MAT 645: Symplectic Topology
Spring 2014 Supplementary Notes

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Contents

1 Local Properties

Most of the properties of J-holomorphic maps to the almost complex manifold (M,J) described in this section do not depend on M being compact. The exceptions are Corollaries 1.19, 1.20, and 1.27, which are direct consequences of Propositions 1.18 and 1.26. The main statements in this section are Proposition 1.1, Theorem 1.11, Corollary 1.19, and Proposition 1.26.

1.1 Local structure of J-holomorphic maps

Proposition 1.1 below is a local description of solutions of a non-linear differential equation which generalizes the J-holomorphic curves equation. It is used in the proof of Theorem 1.11 as well as to describe the general structure of J-holomorphic maps.

For each $R\in\mathbb{R}^+$, denote by $B_R\subset\mathbb{C}$ the open ball of radius R around the origin and let $B_R^*=B_R-\{0\}$.

Proposition 1.1 ([1, Theorem 2.2]). *Suppose $p,\epsilon\in\mathbb{R}^+$, with $p>2$, $u\in L^p_1(B_\epsilon;\mathbb{C}^n)$ for some $n\in\mathbb{Z}^+$, $J\in L^p_1(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$, and $C\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)$ are such that*

$$u(0)=0, \quad J(z)^2=-\mathrm{Id}_{\mathbb{C}^n}, \quad u_s(z)+J(z)u_t(z)+C(z)u(z)=0 \quad \forall\ z=s+\mathfrak{i}t\in B_\epsilon\,. \tag{1.1}$$

Then, there exist $\delta\in(0,\epsilon)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$\sigma(0)=0, \quad J(z)\Phi(z)=\Phi(z)J_{\mathbb{C}^n}, \quad \Phi(z)\sigma(z)=u(z) \quad \forall\ z\in B_\delta\,, \tag{1.2}$$

where $J_{\mathbb{C}^n}=\mathfrak{i}$ is the standard complex structure on $\mathbb{C}^n$.

By the Sobolev Embedding Theorem, the assumption $p>2$ implies that u is a continuous function and in particular the first two equations in (1.1) and in (1.2) make sense. This assumption also implies that the left-hand side of the third equation in (1.1) lies in L^p and that the left-hand sides of the second and third equations in (1.2) lie in L^p_1. Proposition 1.1 is proved at the end of this section.

Example 1.2. Let $\mathfrak{c}\colon \mathbb{C}\longrightarrow\mathbb{C}$ denote the usual conjugate. Define

$$\widehat{J}(z_1,z_2)=\begin{pmatrix}\mathfrak{i}&0\\-2\mathfrak{i}s_1\mathfrak{c}&\mathfrak{i}\end{pmatrix}=\begin{pmatrix}1&0\\s_1\mathfrak{c}&1\end{pmatrix}J_{\mathbb{C}^2}\begin{pmatrix}1&0\\s_1\mathfrak{c}&1\end{pmatrix}^{-1}\colon\mathbb{C}^2\longrightarrow\mathbb{C}^2\quad\forall\,z_i=s_i+\mathfrak{i}t_i,$$
$$u\colon\mathbb{C}\longrightarrow\mathbb{C}^2,\qquad u(s+\mathfrak{i}t)=\big(z,s^2\big).$$

Thus, $\widehat{J}$ is an almost complex structure on $\mathbb{C}^2$ and u is a $\widehat{J}$-holomorphic map, i.e. it satisfies the last condition in (1.1) with $J(z)=\widehat{J}(u(z))$ and $C(z)=0$. The functions

$$\sigma\colon\mathbb{C}\longrightarrow\mathbb{C}^2,\quad \sigma(z)=(z,0),\qquad \Phi\colon\mathbb{C}\longrightarrow\mathrm{GL}_4\mathbb{R},\quad \Phi(s+\mathfrak{i}t)=\begin{pmatrix}1&0\\s\mathfrak{c}+\frac{\mathfrak{i}st}{z}&1\end{pmatrix},$$

satisfy (1.2).

Corollary 1.3. *With the assumptions as in Proposition 1.1, either $u\equiv0$ or there exist $\ell\in\mathbb{Z}^+$ and $\alpha\in\mathbb{C}^n-0$ such that*

$$\lim_{z\longrightarrow0}\frac{u(z)-\alpha z^\ell}{z^\ell}=0\,.$$

Corollary 1.4. *If (M,J) is an almost complex manifold and $u\colon(\Sigma,\mathfrak{j})\longrightarrow(M,J)$ is a non-constant J-holomorphic map from a connected Riemann surface, then the subset*

$$u^{-1}\big(\{u(z)\colon\ z\in\Sigma,\ \mathrm{d}_zu=0\}\big)\subset\Sigma$$

is discrete. If in addition $x\in M$, the subset $u^{-1}(x)\subset\Sigma$ is also discrete.

Corollary 1.5. *Suppose (M,J) is an almost complex manifold,*

$$u,u'\colon(\Sigma,\mathfrak{j}),(\Sigma',\mathfrak{j}')\longrightarrow(M,J)$$

are J-holomorphic maps, $z_0\in\Sigma$ is such that $\mathrm{d}_{z_0}u\neq0$, and $z_0'\in\Sigma'$ is such that $u'(z_0')=u(z_0)$. If there exist sequences $z_i\in\Sigma-z_0$ and $z_i'\in\Sigma'-z_0'$ such that

$$\lim_{i\longrightarrow\infty}z_i=z_0\,,\qquad\lim_{i\longrightarrow\infty}z_i'=z_0'\,,\quad\textit{and}\quad u(z_i)=u'(z_i)\ \ \forall\ i\in\mathbb{Z}^+\,,$$

then there exists a holomorphic map $\sigma\colon U'\longrightarrow\Sigma$ from a neighborhood of z_0' in Σ' such that $\sigma(z_0')=z_0$ and $u'|_{U'}=u\circ\sigma$.

Proof. It can be assumed that $(\Sigma,\mathfrak{j},z_0),(\Sigma',\mathfrak{j}',z_0')=(B_1,\mathfrak{j}_0,0)$, where $B_1\subset\mathbb{C}$ is the unit ball with the standard complex structure. Since $\mathrm{d}_{z_0}u\neq0$ and u is J-holomorphic, u is an embedding near $0\in B_1$ and so is a slice in a coordinate system. Thus, we can assume that

$$u,u'\equiv(v,w)\colon(B_1,0)\longrightarrow(\mathbb{C}\times\mathbb{C}^{n-1},0),\qquad u(z)=(z,0)\in\mathbb{C}\times\mathbb{C}^{n-1}\,,$$

and u,u' are J-holomorphic with respect to some almost complex structure

$$J(x,y)=\begin{pmatrix}J_{11}(x,y)&J_{12}(x,y)\\J_{21}(x,y)&J_{22}(x,y)\end{pmatrix}\colon\mathbb{C}\times\mathbb{C}^{n-1}\longrightarrow\mathbb{C}\times\mathbb{C}^{n-1}\,,\qquad(x,y)\in\mathbb{C}\times\mathbb{C}^{n-1}\,.$$

Since u is J-holomorphic,

$$J_{21}(x,0)=0, \quad J_{22}(x,0)^2=-\mathrm{Id} \qquad \forall\ x\in B_1\subset\mathbb{C}. \tag{1.3}$$

Since u' is J-holomorphic,

$$\partial_s w + J_{22}\big(v(z),w(z)\big)\partial_t w + J_{21}\big(v(z),w(z)\big)\partial_t v = 0.$$

Combining this with

$$J_{ij}(x,y) = J_{ij}(x,0) + \int_0^1 \frac{\mathrm{d}J_{ij}(x,ty)}{\mathrm{d}t}\mathrm{d}t = J_{ij}(x,0) + \sum_{i=1}^{n-1} y_i \int_0^1 \frac{\partial J_{ij}}{\partial y_i}\bigg|_{(x,ty)}\mathrm{d}t$$

and the first equation in (1.3), we find that

$$\partial_s w + J_{22}\big(v(z),0\big)\partial_t w + C(z)w(z) = 0, \qquad \text{where} \quad C\in L^p\big(B_1;\mathrm{End}_{\mathbb{R}}\mathbb{C}^{n-1}\big),$$
$$C(z)y = \sum_{i=1}^{n-1} y_i \Bigg(\bigg(\int_0^1 \frac{\partial J_{22}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_t w|_z + \bigg(\int_0^1 \frac{\partial J_{21}}{\partial y_i}\bigg|_{(v(z),tw(z))}\mathrm{d}t\bigg)\partial_s v|_z\Bigg).$$

Thus, by Proposition 1.1 and the second identity in (1.3),

$$w(z) = \Phi(z)\widetilde{w}(z) \qquad \forall\ z\in B_\delta\,,$$

for some $\delta\in(0,1)$, $\Phi\in L^p_1(B_\delta;\mathrm{GL}_{2n-2}\mathbb{R})$, and holomorphic map $\widetilde{w}\colon B_\delta\longrightarrow\mathbb{C}^{n-1}$. Since $u'(z_i')=u(z_i)$, $\widetilde{w}(z_i')=0$ for all $i\in\mathbb{Z}^+$. Since $z_i'\longrightarrow 0$ and $z_i'\neq 0$, it follows that $w=0$. This implies the claim with $U'=B_\delta$ and $\sigma=v$. □

Corollary 1.6. *Let (M,J) be an almost complex manifold with a Riemannian metric g compatible with J. If $x\in M$ and $u\colon\Sigma\longrightarrow M$ is a J-holomorphic map from a compact Riemann surface with boundary such that $x\in u(\Sigma)-u(\partial\Sigma)$, then*

$$\lim_{\delta\longrightarrow 0}\frac{1}{\pi\delta^2}E\big(u|_{u^{-1}(B^g_\delta(x))}\big)\in\mathbb{Z}^+\,,$$

where $B^g_\delta(x)\subset M$ is the ball of radius δ around x in M with respect to the metric g.

Proof. By the continuity of u, we can assume that $M=\mathbb{C}^n$, J agrees with the standard complex structure $J_{\mathbb{C}^n}$ at the origin, g agrees with the standard metric $g_{\mathbb{C}^n}$ at the origin, $\Sigma=\overline{B_\epsilon}$ for some $\epsilon\in\mathbb{R}^+$, and $u(0)=0$. In particular, there exists $C\geq 1$ such that

$$\big|J_x-J_{\mathbb{C}^n}\big|\leq C|x|, \quad \big|g_x-g_{\mathbb{C}^n}\big|\leq C|x| \qquad \forall\ x\in\mathbb{C}^n \text{ s.t. } |x|\leq 1, \tag{1.4}$$

where $|\cdot|$ denotes the usual norm of x (i.e. the distance to the origin with respect to $g_{\mathbb{C}^n}$).

