Instructor

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Course Schedule

Because of related activities at SCGP (more below), the schedule of this class will be modified to

Tu 9:55-11:05am, Th: 9:55-11:25am.

On Thursdays, I will aim to end by 11:20am and will end no later than 11:25am.

Related Activities at SCGP

One of the two SCGP programs this semester, *Moduli Spaces of Pseudo-Holomorphic Curves and Their Applications to Symplectic Topology*, which is organized by D. McDuff and K. Fukaya, is closely related to this course. There will be two seminars associated with this program, on Tuesdays 11:15-12:15 and 2:30-3:30, of different style. The morning seminar will generally feature several talks on one research topic; in some cases, the first talk may describe the results and then be followed by more accessible talks. This seminar can be a great way to learn about modern techniques in symplectic topology and would benefit from lots of questions (however basic) from the audience. The afternoon seminar will be a typical seminar in symplectic topology and a great way to learn about current research in this area. The mini-course by K. Fukaya, Fridays at noon, will be concerned with defining invariants out of moduli spaces of J-holomorphic maps. There will likely be a mini-course by K. Wehrheim, on Wednesday afternoons, describing an alternative approach. Four of the talks (by H. Hofer, D. Salamon, P. Seidel, and M. Tehrani) during the Spring Break workshop will be introductory.

Course Website

All updates, including schedule and references, will be posted on the course website,

`http://math.sunysb.edu/~azinger/mat645`.

Please visit this website regularly.

Prerequisites

Passing the comps is required. You also should have some understanding of complex geometry (MAT 545) and algebraic topology (MAT 539). Feel free to contact me with any questions.

Grading

Your grade will be based on class participation, in all possible forms.

Course Description

The aim of this course is to establish the foundations of J-holomorphic curves techniques in symplectic topology. These techniques, which intertwine the softness of topology and the rigidity of algebraic geometry, are central to modern symplectic topology, have found applications in algebraic geometry (in areas such as birational and enumerative geometry), and have connections to string theory. We will discuss the structure of J-holomorphic maps, removal of singularities, Gromov's compactness theorem, and Gromov-Witten invariants in special case, covering roughly Chapters 1-7 and some appendices of

> [MS04] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, AMS Colloquium Publications 52, 2004/2012.

Time-permitting, we will discuss such applications as Kontsevich's formula enumerating rational curves in $\mathbb{CP}^2$, Gromov's non-squeezing theorem, and other (non-)embedding theorems such as in

> [Gr] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no. 2, 307-347.

> [Gu] L. Guth, *Symplectic embeddings of polydisks*, Invent. Math. 172 (2008), no. 3, 477-489.

> [MSc] D. McDuff and F. Schlenk, *The embedding capacity of 4-dimensional symplectic ellipsoids*, Ann. of Math. (2) 175 (2012), no. 3, 1191-1282.

The book [MS04] is a well-written, thorough introduction to J-holomorphic techniques. It involves quite a bit of analysis (Sobolev spaces, elliptic operators), but it is done only for the sake of specific geometric applications, not for its own sake. The organization of [MS04] also makes it easy to skip technical arguments, taking their conclusions on faith. Proofs of some basic facts in symplectic geometry not appearing in [MS04], such as Darboux's and Moser's Theorems, can be found in

> [MS98] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford Mathematical Monographs, 1998.

This book is more elementary, but barely touches on J-holomorphic maps. An even more elementary treatment of basic symplectic geometry appears in

> [C] A. Cannas de Silva, *Lecture on Symplectic Geometry*, Lecture Notes in Mathematics 1764, Springer 2001.

There are still fundamental mysteries concerning how soft/rigid the symplectic category actually is. Here is one example; it is motivated by important problems in birational algebraic geometry.

> Let (M,ω) be a connected symplectic manifold. Suppose J is an almost complex structure on M compatible with ω (or tamed by ω) and $U\subset M$ is a nonempty open subset such that each point in U is contained in the image of a J-holomorphic map from S^2 to M. *Is every point of X contained in the image of a J-holomorphic map from S^2 to M? If J' is another ω-compatible (or ω-tamed) almost complex structure on M, is every point of X contained in the image of a J'-holomorphic map from S^2 to M?*

In the case (M,ω,J) is projective and admits a nonempty open subset U as above, a theorem of Kollár and Ruan (independently) implies that (M,ω) admits a nonzero genus 0 GW-invariant of the form $\mathrm{GW}_{0,A}(\mathrm{pt},\cdot,\ldots,\cdot)$. This in turn implies that every point of X is contained in the image of a J'-holomorphic map from S^2 to M, for any pre-specified J'. However, the general case is open.