By Corollary 1.3,

$$u(z)=\alpha\cdot\big(z^\ell+f(z)\big) \tag{1.5}$$

after possibly shrinking ϵ, for some $\ell\in\mathbb{Z}^+$, $\alpha\in\mathbb{C}^{n-1}-0$, and a smooth function f on B_ϵ such that

$$\big|f(z)\big|\leq C|z|^{\ell+1} \qquad \forall\ z\in B_\epsilon\,. \tag{1.6}$$

Let $z=s+\mathfrak{i}t$ as before. By (1.5) and (1.6),

$$u_s(z) = \alpha\ell\cdot\big(z^{\ell-1}+\tilde{f}(z)\big) \tag{1.7}$$

for a smooth function $\tilde{f}$ on B_ϵ such that

$$\big|\tilde{f}(z)\big| \le C|z|^\ell \qquad \forall\ z\in B_\epsilon\,. \tag{1.8}$$

We can also assume that the three constants C in (1.4), (1.6), and (1.8) are the same, $C\ge 1$,

$$C_\alpha\epsilon \equiv \big(C+C|\alpha|+C^2|\alpha|\big)\epsilon \le 1\,,$$

and $|u(z)|\le 1$ for all $z\in B_\epsilon$. By (1.4)-(1.8),

$$\left|\frac{|u(z)|_g}{|\alpha||z|^\ell}-1\right|,\left|\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}}-1\right| \le C|z|+C|\alpha||z|^\ell+C^2|\alpha||z|^{\ell+1}\le C_\alpha|z| \quad \forall\ z\in B_\epsilon\subset B_1, \tag{1.9}$$

where $|\cdot|_g$ denotes the distance to the origin in $\mathbb{C}^n$ with respect to the metric g.

Given $r\in(0,1)$, let $\delta_r\in(0,\epsilon)$ be such that

$$C_\alpha\left(\frac{2\delta_r}{(1-r)|\alpha|}\right)^{1/\ell} \le r\,. \tag{1.10}$$

For any $\delta\in[0,\delta_r]$, (1.9) and (1.10) give

$$\begin{aligned}
|z| \le \left(\frac{\delta}{(1+r)|\alpha|}\right)^{1/\ell} \qquad &\Longrightarrow \qquad u(z)\in B_\delta^g(0)\,,\\
u(z)\in B_\delta^g(0) \qquad &\Longrightarrow \qquad |z|\le\left(\frac{\delta}{(1-r)|\alpha|}\right)^{1/\ell},\\
|z| \le \left(\frac{\delta}{(1-r)|\alpha|}\right)^{1/\ell} \qquad &\Longrightarrow \qquad 1-r\le\frac{|u_s(z)|_g}{|\alpha|\ell|z|^{\ell-1}}\le 1+r.
\end{aligned}$$

Combining these, we obtain

$$\int_{|z|\le\left(\frac{\delta}{(1+r)|\alpha|}\right)^{\frac{1}{\ell}}}(1-r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2 \le \int_{u^{-1}(B_\delta^g(0))}|u_s|_g^2 \le \int_{|z|\le\left(\frac{\delta}{(1-r)|\alpha|}\right)^{\frac{1}{\ell}}}(1+r)^2\big(|\alpha|\ell|z|^{\ell-1}\big)^2\,.$$

Evaluating the outer integrals, we find that

$$\left(\frac{1-r}{1+r}\right)^2\ell\pi\delta^2 \le E\big(u|_{u^{-1}(B_\delta^g(0))}\big) \le \left(\frac{1+r}{1-r}\right)^2\ell\pi\delta^2\,.$$

These inequalities hold for all $r\in(0,1)$ and $\delta\in(0,\delta_r)$; the claim is obtained by sending $r\longrightarrow 0$. □

Before establishing the full statement of Proposition 1.1, we consider a special case.

Lemma 1.7. *Suppose $p,\epsilon\in\mathbb{R}^+$, with $p>2$, $u\in L_1^p(B_\epsilon;\mathbb{C}^n)$ for some $n\in\mathbb{Z}^+$, and $A\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$ are such that*

$$u(0)=0, \qquad u_s+J_{\mathbb{C}^n}u_t(z)+A(z)u(z)=0\ \ \forall\ z=s+\mathfrak{i}t\in B_\epsilon\,, \tag{1.11}$$

where $J_{\mathbb{C}^n}=\mathfrak{i}$ is the standard complex structure on $\mathbb{C}^n$. Then, there exist $\delta\in(0,\epsilon)$, $\Phi\in L_1^p(B_\delta;\mathrm{GL}_n\mathbb{C})$, a $J_{\mathbb{C}^n}$-holomorphic map $\sigma\colon B_\delta\longrightarrow\mathbb{C}^n$ such that

$$\sigma(0)=0, \qquad \Phi(0)=\mathrm{Id}_{\mathbb{C}^n}, \qquad \Phi(z)\sigma(z)=u(z)\ \ \forall\ z\in B_\delta\,. \tag{1.12}$$

Proof. For each $\delta\in[0,\epsilon]$, we define

$$A_\delta \in L^p(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad A_\delta(z)=\begin{cases} A(z), & \text{if } z\in B_\delta;\\ 0, & \text{otherwise;}\end{cases}$$

$$D_\delta : L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \qquad \text{by} \quad D_\delta\Theta = \big(\Theta_s+J_{\mathbb{C}^n}\Theta_t+A_\delta\Theta\big)\mathrm{d}\bar z\,.$$

Since the cokernel of $D_0=2\bar\partial$ is isomorphic $H^1(S^2;\mathbb{C})\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n$, D_0 is surjective and the homomorphism

$$\tilde D_0\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus \mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow \big(D_0\Theta,\Theta(0)\big),$$

is an isomorphism. Since

$$\big\|D_\delta\Theta-D_0\Theta\big\|_{L^p}\le \|A_\delta\|_{L^p}\|\Theta\|_{C^0}\le C\|A_\delta\|_{L^p}\|\Theta\|_{L^p_1} \qquad \forall\ \Theta\in L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$$

and $\|A_\delta\|_{L^p}\longrightarrow 0$ as $\delta\longrightarrow 0$, the homomorphism

$$\tilde D_\delta\colon L^p_1(S^2;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n) \longrightarrow L^p(S^2;(T^*S^2)^{0,1}\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)\oplus \mathrm{End}_{\mathbb{C}}\mathbb{C}^n, \qquad \Theta\longrightarrow \big(D_\delta\Theta,\Theta(0)\big),$$

is also an isomorphism for $\delta>0$ sufficient small. Let $\Theta_\delta=D_\delta^{-1}(0,\mathrm{Id}_{\mathbb{C}^n})$. Since D_δ is an isomorphism,

$$\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{C^0}\le C\big\|\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n}\big\|_{L^p_1}\le C'\big\|D_\delta(\Theta_\delta-\mathrm{Id}_{\mathbb{C}^n})\big\|_{L^p}=C'\big\|A_\delta\big\|_{L^p}\,.$$

Since $\|A_\delta\|_{L^p}\longrightarrow 0$ as $\delta\longrightarrow 0$, $\Theta_\delta\in L^p_1(B_\delta;\mathrm{GL}_n\mathbb{C})$. By the third equation in (1.11), the function $\sigma=\Theta_\delta^{-1}u$ then satisfies

$$\sigma(0)=0, \qquad \sigma_s+J_{\mathbb{C}^n}\sigma_t=0 \quad \forall\ z\in B_\delta,$$

i.e. σ is $J_{\mathbb{C}^n}$-holomorphic, as required. □

Proof of Proposition 1.1. (1) Since B_ϵ is contractible, the complex vector bundles $u^*(T\mathbb{C}^n,J_{\mathbb{C}^n})$ and $u^*(T\mathbb{C}^n,J)$ over B_ϵ are isomorphic. Thus, there exists

$$\Psi\in L^p_1(B_\epsilon;\mathrm{GL}_{2n}\mathbb{R}) \qquad \text{s.t.} \quad J(z)\Psi(z)=\Psi(z)J_{\mathbb{C}^n} \quad \forall\ z\in B_\epsilon\,.$$

Let $v=\Psi^{-1}u$. By the assumptions on u, $v\in L^p_1(B_\epsilon;\mathbb{C}^n)$ and

$$\begin{gathered} v(0)=0, \qquad v_s(z)+J_{\mathbb{C}^n}v_t(z)+\tilde C(z)v(z)=0 \quad \forall\ z=s+\mathfrak{i}t\in B_\epsilon,\\ \text{where} \quad \tilde C=\Psi^{-1}\cdot\big(\Psi_s+J\Psi_t+C\Psi\big)\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n)\,. \end{gathered} \tag{1.13}$$

Thus, we have reduced the problem to the case $J=J_{\mathbb{C}^n}$.

(2) Let $\tilde C^{\pm}=\frac{1}{2}(\tilde C\mp J_{\mathbb{C}^n}\tilde C J_{\mathbb{C}^n})$ be the $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts of $\tilde C$, i.e. $\tilde C^{\pm}J_{\mathbb{C}^n}=\pm J_{\mathbb{C}^n}\tilde C^{\pm}$. With $\langle\cdot,\cdot\rangle$ denoting the Hermitian inner-product on $\mathbb{C}^n$ which is $\mathbb{C}$-antilinear in the second input, define

$$D\in L^\infty(B_\epsilon;\mathrm{End}_{\mathbb{R}}\mathbb{C}^n), \quad D(z)w=\begin{cases} |v(z)|^{-2}\langle v(z),w\rangle v(z), & \text{if } v(z)\neq 0;\\ 0, & \text{otherwise;}\end{cases} \qquad A=\tilde C^{+}+\tilde C^{-}D\,.$$

Since $DJ_{\mathbb{C}^n}=-J_{\mathbb{C}^n}D$ and $Dv=v$, $A\in L^p(B_\epsilon;\mathrm{End}_{\mathbb{C}}\mathbb{C}^n)$ and $Av=\tilde C v$. Thus, by (1.13),

$$v_s+J_{\mathbb{C}^n}v_t+Av=0\,.$$

The claim now follows from Lemma 1.7. □

Corollary 1.8. *Suppose $n\in\mathbb{Z}^+$, $\epsilon\in\mathbb{R}^+$, J is a smooth almost complex structure on $\mathbb{C}^n$ with $J_0=J_{\mathbb{C}^n}$, and $u\colon B_\epsilon\longrightarrow\mathbb{C}^n$ is a J-holomorphic map with $u(0)=0$. Then, there exist $\delta\in(0,\epsilon)$, $C\in\mathbb{R}^+$, $\Phi\in C^0(B_\delta;\mathrm{GL}_{2n}\mathbb{R})$, and a $J_{\mathbb{C}^n}$-holomorphic map $\sigma: B_\delta\longrightarrow\mathbb{C}^n$ such that Φ is smooth on $B_\delta-0$,*

$$\sigma(0)=0,\quad \Phi(0)=\mathrm{Id}_{\mathbb{C}^n},\quad J(u(z))\Phi(z)=\Phi(z)J_{\mathbb{C}^n},\quad u(z)=\Phi(z)\sigma(z),\quad \big|\mathrm{d}_z\Phi\big|\le C\ \forall\, z\in B_\delta-0\,.$$

Proof. We can assume that u is not identically 0 on some neighborhood of $0\in B_\epsilon$. Similarly to (1) in the proof of Proposition 1.1, there exists

$$\Psi\in C^\infty(\mathbb{C}^n;\mathrm{GL}_{2n}\mathbb{R})\qquad \text{s.t.}\qquad \Psi(0)=\mathrm{Id}_{\mathbb{C}^n},\quad J(x)\Psi(x)=\Psi(x)J_{\mathbb{C}^n}\quad \forall\ x\in\mathbb{C}^n\,.$$

Let $v(z)=\Psi(u(z))^{-1}u(z)$. By Proposition 1.1, we can choose complex linear coordinates on $\mathbb{C}^n$ so that

$$v(z)=\big(f(z),g(z)\big)h(z)\in\mathbb{C}\oplus\mathbb{C}^{n-1}\qquad \forall\ z\in B_{\epsilon'}$$

for some $\epsilon'\in(0,\epsilon)$, holomorphic function h on $B_{\epsilon'}$ with $h(0)=0$, and continuous functions f and g on $B_{\epsilon'}$ with $f(0)=1$ and $g(0)=0$. By Lemma 1.9 below, there exists $\delta\in(0,\epsilon')$ so that the function

$$\Phi\colon B_\delta\longrightarrow\mathrm{GL}_{2n}\mathbb{R},\qquad \Phi(z)=\Psi\big(u(z)\big)\begin{pmatrix}f(z)&0\\ g(z)&1\end{pmatrix},$$

is continuous on B_δ and smooth on $B_\delta-0$ with $|\mathrm{d}_z\Phi|$ uniformly bounded on $B_\delta-0$. Taking $\sigma(z)=(h(z),0)$, we conclude the proof. $\square$

Lemma 1.9. *Suppose $\epsilon\in\mathbb{R}^+$, and $f,h\colon B_\epsilon\longrightarrow\mathbb{C}$ are continuous functions such that h is holomorphic, $h(z)\neq z$ for some $z\in B_\epsilon$, and the function*

$$B_\epsilon\longrightarrow\mathbb{C},\qquad z\longrightarrow f(z)h(z), \tag{1.14}$$

is smooth. Then there exist $\delta,C\in\mathbb{R}^+$ such that f is differentiable on $B_\epsilon-0$ and

$$\big|\mathrm{d}_zf\big|\le C\qquad \forall\ z\in B_\delta-0\,. \tag{1.15}$$

Proof. After a holomorphic change of coordinate on $B_{2\delta}\subset B_\epsilon$, we can assume that $h(z)=z^\ell$ for some $\ell\in\mathbb{Z}^{\ge0}$. Define

$$g\colon B_{2\delta}\longrightarrow\mathbb{C},\qquad g(z)=f(z)z^\ell-f(0)z^\ell\,.$$

By Taylor's Theorem and the assumptions on the function (1.14), there exists $C>0$ such that the smooth function g satisfies

$$\big|g(z)\big|\le C|z|^{\ell+1}\qquad \forall\ z\in B_\delta\,.$$

Dividing by g by z^ℓ, we thus obtain (1.15). $\square$

Remark 1.10. Corollary 1.8 refines the conclusion of Proposition 1.1 for J-holomorphic maps. In contrast to the output (Φ,σ) of Proposition 1.1, the output of Corollary 1.8 does not depend continuously on the input u with respect to the L^p_1-norms. This makes Corollary 1.8 less suitable for applications in settings involving families of J-holomorphic maps.

1.2 The Monotonicity Lemma

Theorem 1.11 below is a key step in the continuity part of the proof of the Removal of Singularity Theorem 2.1. The precise nature of the lower energy bound in this theorem, i.e. of the function on the right hand-side of (1.16), does not matter, as long as it is positive for $\delta > 0$.

Theorem 1.11 (Monotonicity Lemma)**.** *If (M, J) is an almost complex manifold and g is a Riemannian metric on M, there exists a continuous function $C\colon M \longrightarrow \mathbb{R}^+$ with the following property. If $u\colon \Sigma \longrightarrow M$ is a J-holomorphic map from a compact Riemann surface with boundary, $x \in u(\Sigma)$, and $\delta \in \mathbb{R}^+$ is such that $u(\partial\Sigma) \cap B^g_\delta(x) = \emptyset$, then*

$$E_g(u) \geq \frac{\pi\delta^2}{(1+C(x)\delta)^4}\,. \tag{1.16}$$

If $\omega = g(J\cdot, \cdot)$ is a symplectic form on M, then the above fraction can be replaced by $\pi\delta^2 \mathrm{e}^{-C(x)\delta^2}$.

According to this theorem, "completely getting out" of the ball $B_\delta(x)$ via a J-holomorphic map requires an energy bounded below by a little less than $\pi\delta^2$. Thus, the L^2_1-norm of a J-holomorphic map u exerts some control over the C^0-norm of u. If $p > 2$, the L^p_1-norm of any smooth map f from a two-dimensional manifold controls the C^0-norm of f. However, this is not the case of the L^2_1-norm, as illustrated by the example of [5, Lemma 10.4.1]: the function

$$f_\delta\colon \mathbb{R}^2 \longrightarrow [0,1], \qquad f_\delta(z) = \begin{cases} 1, & \text{if } |z| \leq \delta; \\ \frac{\ln|z|}{\ln\delta}, & \text{if } \delta \leq |z| \leq 1; \\ 0, & \text{if } |z| \geq 1; \end{cases}$$

with any $\delta \in (0,1)$ is continuous and satisfies

$$\int_{\mathbb{R}^2} |\mathrm{d}f_\delta|^2 = -\frac{2\pi}{\ln\delta}\,.$$

It is arbitrarily close in the L^2_1-norm to a smooth function $\tilde{f}_\delta$. Thus, it is possible to "completely get out" of $B_\delta(x)$ using a smooth function with arbitrarily small energy ($\tilde{f}_\delta$ does this for $x = 1$ in $\mathbb{R}$).

Proof of Theorem 1.11. It is sufficient to establish the claim for $\delta \leq \delta_g(x)$ for some continuous function $\delta_g\colon M \longrightarrow \mathbb{R}^+$ smaller than half the injectivity radius function $r_g\colon M \longrightarrow \mathbb{R}^+$. Furthermore, we can assume that the metric g on $B^g_{\delta_g(x)}(x)$ is determined by J and some symplectic form ω so that J is ω-tame on $B^g_{\delta_g(x)}(x)$ and ω-compatible at x (the form ω may depend on x).

Choose a C^∞-function $\eta\colon \mathbb{R} \longrightarrow [0,1]$ such that

$$\eta(\tau) = \begin{cases} 1, & \text{if } \tau \leq \frac{1}{2}; \\ 0, & \text{if } \tau \geq 1; \end{cases} \qquad \eta'(\tau) \leq 0.$$

Let ζ_x be the vector field on $B^g_{\delta_g(x)}(x)$ given by $\zeta_x(y) = \exp_y^{-1}(x)$. Given $\delta \in (0, \delta_g(x))$ and a C^∞-map $u\colon \Sigma \longrightarrow M$ from a compact Riemann surface, define

$$\xi \in \Gamma(\Sigma; u^*TM) \qquad \text{by} \quad \xi(z) = -\eta\biggl(\frac{d_g(x, u(z))}{\delta}\biggr)\zeta_x\bigl(u(z)\bigr);$$

the vanishing assumption on η implies that ξ is well-defined. If $z=s+\mathfrak{i}t$ is a coordinate on Σ,

$$\nabla_s\xi=\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{1}{\delta\, d_g(x,u(z))}\langle u_s,\zeta_x(u(z))\rangle\zeta_x(u(z))-\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\nabla_s\zeta_x(u(z)), \quad (1.17)$$

where ∇ is the Levi-Civita connection of the metric g; see Lemma 1.15. Combining Lemma 1.14 with the ω-compatibility assumption at x, (1.17), and Corollary 1.17, we find that

$$\begin{aligned}\bigg|\int_\Sigma\big(\langle u_s,\nabla_s\xi\rangle+\langle u_t,\nabla_t\xi\rangle\big)\mathrm{d}s\wedge\mathrm{d}t\bigg|&\\ \le C(x)\int_\Sigma\big(|\xi||u_s||u_t|+d_g(x,u(z))(|\nabla_s\xi||u_t|+|u_s||\nabla_t\xi|)\big)\mathrm{d}s\wedge\mathrm{d}t&\\ \le \tilde{C}(x)\delta\Bigg(\int_\Sigma\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\big(|u_s|^2+|u_t|^2\big)\mathrm{d}s\wedge\mathrm{d}t&\\ -\int_\Sigma\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{d_g(x,u(z))}{\delta}\big(|u_s|^2+|u_t|^2\big)\mathrm{d}s\wedge\mathrm{d}t\Bigg),&\end{aligned} \quad (1.18)$$

if u is J-holomorphic.

On the other hand, (1.17) gives

$$\begin{aligned}\langle u_s,\nabla_s\xi\rangle=\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{1}{\delta\, d_g(x,u(z))}\langle u_s,\zeta_x(u(z))\rangle^2&\\ +\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\langle u_s,\nabla_s(-\zeta_x(u(z)))\rangle.&\end{aligned} \quad (1.19)$$

By Corollary 1.17,

$$\langle u_s,\nabla_s(-\zeta_x(u(z)))\rangle\ge|u_s|^2-C(x)d_g(x,u(z))^2|u_s|^2\,. \quad (1.20)$$

If u is J-holomorphic, then $|u_s|=|u_t|$, $\langle u_s,u_t\rangle=0$, and

$$\langle u_s,\zeta_x(u(z))\rangle^2+\langle u_t,\zeta_x(u(z))\rangle^2\le|u_s|^2|\zeta_x(u(z))|^2=\frac{1}{2}\big(|u_s|^2+|u_t|^2\big)d_g(x,u(z))^2\,. \quad (1.21)$$

Since $\eta'\le 0$, (1.19)-(1.21) give

$$\begin{aligned}\frac{1}{2}\eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{d_g(x,u(z))}{\delta}\big(|u_s|^2+|u_t|^2\big)+\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\big(|u_s|^2+|u_t|^2\big)&\\ \le C(x)\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)d_g(x,u(z))^2\big(|u_s|^2+|u_t|^2\big)+\langle u_s,\nabla_s\xi\rangle+\langle u_t,\nabla_t\xi\rangle&\\ \le C(x)\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\delta^2\big(|u_s|^2+|u_t|^2\big)+\langle u_s,\nabla_s\xi\rangle+\langle u_t,\nabla_t\xi\rangle,&\end{aligned} \quad (1.22)$$

whenever u is J-holomorphic and $u(\partial\Sigma)\cap B_\delta^g(x)=\emptyset$.

Let $u\colon\Sigma\longrightarrow M$ be a J-holomorphic map such that $x\in u(\Sigma)$,

$$A_\eta(\delta)=\frac{1}{2}\int_\Sigma\eta\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\big(|u_s|^2+|u_t|^2\big)\mathrm{d}s\wedge\mathrm{d}t\,,\quad A(\delta)=\frac{1}{2}\int_{u^{-1}(B_\delta^g(x))}\big(|u_s|^2+|u_t|^2\big)\mathrm{d}s\wedge\mathrm{d}t\,.$$

Thus,

$$A'_\eta(\delta) = -\frac{1}{2}\int_\Sigma \eta'\bigg(\frac{d_g(x,u(z))}{\delta}\bigg)\frac{d_g(x,u(z))}{\delta^2}\big(|u_s|^2+|u_t|^2\big)\mathrm{d}s\wedge\mathrm{d}t\,.$$

Combining this identity with (1.22) and (1.18), we find that

$$-\frac{1}{2}\delta A'_\eta(\delta) + A_\eta(\delta) \le C(x)\delta^2 A_\eta(\delta) + C(x)\delta A_\eta(\delta) + C(x)\delta^2 A'_\eta(\delta),$$

for all $\delta\in\mathbb{R}^+$ such that $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. The last inequality is equivalent to

$$\bigg(A_\eta(\delta)\Big/\frac{\delta^2}{(1+C(x)\delta)^4}\bigg)' \ge 0. \tag{1.23}$$

By Lebesgue's Dominated Convergence Theorem, $A_\eta(\delta)\longrightarrow A(\delta)$ from below as $\eta\longrightarrow\chi_{(-\infty,1)}$ (the characteristic function of $(-\infty,1)$). Thus, by (1.23),

$$\delta \longrightarrow A(\delta)\Big/\frac{\delta^2}{(1+C(x)\delta)^4}$$

is a non-decreasing function of δ, as long as $u(\partial\Sigma)\cap B^g_\delta(x)=\emptyset$. By Corollary 1.6,

$$\lim_{\delta\longrightarrow 0}\bigg(A(\delta)\Big/\frac{\delta^2}{(1+C(x)\delta)^4}\bigg) = \lim_{\delta\longrightarrow 0}\frac{A(\delta)}{\delta^2} \ge \pi.$$

This implies the claim. $\square$

Exercise 1.12. Suppose (M,ω) is a symplectic manifold, J is an ω-tame almost complex structure on M,

$$\begin{aligned} g_J(v,v') &= \frac{1}{2}\big(\omega(v,Jv')-\omega(Jv,v')\big),\\ \omega_J(v,v') &= \frac{1}{2}\big(\omega(Jv,Jv')-\omega(v,v')\big) \end{aligned} \qquad \forall\ v,v'\in T_xM,\ x\in M, \tag{1.24}$$

and $f\colon\Sigma\longrightarrow M$ is a C^1-map. Show that

$$g_J(f_s,f_s)+g_J(f_t,f_t) = 2\omega(f_s,f_t) + g_J(f_s+Jf_t,f_s+Jf_t) + 2\omega_J(f_s,f_t),$$

if $z=s+\mathfrak{i}t$ is a local coordinate on Σ.

Exercise 1.13. Let (M,ω,J), g_J, ω_J, and f be as in Exercise 1.12, and $\xi\in\Gamma(\Sigma;u^*TM)$. Show that the 2-forms

$$\big(g_J(f_s,\nabla_s\xi)+g_J(f_t,\nabla_t\xi)\big)\mathrm{d}s\wedge\mathrm{d}t,\ \big(\omega_J(\nabla_s\xi,f_t)+\omega_J(f_s,\nabla_t\xi)\big)\mathrm{d}s\wedge\mathrm{d}t$$

are independent of the choice of local coordinate $z=s+\mathfrak{i}t$.

Lemma 1.14. *Suppose (M,ω) is a symplectic manifold, J is an ω-compatible almost complex structure on M, and ∇ is the Levi-Civita connection of the metric g_J. If $(\Sigma,\mathfrak{j})$ is a compact Riemann surface with boundary and $u\colon\Sigma\longrightarrow M$ is a J-holomorphic map, then*

$$\int_\Sigma\big(g_J(u_s,\nabla_s\xi)+g_J(u_t,\nabla_t\xi)\big)\mathrm{d}s\wedge\mathrm{d}t = \int_\Sigma\big(\{\nabla_\xi\omega_J\}(u_s,u_t)+\omega_J(\nabla_s\xi,u_t)+\omega_J(u_s,\nabla_t\xi)\big)\mathrm{d}s\wedge\mathrm{d}t$$

for all $\xi\in\Gamma(\Sigma;u^*TM)$ such that $\xi|_{\partial\Sigma}=0$.

Proof. Let $u_\tau(z)=\exp_{u(z)}(\tau\xi(z))$ for $z\in\Sigma$ and $\tau\in\mathbb{R}$ close to 0. Denote by $\widehat{\Sigma}$ the closed oriented surface obtained by gluing two copies of Σ along the common boundary and reversing the orientation on the second copy and by $\widehat{u}_t$ the map restricting to u_t on the first copy of Σ and to u on the second. By Exercise 1.12,

$$\begin{aligned} &E_{g_J}(u_\tau)-\int_\Sigma \omega_J\big((u_\tau)_s,(u_\tau)_t\big)\mathrm{d}s\wedge\mathrm{d}t - E_{g_J}(u)\\ &\quad=\int_{\widehat{\Sigma}}\widehat{u}_t^*\omega+\frac{1}{2}\int_\Sigma g_J\big((u_\tau)_s+J(u_\tau)_t,(u_\tau)_s+J(u_\tau)_t\big)\mathrm{d}s\wedge\mathrm{d}t\ge 0 \quad \forall\tau. \end{aligned}\tag{1.25}$$

The first integral on the right-hand side of (1.25) vanishes, because ω is closed and $\widehat{u}_*$ represents the zero class in $H_2(M;\mathbb{Z})$. Thus, the function

$$\tau\longrightarrow E_{g_J}(u_\tau)-\int_\Sigma \omega_J\big((u_\tau)_s,(u_\tau)_t\big)\mathrm{d}s\wedge\mathrm{d}t - E_{g_J}(u)$$

is minimized at $\tau=0$ (when it equals 0) and so

$$\begin{aligned} 0&=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(E_{g_J}(u_\tau)-\int_\Sigma \omega_J\big((u_\tau)_s,(u_\tau)_t\big)\mathrm{d}s\wedge\mathrm{d}t\bigg)\bigg|_{\tau=0}\\ &=\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg(\frac{1}{2}\int_\Sigma\!\big(g_J((u_\tau)_s,(u_\tau)_s)+g_J((u_\tau)_t,(u_\tau)_t)\big)-\int_\Sigma \omega_J\big((u_\tau)_s,(u_\tau)_t\big)\mathrm{d}s\wedge\mathrm{d}t\bigg)\bigg|_{\tau=0}. \end{aligned}\tag{1.26}$$

Since ∇ is g-compatible and torsion-free,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\big(g_J((u_\tau)_s,(u_\tau)_s)+g_J((u_\tau)_t,(u_\tau)_t)\big)\Big|_{\tau=0}&=g_J(u_s,\nabla_\tau(u_\tau)_s|_{\tau=0})+g_J(u_t,\nabla_\tau(u_\tau)_t|_{\tau=0})\\ &=g_J(u_s,\nabla_s\xi)+g_J(u_t,\nabla_t\xi)\,,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}\omega_J\big((u_\tau)_s,(u_\tau)_t\big)\Big|_{\tau=0}&=\{\nabla_\xi\omega_J\}(u_s,u_t)+\omega_J\big(\nabla_\xi(u_\tau)_s,u_t\big)+\omega_J\big((u_\tau)_s,\nabla_\xi(u_\tau)_t\big)\\ &=\{\nabla_\xi\omega_J\}(u_s,u_t)+\omega_J(\nabla_s\xi,u_t)+\omega_J(u_s,\nabla_t\xi)\,. \end{aligned}\tag{1.27}$$

Combining (1.26) and (1.27), we obtain the claim. □

Lemma 1.15. *Let (M,g) be a Riemannian manifold and $x,y\in M$ be such that $2d_g(x,y)<r_g(x),r_g(y)$, where d_g is the distance function with respect to g and $r_g(\cdot)$ is the injectivity radius of g at the specified point. If $\alpha\colon(-\epsilon,\epsilon)\longrightarrow M$ is a smooth curve such that $\alpha(0)=y$, then*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\Big|_{\tau=0}=-\big\langle\alpha'(0),\exp_y^{-1}x\big\rangle\,.$$

Proof. The smoothness of $\tau\longrightarrow d_g(x,\alpha(\tau))^2$ is immediate, since $\exp_x$ is a diffeomorphism onto the ball $B^g_{r_g(x)}(x)$. If $\beta(\tau)=\exp_x^{-1}\alpha(\tau)$,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}d_g\big(x,\alpha(\tau)\big)^2\Big|_{\tau=0}&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}|\beta(\tau)|^2\Big|_{\tau=0}=\big\langle\beta'(0),\beta(0)\big\rangle_g\\ &=\big\langle\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta'(0)),\{\mathrm{d}_{\beta(0)}\exp_x\}(\beta(0))\big\rangle=\big\langle\alpha'(0),-\exp_y^{-1}x\big\rangle\,; \end{aligned}$$

the third equality holds by Gauss's Lemma. □

Lemma 1.16. *If (M,g) is a Riemannian manifold, there exists a continuous function $C\colon M\longrightarrow\mathbb{R}^+$ with the following property. If $x\in M$, $v\in T_xM$, and $\tau\longrightarrow J(\tau)$ is a Jacobi vector field along the geodesic $\gamma(\tau)=\exp_x(\tau v)$ with $C(x)|v|<1$ and $J(0)=0$, then*

$$\big|J'(1)-J(1)\big|\le C(x)|v|^2|J(1)|\,.$$

Proof. If $f(\tau)=|\tau J'(\tau)-J(\tau)|$ and R_g is the Riemann curvature tensor of g, then $f(0)=0$ and

$$\begin{aligned} f(\tau)f'(\tau)&=\big\langle \tau J''(\tau),\tau J'(\tau)-J(\tau)\big\rangle=\tau\big\langle R(\gamma'(\tau),J(\tau))\gamma'(\tau),\tau J'(\tau)-J(\tau)\big\rangle\\ &\le C(x)|v|^2|J(\tau)|\tau f(\tau)\le 2C(x)|v|^2|J(1)|\tau f(\tau)\,; \end{aligned}$$

the last inequality holds if $|v|$ is sufficiently small. Thus,

$$f'(\tau)\le 2C(x)|v|^2|J(1)|\tau,$$

which implies the claim. □

Corollary 1.17. *If (M,g) is a Riemannian manifold, there exists a continuous function $C\colon M\longrightarrow\mathbb{R}^+$ with the following property. If $x\in M$ and ζ_x is the vector field on $B_{r_g(x)/2}(x)$ given by $\zeta_x(y)=\exp_y^{-1}(x)$, then*

$$\big|\nabla_w\zeta_x+w\big|\le C(x)d_g(x,y)^2|w|\qquad\forall\ w\in T_yM,\ y\in B_{r_g(x)/2}(x),$$

where ∇ is the Levi-Civita connection of g.

Proof. Let $\tau\longrightarrow u(s,\tau)$ be a family of geodesics such that

$$u(s,0)=x,\qquad u(0,1)=y,\qquad \frac{\mathrm{d}}{\mathrm{d}s}u(s,1)\bigg|_{s=0}=w.$$

Then, $J(\tau)=\frac{\mathrm{d}}{\mathrm{d}s}u(s,\tau)\big|_{s=0}$ is a Jacobi vector field along the geodesic $\tau\longrightarrow u(0,\tau)$ with

$$\begin{gathered} J(0)=0,\qquad J(1)=w,\qquad \zeta_x\big(u(s,1)\big)=-\frac{\mathrm{d}}{\mathrm{d}\tau}u(s,\tau)\bigg|_{\tau=1},\\ -\nabla_w\zeta_x=\frac{\mathrm{D}}{\mathrm{d}s}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}\tau}\bigg|_{(s,\tau)=(0,1)}=\frac{\mathrm{D}}{\mathrm{d}\tau}\frac{\mathrm{d}u(s,\tau)}{\mathrm{d}s}\bigg|_{(s,\tau)=(0,1)}=J'(1). \end{gathered}$$

Thus, the claim follows from Lemma 1.16. □

1.3 The Mean Value Inequality

Proposition 1.18 (Mean Value Inequality)**.** *If (M,J) is an almost complex manifold and g is a Riemannian metric on M compatible with J, there exists a continuous function $\hbar_{J,g}\colon M\times\mathbb{R}\longrightarrow\mathbb{R}^+$ with the following property. If $u\colon B_R\longrightarrow M$ is a J-holomorphic map such that*

$$u(B_R)\subset B_r^g(x)\qquad\textit{and}\qquad E_g(u)<\hbar_{J,g}(x,r)$$

for some $x\in M$ and $r\in\mathbb{R}$, then

$$|\mathrm{d}_0u|^2<\frac{16}{\pi R^2}E_g(u)\,. \tag{1.28}$$

According to Proposition 1.18, the norms of the differentials of J-holomorphic maps away from the boundary of the domain are "uniformly" bounded by their L^2-norms (the integral of the square of the norm). In general, one would not expect the value of a function to be bounded by its integral. Proposition 1.18 immediately implies that the energy of J-holomorphic maps from the Riemann sphere S^2 is bounded below.

Proof of Proposition 1.18. Let $\phi(z)=\frac{1}{2}|\mathrm{d}_z u|^2$. By Lemma 1.25 below, $\Delta\phi\ge -A_{J,g}\phi^2$ with $A_{J,g}\colon M\times\mathbb{R}\longrightarrow\mathbb{R}^+$ determined by (M,J,g). The claim with $\hbar_{J,g}=\pi/8A_{J,g}$ thus follows from Proposition 1.24. □

Corollary 1.19 (Lower Energy Bound)**.** *If (M,J) is a compact almost complex manifold and g is a Riemannian metric on M, then there exists $\hbar_{J,g}\in\mathbb{R}^+$ such that $E_g(u)\ge\hbar_{J,g}$ for every non-constant J-holomorphic map $u\colon S^2\longrightarrow X$.*

Proof. By the compactness of M, we can assume that g is compatible with J. Let $\hbar_{J,g}>0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 1.18 on the compact space $M\times[0,\mathrm{diam}_g(M)]$. If $u\colon S^2\longrightarrow X$ is J-holomorphic map with $E_g(u)<\hbar_{J,g}$,

$$|\mathrm{d}_z u|^2<\frac{16}{\pi R^2}E_g\big(u|_{B_R(z)}\big)\le\frac{16}{\pi R^2}E_g(u)\qquad\forall\, z\in\mathbb{C},\;R\in\mathbb{R}^+$$

by Proposition 1.18, since $B_R(z)\subset\mathbb{C}$ as Riemann surfaces. Thus, $\mathrm{d}_z u=0$ for all $z\in\mathbb{C}$, and so u is constant. □

If $\phi\colon U\longrightarrow\mathbb{R}$ is a C^2-function on an open subset of $\mathbb{R}^2$, let

$$\Delta\phi=\frac{\partial^2\phi}{\partial s^2}+\frac{\partial^2\phi}{\partial t^2}\equiv\phi_{ss}+\phi_{tt}$$

denote the Laplacian of ϕ.

Corollary 1.20. *If (M,J) is a compact almost complex manifold and g is a Riemannian metric on M, there exists a continuous function $\epsilon_{J,g}\colon\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that*

$$\mathrm{diam}_g\big(u\big([-R+1,R-1]\times S^1\big)\big)\le\delta$$

whenever $u\colon(-R,R)\times S^1\longrightarrow M$ is a J-holomorphic map with $E_g(u)<\epsilon_{J,g}(\delta)$ and $\delta\in\mathbb{R}^+$.

Proof. Let $\hbar_{J,g}>0$ be the minimal value of the function $\hbar_{J,g}$ in the statement of Proposition 1.18 on the compact space $M\times[0,\mathrm{diam}_g(M)]$. If $E_g(u)<\hbar_{J,g}$, then

$$|d_z u|^2\le 8E_g(u)\qquad\forall\; z\in[-R+1,R-1]\times S^1\,.$$

Thus, $\mathrm{diam}_g(u(r\times S^1))\le 16\sqrt{E_g(u)}$ for every $r\in[-R+1,R-1]$. If

$$\delta_u\equiv\mathrm{diam}_g(u([-R+1,R-1]\times S^1))>64\sqrt{E_g(u)},$$

there exist

$$r_-,r_0,r_+\in[-R+1,R-1],\quad \theta_-,\theta_0,\theta_+\in S^1\qquad\text{s.t.}$$
$$r_-<r_0<r_+,\quad d_g\big(u(r_0,\theta_0),u(r_\pm,\theta_\pm)\big)\ge\frac{1}{2}\delta_u\,.$$

Applying Theorem 1.11 with

$$\Sigma = [r_-, r_+]\times S^1, \qquad x = u(r_0,\theta_0), \quad \text{and} \quad \delta = \frac{1}{4}\delta_u,$$

we conclude that

$$E_g(u) \ge \frac{\pi\delta_u^2}{16(1+C_{J,g}\delta_u)^4}\,,$$

for some $C_{J,g}\in\mathbb{R}^+$ dependent only on (M,J,g). It follows that the function

$$\epsilon_{J,g} = \min\left(\frac{\delta^2}{64^2}, \frac{\pi\delta^2}{16(1+C_{J,g}\delta)^4}\right)$$

has the claimed property. □

Exercise 1.21. Show that in the polar coordinates (r,θ) on $\mathbb{R}^2$,

$$\Delta\phi = \phi_{rr} + r^{-1}\phi_r + r^{-2}\phi_{\theta\theta}\,. \tag{1.29}$$

Lemma 1.22. *If $\phi\colon \overline{B_R}\longrightarrow\mathbb{R}$ is C^2, then*

$$2\pi R\,\phi(0) = -R\int_{(r,\theta)\in B_R}(\ln R - \ln r)\Delta\phi + \int_{\partial B_R}\phi\,. \tag{1.30}$$

Proof. By Stokes' Theorem applied to $\phi\mathrm{d}\theta$ on $\overline{B_R}-B_\epsilon$,

$$\begin{aligned}\int_{\partial B_R}\phi\,\mathrm{d}\theta - \int_{\partial B_\delta}\phi\,\mathrm{d}\theta &= \int_{\overline{B_R}-B_\delta}\phi_r\,\mathrm{d}r\wedge\mathrm{d}\theta = \int_0^{2\pi}\!\!\int_\delta^R (r\phi_r)r^{-1}\,\mathrm{d}r\mathrm{d}\theta\\ &= \int_0^{2\pi}(\ln R - \ln\delta)\delta\,\phi_r(\delta,\theta)\mathrm{d}\theta + \int_0^{2\pi}\!\!\int_\delta^R(\ln R-\ln r)(\phi_{rr}+r^{-1}\phi_r)r\,\mathrm{d}r\mathrm{d}\theta\,;\end{aligned}$$

the last equality above is obtained by applying integration by parts to the functions $\ln r - \ln R$ and $r\phi_r$. Sending $\delta\longrightarrow 0$ and using (1.29), we obtain

$$\frac{1}{R}\int_{\partial B_R}\phi - 2\pi\,\phi(0) = 0 + \int_{(r,\theta)\in B_R}(\ln R-\ln r)\Delta\phi\,,$$

which is equivalent to (1.30). □

Corollary 1.23. *If $\phi\colon\overline{B_R}\longrightarrow\mathbb{R}$ is C^2 and $\Delta\phi\ge -C$ for some $C\in\mathbb{R}^+$, then*

$$\phi(0) \le \frac{1}{8}CR^2 + \frac{1}{\pi R^2}\int_{B_R}\phi\,. \tag{1.31}$$

Proof. By (1.30),

$$2\pi r\,\phi(0) \le Cr\int_0^{2\pi}\!\!\int_0^r(\ln r - \ln\rho)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_{\partial B_r}\phi = Cr\cdot 2\pi\cdot\frac{r^2}{4} + \int_{\partial B_r}\phi \qquad \forall\, r\in(0,R).$$

Integrating the above in $r\in(0,R)$, we obtain

$$2\pi\phi(0)\cdot\frac{R^2}{2} \le 2\pi C\cdot\frac{R^4}{16} + \int_{B_R}\phi.$$

This inequality is equivalent to (1.31). □

Proposition 1.24. *If* $\phi\colon B_R \longrightarrow \mathbb{R}^{\ge 0}$ *is* C^2 *and there exists* $A\in\mathbb{R}^+$ *such that* $\Delta\phi \ge -A\phi^2$ *and* $\int_{B_R}\phi < \frac{\pi}{8A}$, *then*

$$\phi(0) \le \frac{8}{\pi R^2}\int_{B_R}\phi\,. \tag{1.32}$$

Proof. Replacing A by $\tilde{A}=R^2A$ and ϕ by

$$\tilde{\phi}\colon B_1 \longrightarrow \mathbb{R}, \qquad \tilde{\phi}(z)=\phi(Rz),$$

we can assume that $R=1$, as well as that ϕ is defined on $\overline{B_1}$.

(1) Define

$$f\colon [0,1)\longrightarrow \mathbb{R} \qquad \text{by} \qquad f(r)=(1-r)^2\sup_{B_r}\phi\,;$$

in particular, $f(0)=\phi(0)$ and $f(1)=0$. Choose $r^*\in[0,1)$ and $z^*\in B_{r^*}$ such that

$$f(r^*)=\sup f \qquad \text{and} \qquad \phi(z^*)=\sup_{B_{r^*}}\phi\equiv c^*\,.$$

Let $\delta=\frac{1}{2}(1-r^*)>0$; see Figure 1. Thus,

$$\sup_{B_\delta(z^*)}\phi \le \sup_{B_{r^*+\delta}}\phi = \frac{f(r^*+\delta)}{(1-(r^*+\delta))^2} \le \frac{f(r^*)}{\frac{1}{4}(1-r^*)^2} = 4\phi(z^*) = 4c^*\,.$$

In particular, $\Delta\phi \ge -A\phi^2 \ge -16Ac^{*2}$ on $B_\delta(z^*)$.

(2) Using Corollary 1.23, we thus find that

$$c^* = \phi(z^*) \le \frac{1}{8}\cdot 16Ac^{*2}\cdot\rho^2 + \frac{1}{\pi\rho^2}\int_{B_\rho(z^*)}\phi \le 2Ac^{*2}\rho^2 + \frac{1}{\pi\rho^2}\int_{B_1}\phi \qquad \forall\ \rho\in[0,\delta]\,. \tag{1.33}$$

If $2Ac^*\delta^2 \le \frac{1}{2}$, the $\rho=\delta$ case of the above inequality gives

$$\frac{1}{2}c^* \le \frac{1}{\pi\delta^2}\int_{B_1}\phi\,, \qquad \phi(0)=f(0)\le f(r^*)=4c^*\cdot\delta^2 \le \frac{8}{\pi}\int_{B_1}\phi\,,$$

as claimed. If $2Ac^*\delta^2 \ge \frac{1}{2}$, $\rho\equiv(4Ac^*)^{-\frac{1}{2}} \le \delta$ and (1.33) gives

$$c^* \le 2Ac^{*2}\cdot\frac{1}{4Ac^*} + \frac{4Ac^*}{\pi}\int_{B_1}\phi\,.$$

Thus, $\frac{\pi}{8A} \le \int_{B_1}\phi$, contrary to the assumption. □

Lemma 1.25. *If* (M,J) *is an almost complex manifold and* g *is a Riemannian metric on* M *compatible with* J, *there exists a continuous function* $A_{J,g}\colon M\times\mathbb{R}\longrightarrow\mathbb{R}^+$ *with the following property. If* $\Omega\subset\mathbb{C}$ *is an open subset,* $u\colon\Omega\longrightarrow M$ *is a* J*-holomorphic map, and* $u(\Omega)\subset B_r^g(x)$ *for some* $x\in M$ *and* $r\in\mathbb{R}$, *the function* $\phi(z)\equiv\frac{1}{2}|\mathrm{d}_z u|_g^2$ *satisfies* $\Delta\phi \ge -A_{J,g}(x,r)\phi^2$.

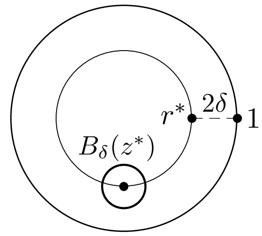

Figure 1: Setup for the proof of Proposition 1.24

Proof. Let $z=s+\mathrm{i}t$ be the standard coordinate on $\mathbb{C}$ and denote by u_s and u_t the s and t-partials of u, respectively. Since u is J-holomorphic, i.e. $u_s=-Ju_t$, and g is J-compatible, i.e. $g(J\cdot,J\cdot)=g(\cdot,\cdot)$, $|u_s|^2=|u_t|^2$, where $|\cdot|$ is the norm with respect to the metric g. Since the Levi-Civita connection ∇ of g is g-compatible,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2 t}|u_s|^2 = |\nabla_t u_s|^2 + \langle \nabla_t\nabla_t u_s, u_t\rangle = |\nabla_t u_s|^2 + \langle \nabla_t\nabla_s u_t, u_s\rangle\,; \tag{1.34}$$

the last equality holds because ∇ is torsion-free. Similarly,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}^2 s}|u_t|^2 = |\nabla_s u_t|^2 + \langle \nabla_s\nabla_t u_s, u_t\rangle. \tag{1.35}$$

Since $u_s=-Ju_t$,

$$\begin{aligned}\langle \nabla_s\nabla_t u_s, u_t\rangle &= -\langle \nabla_s\nabla_t(Ju_t), u_t\rangle \\ &= -\langle J\nabla_s\nabla_t u_t, u_t\rangle - \langle(\nabla_s J)\nabla_t u_t, u_t\rangle - \langle \nabla_s((\nabla_t J)u_t), u_t\rangle \\ &= -\langle \nabla_s\nabla_t u_t, u_s\rangle - \langle(\nabla_s J)\nabla_t u_t, u_t\rangle - \langle \nabla_s((\nabla_t J)u_t), u_t\rangle\,.\end{aligned} \tag{1.36}$$

Putting (1.34)-(1.36), we find that

$$\frac{1}{2}\Delta\phi = |\nabla_t u_s|^2 + |\nabla_s u_t|^2 + \langle R_g(u_t,u_s)u_t,u_s\rangle - \langle(\nabla_s J)\nabla_t u_t, u_t\rangle - \langle \nabla_s((\nabla_t J)u_t), u_t\rangle\,, \tag{1.37}$$

where R_g is the curvature tensor of the connection ∇. Since $u(\Omega)\subset B_r^g(x)$,

$$\begin{aligned}\big|\langle R_g(u_t,u_s)u_t,u_s\rangle\big| &\le C_g(x,r)|u_s|^2|u_t|^2\,, \\ \big|\langle(\nabla_s J)\nabla_t u_t, u_t\rangle\big| &\le C_{J,g}(x,r)|u_s||u_t||\nabla_t(Ju_s)| \le C_{J,g}(x,r)|u_s||u_t|\big(|u_s||u_t|+|\nabla_t u_s|\big) \\ &\le (C_{J,g}(x,r)+C_{J,g}(x,r)^2)|u_s|^2|u_t|^2 + |\nabla_t u_s|^2\,, \\ \big|\langle \nabla_s((\nabla_t J)u_t), u_t\rangle\big| &\le C_{J,g}(x,r)|u_t|^2\big(|u_s||u_t|+|\nabla_s u_t|\big) \\ &\le C_{J,g}(x,r)|u_s||u_t|^3 + C_{J,g}(x,r)^2|u_t|^4 + |\nabla_s u_t|^2.\end{aligned} \tag{1.38}$$

Combining (1.37) and (1.38), we find that

$$\frac{1}{2}\Delta\phi \ge -C(x,r)\big(|u_s|^2|u_t|^2+|u_s||u_t|^3+|u_t|^4\big) \ge -8C(x,r)\phi^2,$$

as claimed. □

1.4 Energy bound on long cylinders

Proposition 1.26. *If (M,J) is a symplectic manifold and g is a Riemannian metric on M, then there exist continuous functions $\delta_{J,g},\hbar_{J,g},C_{J,g}: M\longrightarrow\mathbb{R}^+$ with the following properties. If $u\colon[-R,R]\times S^1\longrightarrow M$ is a J-holomorphic map such that $\operatorname{Im}u\subset B^g_{\delta_{J,g}(u(0,1))}(u(0,1))$, then*

$$E_g\big(u;[-R+T,R-T]\times S^1\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T}E_g(u)\qquad\forall\,T\ge 0\,. \tag{1.39}$$

If in addition $E_g(u)<\hbar_{J,g}\big(u(0,1)\big)$, then

$$\operatorname{diam}_g\big(u([-R+T,R-T]\times S^1)\big)\le C_{J,g}\big(u(1,0)\big)\mathrm{e}^{-T/2}\sqrt{E_g(u)}\quad\forall\,T\ge 1\,. \tag{1.40}$$

Corollary 1.27. *If (M,J) is a compact almost complex manifold and g is a Riemannian metric on M, there exist $\hbar_{J,g},C_{J,g}\in\mathbb{R}^+$ with the following property. If $u:[-R,R]\times S^1\longrightarrow M$ is a J-holomorphic map such that $E_g(u)<\hbar_{J,g}$, then*

$$\begin{aligned}E_g\big(u;[-R+T,R-T]\times S^1\big)&\le C_{J,g}\mathrm{e}^{-T}E_g(u)&&\forall\,T\ge 0,\\ \operatorname{diam}_g\big(u([-R+T,R-T]\times S^1)\big)&\le C_{J,g}\mathrm{e}^{-T/2}\sqrt{E_g(u)}&&\forall\,T\ge 2\,.\end{aligned}$$

Proof. Let $\delta\in\mathbb{R}^+$ be the minimum of the function $\delta_{J,g}$ in Proposition 1.26. Take $C_{J,g}\ge1$ to be at least as the big as the maximum of the function $C_{J,g}$ in Proposition 1.26 and $\hbar_{J,g}\in\mathbb{R}^+$ to be smaller than the minimum of the function $\hbar_{J,g}$ in Proposition 1.26 and the number $\varepsilon_{J,g}(\delta)$ with $\varepsilon_{J,g}(\cdot)$ as in Corollary 1.20. □

As an example, the energy of the injective map

$$[-R,R]\times S^1\longrightarrow\mathbb{C},\qquad(s,\theta)\longrightarrow s\mathrm{e}^{\mathrm{i}\theta}\,,$$

is the area of its image, i.e. $\pi\big(\mathrm{e}^{2R}–\mathrm{e}^{-2R}\big)$. Thus, the exponent e^{-T} in (1.39) can be replaced by e^{-2T} in this case. The proof of Proposition 1.26 shows that in general the exponent can be taken to be $\mathrm{e}^{-\mu T}$ with μ arbitrarily close to 2, but at the cost of increasing $C_{J,g}$ and reducing $\delta_{J,g}$.

Lemma 1.28 (Poincare Inequality)**.** *If $f:S^1\longrightarrow\mathbb{R}^N$ is a smooth function such that $\int_0^{2\pi}f(\theta)\mathrm{d}\theta=0$,*

$$\int_0^{2\pi}|f(\theta)|^2\mathrm{d}\theta\le\int_0^{2\pi}|f'(\theta)|^2\mathrm{d}\theta.$$

Proof: We can write $f(\theta)=\sum\limits_{k>-\infty}^{k<\infty}a_k\mathrm{e}^{\mathrm{i}k\theta}$. Since $\int_0^{2\pi}f(\theta)\mathrm{d}\theta=0$, $a_0=0$. Thus,

$$\int_0^{2\pi}|f(\theta)|^2\mathrm{d}\theta=\sum_{k>-\infty}^{k<\infty}|a_k|^2\le\sum_{k>-\infty}^{k<\infty}|ka_k|^2=\int_0^{2\pi}|f'(\theta)|^2\mathrm{d}\theta.$$

Proof of Proposition 1.26. It is sufficient to establish the first statement under the assumption that (M,g) is $\mathbb{C}^n$ with the standard Riemannian metric, J agrees with the standard complex structure J_0 at $0\in\mathbb{C}^n$, and $u(0,1)=0$. Let

$$\bar\partial u=\frac12\big(u_s+J_0u_\theta\big)\,.$$

By our assumptions, there exist $\delta', C>0$ (dependent on $u(0,1)$) such that

$$\big|\bar{\partial}_z u\big| \le C\delta\big|\mathrm{d}_z u\big| \qquad \forall\, z\in u^{-1}\big(B_\delta(0)\big),\ \delta\le\delta'\,. \tag{1.41}$$

Write $u=f+\mathrm{i}g$, with f,g taking values in $\mathbb{R}^n$ and assume that $\mathrm{Im}\,u\subset B_\delta(0)$. By Exercise 1.12 (or a direct computation), (1.41), and Stokes' Theorem,

$$\begin{aligned}\int_{[-t,t]\times S^1}|\mathrm{d}u|^2 &= 4\int_{[-t,t]\times S^1}\big|\bar{\partial}u\big|^2+2\int_{[-t,t]\times S^1}\mathrm{d}(f\cdot\mathrm{d}g)\\ &\le 4C^2\delta^2\int_{[-t,t]\times S^1}\big|\bar{\partial}u\big|^2+2\int_{\{t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta-2\int_{\{-t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta\,.\end{aligned}\tag{1.42}$$

Let $\tilde{f}=f-\frac{1}{2\pi}\int_0^{2\pi}f\mathrm{d}\theta$. By Hölder's inequality and Lemma 1.28,

$$\begin{aligned}\int_{\{\pm t\}\times S^1}f\cdot g_\theta\,\mathrm{d}\theta &= \int_{\{\pm t\}\times S^1}\tilde{f}\cdot g_\theta\,\mathrm{d}\theta \le \bigg(\int_{\{\pm t\}\times S^1}|\tilde{f}|^2\mathrm{d}\theta\bigg)^{\frac12}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac12}\\ &\le \bigg(\int_{\{\pm t\}\times S^1}|\tilde{f}_\theta|^2\mathrm{d}\theta\bigg)^{\frac12}\bigg(\int_{\{\pm t\}\times S^1}|g_\theta|^2\mathrm{d}\theta\bigg)^{\frac12}\le\frac12\int_{\{\pm t\}\times S^1}|u_\theta|^2\mathrm{d}\theta\,.\end{aligned}\tag{1.43}$$

Since

$$3|u_\theta|^2=2|u_\theta|^2+\big|u_t-2\bar{\partial}u\big|^2\le 2|\mathrm{d}u|^2+8\big|\bar{\partial}u\big|^2\,,$$

the inequalities (1.41)-(1.43) give

$$\big(1-4C^2\delta^2\big)\int_{[-t,t]\times S^1}|\mathrm{d}u|^2\le\frac23\big(1+4C^2\delta^2\big)\bigg(\int_{\{t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta+\int_{\{-t\}\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\bigg).$$

Thus, the function

$$\varepsilon(T)\equiv E_g\big(u;[-R+T,R-T]\big)\equiv\frac12\int_{[-R+T,R-T]\times S^1}|\mathrm{d}u|^2\mathrm{d}\theta\mathrm{d}s$$

satisfies $\varepsilon(T)\le-\varepsilon'(T)$ for all $T\in[-R,R]$, if δ is sufficiently small (depending on C). This implies (1.39).

Let $h_{J,g}(x)=(x,\delta_{J,g}(x))$, with $h_{J,g}(\cdot,\cdot)$ as in Proposition 1.18 and $\delta_{J,g}(\cdot)$ as provided by the previous paragraph. Suppose u also satisfies the last condition in Proposition 1.26. By Proposition 1.18 and (1.39),

$$|\mathrm{d}_{(s,\theta)}u|\le 3\sqrt{E_g(u;[-|s|-1,|s|+1]\times S^1)}\le 3\sqrt{C_{J,\omega}(u(0,1))}\mathrm{e}^{(1+|s|-R)/2}\sqrt{E_g(u)}$$

for all $s\in[-R+1,R-1]$ and $\theta\in S^1$. Thus, for any $s_1,s_2\in[-R+T,R-T]$ with $T\ge1$ and $\theta_1,\theta_2\in S^1$,

$$\begin{aligned}d_g\big(u(s_1,\theta_1),u(s_2,\theta_2)\big)&\le 3\sqrt{C_{J,\omega}(u(0,1))}\mathrm{e}^{(1+|s_1|-R)/2}\sqrt{E_g(u)}\bigg(\pi+\bigg|\int_{s_1}^{s_2}\mathrm{e}^{(1+|s|-R)/2}\mathrm{d}s\bigg|\bigg)\\ &\le\big(3\pi+12\big)\sqrt{C_{J,\omega}(u(0,1))}\sqrt{E_g(u)}\,\mathrm{e}^{(1-T)/2}\,.\end{aligned}$$

This establishes (1.40). $\square$

2 Global Properties

The properties of J-holomorphic maps to the almost complex manifold (M,J) described in this section depend on M being compact.

For each $R\in\mathbb{R}^+$, denote by $B_R\subset\mathbb{C}$ the open ball of radius R around the origin and let $B_R^*=B_R-\{0\}$, as before.

Theorem 2.1 (Removal of Singularity)**.** *Let (M,J) be a compact almost complex manifold and $u\colon B_R^*\longrightarrow M$ be a J-holomorphic map with respect to the standard complex structure $\mathfrak{i}$ on $\mathbb{C}$. If the energy $E(u)$ of u, with respect to any metric on B_R and on M, is finite, then u extends to a J-holomorphic map $\tilde{u}\colon B_R\longrightarrow M$.*

A basic example of a holomorphic function $u\colon\mathbb{C}^*\longrightarrow\mathbb{C}$ that does not extend over the origin $0\in\mathbb{C}$ is $z\longrightarrow 1/z$. The energy of $u|_{B_R^*}$ with respect to the standard metric on $\mathbb{C}$ is given by

$$E\big(u|_{B_R^*}\big)=\frac{1}{2}\int_{B_R}|\mathrm{d}u|^2=\int_{B_R}\frac{1}{|z|^2}=\int_0^{2\pi}\!\!\int_0^R r^{-1}\mathrm{d}r\mathrm{d}\theta\not<\infty.$$

The above integral would have been finite if $|\mathrm{d}u|^2$ were replaced by $|\mathrm{d}u|^{2-\epsilon}$ for *any* $\epsilon>0$. This observation illustrates the crucial role played by the energy in the theory of J-holomorphic maps.

It is a standard fact in complex analysis that a bounded holomorphic map $u\colon B_R^*\longrightarrow\mathbb{C}^n$ extends to a holomorphic map $\tilde{u}\colon B_R\longrightarrow\mathbb{C}^n$. This implies the conclusion of Theorem 2.1 whenever J is an integrable almost complex structure and $u(B_\delta^*)$ is contained in a complex coordinate chart for some $\delta\in(0,R)$. We will use the finiteness of the energy of u to show that the latter is the case; the integrability of J turns out to be irrelevant here.

Proof of Theorem 2.1. We can assume that $R=1$. The first step is to show that u extends continuously over the origin.

(1) The map

$$v\colon\mathbb{R}^-\times S^1\longrightarrow M,\qquad v(r,\theta)=u\big(\mathrm{e}^{r+\mathfrak{i}\theta}\big),$$

is J-holomorphic and satisfies $E(v)=E(u)<\infty$. For each $i\in\mathbb{Z}^+$, define

$$v_i\colon\overline{\mathbb{R}}^-\times S^1,\qquad v_i(r,\theta)=v(r-i,\theta).$$

This map is again J-holomorphic and $E(v_i)=E(v|_{(-\infty,-i)\times S^1})$ approaches zero as $i\longrightarrow\infty$, since $E(v)<\infty$. Proposition 1.18 then implies that $|\mathrm{d}v_i|_{L^\infty}\longrightarrow 0$. Since M is a compact, v_i contains a subsequence which converges uniformly on compact subsets to a C^1-function $v_\infty\colon\overline{\mathbb{R}}^-\times S^1\longrightarrow M$ with $\mathrm{d}v_\infty=0$. Thus, v_∞ is the constant map to a point $x\in M$.

We next show that

$$\lim_{r\longrightarrow-\infty}v(r,\theta)=x\qquad\forall\,\theta\in S^1,$$

and so the extension of u defined by $\tilde{u}(0)=x$ is continuous. Suppose instead that there exist $\delta>0$ and a sequence $(r_k,\theta_k)\in\mathbb{R}^-\times S^1$ such that $r_k\longrightarrow-\infty$ and $v(r_k,\theta_k)\notin B_{3\delta}(x)$. By the same reasoning as in the previous paragraph, we can assume that the functions

$$\tilde{v}_k\colon\overline{\mathbb{R}}^-\times S^1,\qquad\tilde{v}_k(r,\theta)=v(r+r_k,\theta),$$

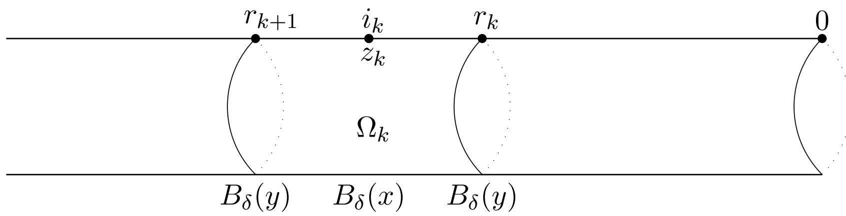

Figure 2: Setup for the proof of Theorem 2.1

converge uniformly on compact subsets to the constant function to some $y\in M-B_{3\delta}(x)$. By the uniform convergence of v and $\tilde{v}$, we can choose sequences r_k and $i_k\in\mathbb{Z}^-$ such that

$$r_{k+1} < i_k < r_k, \qquad v\big(\{i_k\}\times S^1\big) \subset B_\delta(x), \qquad v\big(\{r_k\}\times S^1\big) \subset B_\delta(y).$$

Let $\overline{\Omega}_k = [r_{k+1}, r_k]\times S^1$ and $z_k \in \overline{\Omega}_k$ be such that $v(z_k) \in B_\delta(x)$; see Figure 2. Since $v(\partial\overline{\Omega}_k)\cap B_\delta(v(z_k))=\emptyset$,

$$E(v) \ge \sum_{k=1}^{\infty} E\big(v|_{\overline{\Omega}_k}\big) \ge \sum_{k=1}^{\infty} \pi\frac{\delta^2}{(1+C\delta)^4} = \infty;$$

the second inequality above holds by Theorem 1.11. However, this contradicts the assumption that $E(v)<\infty$.

(2) It remains to show that the extension $\tilde{u}$ is a smooth function. We can now assume that $u\colon (B_1,0)\longrightarrow(\mathbb{C}^n,0)$ is a continuous map such that its restriction to B_1^* is smooth and satisfies

$$u_s + J(u)u_t = 0 \tag{2.1}$$

for some smooth almost complex structure J on $\mathbb{C}^n$ such that $J(0)=\mathfrak{i}$. □

3 Convergence

The next lemma is used to show that no energy is lost under Gromov's convergence and the resulting bubbles connect.

Lemma 3.1. *If (M,J) is a compact almost complex manifold and g is a Riemannian metric on M, then there exists $\hbar_{J,g}\in\mathbb{R}^+$ with the following properties. If $u_i: B_1\longrightarrow M$ is a sequence of J-holomorphic maps converging uniformly in the C^∞-topology on compact subsets of B_1^* to a J-holomorphic map $u\colon B_1\longrightarrow M$ such the limit*

$$\mathfrak{m} \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g(u_i;B_\delta) \tag{3.1}$$

exists and is nonzero, then

(1) $\mathfrak{m}\ge\hbar_{J,g}$;

(2) the limit $\mathfrak{m}(\delta)\equiv\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta)$ exists and is a continuous, non-decreasing function of δ;

(3) for every sequence $z_i\in B_\delta$ converging to 0, $\lim_{i\longrightarrow\infty}E_g(u_i;B_\delta(z_i))=\mathfrak{m}(\delta)$;

(4) for every sequence $z_i \in B_\delta$ converging to 0, $\mu \in (0, \mathfrak{m})$, and $i \in \mathbb{Z}^+$ sufficiently large, there exists a unique $\delta_i(\mu) \in \mathbb{R}^+$ such that $E_g(u_k; B_{\delta_i(\mu)}(z_i)) = \mu$;

(5) for every sequence $z_i \in B_\delta$ converging to 0 and $\mu \in (\mathfrak{m} - \hbar_{J,g}, \mathfrak{m})$,

$$\lim_{R\longrightarrow\infty} \lim_{i\longrightarrow\infty} E_g(u_i; B_{R\delta_i(\mu)}(z_i)) = \mathfrak{m}, \tag{3.2}$$

$$\lim_{R\longrightarrow\infty} \lim_{\delta\longrightarrow 0} \lim_{i\longrightarrow\infty} \mathrm{diam}_g\big(u_i(B_\delta - B_{R\delta_i(\mu)}(z_i))\big) = 0. \tag{3.3}$$

Proof. Let $\hbar_{J,g}$ be the smaller of the constants $\hbar_{J,g}$ in Corollaries 1.19 and 1.27. Let u_i, u, and $\mathfrak{m}$ be as in the statement of the lemma.

(1) By the rescaling procedure at the beginning of [5, Section 4.2], a subsequence of u_i gives rise to a non-constant J-holomorphic map v (bubble at 0) such that

$$E_g(v) \le \lim_{\delta\longrightarrow 0} \lim_{i\longrightarrow\infty} E_g(u_i; B_\delta) \equiv \mathfrak{m}.$$

By Corollary 1.19, $\hbar_{J,g} \le E_g(v)$.

(2) Since $\mathrm{d}u_i$ converges uniformly to $\mathrm{d}u$ on compact subsets of B_1^*,

$$\begin{aligned}\mathfrak{m}(\delta) \equiv \lim_{i\longrightarrow\infty} E_g(u_i; B_\delta) &= \lim_{\delta'\longrightarrow 0} \lim_{i\longrightarrow\infty} E_g(u_i; B_{\delta'}) + \lim_{\delta'\longrightarrow 0} \lim_{i\longrightarrow\infty} E_g(u_i; B_\delta - B_{\delta'}) \\ &= \mathfrak{m} + \lim_{\delta'\longrightarrow 0} E_g(u; B_\delta - B_{\delta'}) = \mathfrak{m} + E_g(u; B_\delta).\end{aligned}$$

Since $E_g(u; B_\delta)$ is a continuous, non-decreasing function of δ, so is $\mathfrak{m}(\delta)$.

(3) For each $\delta' \in \mathbb{R}^+$, $z_i \in B_{\delta'}$ for all $i \in \mathbb{Z}^+$ sufficiently large and so

$$E_g\big(u_i; B_{\delta-\delta'}\big) \le E_g\big(u_i; B_\delta(z_i)\big) \le E_g\big(u_i; B_{\delta+\delta'}\big).$$

This implies that

$$\mathfrak{m}(\delta - \delta') \le \lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) \le \mathfrak{m}(\delta + \delta') \qquad \forall\ \delta' \in \mathbb{R}^+.$$

The claim now follows from (2).

(4) By (3), (2), and (3.1),

$$\Big|\lim_{i\longrightarrow\infty} E_g\big(u_i; B_\delta(z_i)\big) - \mathfrak{m}\Big| < \frac{1}{2}(\mathfrak{m} - \mu)$$

for some $\delta \in (0,1)$. Thus, there exists $i(\mu) \in \mathbb{Z}^+$ such that

$$\big|E_g\big(u_i; B_\delta(z_i)\big) - \mathfrak{m}\big| < \mathfrak{m} - \mu \qquad \forall\ i \ge i(\mu)$$

and thus $E_g(u_i; B_\delta(z_i)) > \mu$ for all $i \ge i(\mu)$. Since each $E_g(u_i; B_\delta(z_i))$ is a continuous, increasing function of δ which vanishes at $\delta = 0$, there exists a unique $\delta_i(\mu) \in (0, \delta)$ such that $E_g(u_i; B_{\delta_i(\mu)}(z_i)) = \mu$.

(5) By (3.1), $\delta_i(\mu) \longrightarrow 0$ as $i \longrightarrow \infty$ for every $\mu \in (0, \mathfrak{m})$. Suppose (3.2) does not hold for some $\mu \in (\mathfrak{m} - \hbar_{J,g}, \mathfrak{m})$. After passing to a subsequence, we can assume that

$$\lim_{R \longrightarrow \infty} \lim_{i \longrightarrow \infty} E_g(u_i; B_{R\delta_i(\mu)}(z_i)) = \mu^* \tag{3.4}$$

for some $\mu^* \in [\mu, \mathfrak{m})$. By (3), (2), and (3.1),

$$\lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_\delta(z_i)\big) = \mathfrak{m}\,. \tag{3.5}$$

Thus, after passing to another subsequence, we can assume that there exists a sequence $\delta_i \longrightarrow 0$ such that

$$\lim_{i \longrightarrow \infty} E_g(u_i; B_{\delta_i}(z_i)) = \mathfrak{m}. \tag{3.6}$$

Since $\delta_i \longrightarrow 0$, (3.5) and (3.6) imply that

$$\lim_{R \longrightarrow \infty} \lim_{i \longrightarrow \infty} E_g(u_i; B_{R\delta_i}(z_i)) = \mathfrak{m}. \tag{3.7}$$

By (3.7) and the definition of $\delta_i(\mu)$ in (4),

$$\lim_{i \longrightarrow \infty} E\big(u; B_{R\delta_i}(z_i) - B_{\delta_i(\mu)}(z_i)\big) = \mathfrak{m} - \mu < \hbar_{J,g}\,.$$

Thus, (1.39) applies with (R, T) replaced by $(\frac{1}{2}\ln(R\delta_i/\delta_i(\mu)), \ln R)$ and u replaced by the J-holomorphic map

$$v(r, \theta) = u\big(z_i + \sqrt{R\delta_i\delta_i(\mu)}\, \mathrm{e}^{r+\mathrm{i}\theta}\big)$$

and gives

$$E\big(u; B_{\delta_i}(z_i)\big) - E\big(u; B_{R\delta_i(\mu)}(z_i)\big) = E\big(u; B_{\delta_i}(z_i) - B_{R\delta_i(\mu)}(z_i)\big) \le \frac{C_{J,g}}{R} E_g(u)$$

for all i sufficiently large (depending on R). However, this contradicts (3.4) and (3.6), since $\mu^* < \mathfrak{m}$. This argument is illustrated in Figure 3. Thus, (3.2) holds.

It remains to establish (3.3). By (3), (2), and (3.1),

$$\lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_{R\delta}(z_i)\big) = \lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_\delta(z_i)\big) = \mathfrak{m}.$$

Combining this with the definition of $\delta_i(\mu)$, we find that

$$\lim_{R \longrightarrow \infty} \lim_{\delta \longrightarrow 0} \lim_{i \longrightarrow \infty} E_g\big(u_i; B_{R\delta}(z_i) - B_{\delta_i(\mu)}(z_i)\big) = \mathfrak{m} - \mu < \hbar_{J,g}.$$

Thus, for all $R > 0$, $\delta > 0$ sufficiently small (depending on R), and

$$E_g\big(u_i; B_{R\delta}(z_i) - B_{\delta_i(\mu)}(z_i)\big) < \hbar_{J,g} \qquad \forall\ i > i(R, \delta).$$

Corollary 1.27 then gives

$$\mathrm{diam}_g\big(u_i(B_\delta(z_i) - B_{R\delta_i(\mu)}(z_i))\big) \le \frac{C_{J,g}}{\sqrt{R}} \hbar_{J,g} \qquad \forall\ i > i(R, \delta).$$

This gives (3.3). □

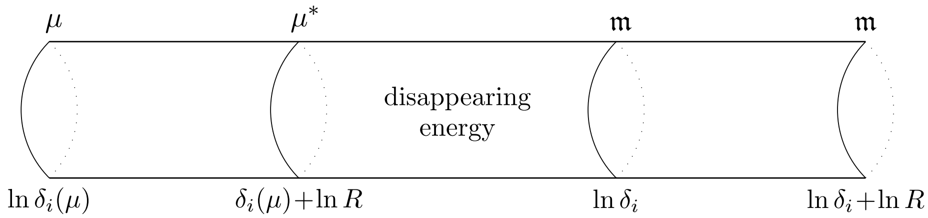

Figure 3: Contradiction in the proof of Lemma 3.1

We next show that a sequence of maps as in Lemma 3.1 gives rise to a continuous map from a tree of spheres attached at $0\in B_1$, i.e. a connected union of spheres that have a distinguished, base component and no loops; the distinguished component will be attached at $\infty\in S^2$ to $0\in B_1$. The combinatorial structure of such a tree is described by a finite rooted linearly ordered set, i.e. a partially ordered set $(I,\prec)$ such that

- there is a minimal element (root) $i_0\in I$, i.e. $i_0\prec h$ for every $h\in I-\{i_0\}$, and
- for all $h_1,h_2,i\in I$ with $h_1,h_2\prec i$, either $h_1=h_2$, or $h_1\prec h_2$, or $h_2\prec h_1$.

For each $i\in I-\{i_0\}$, let $p(i)\in I$ denote the immediate predecessor of i, i.e. $p(i)\in I$ such that $h\prec p(i)\prec i$ for all $h\in I-\{p(i)\}$ such that $h\prec p(i)$; it exists by the first condition above and unique by the second. In the first diagram in Figure 4, the vertices (dots) represent the elements of a rooted linearly ordered set $(I,\prec)$ and the edges run from $i\in I-\{i_0\}$ down to $p(i)$. Given a finite rooted linearly ordered set $(I,\prec)$ with minimal element i_0 and a function

$$z\colon I-\{i_0\}\longrightarrow\mathbb{C},\quad i\longrightarrow z_i,\qquad \text{s.t.}\qquad \big(p(i_1),z_{i_1}\big)\neq\big(p(i_2),z_{i_2}\big)\ \ \forall\, i_1,i_2\in I-\{i_0\},\ i_1\neq i_2,\quad (3.8)$$

let

$$\Sigma=\Big(\bigsqcup_{i\in I}\{i\}\times S^2\Big)\Big/\sim,\qquad (i,\infty)\sim\big(p(i),z_i\big)\quad\forall\, i\in I-\{i_0\};$$

see the second diagram in Figure 4. Thus, the tree of spheres Σ is obtained by attaching ∞ in the sphere indexed by i to z_i in the sphere indexed by $p(i)$. The last condition in (3.8) insures that Σ is a nodal Riemann surface, i.e. each non-smooth point (node) has only two local branches (pieces homeomorphic to $\mathbb{C}$).

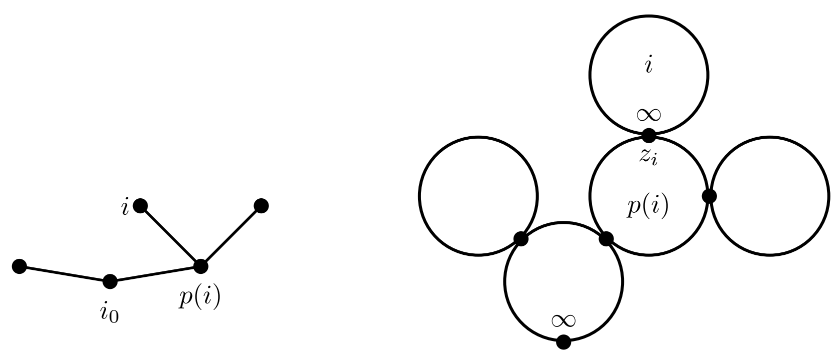

Figure 4: A rooted linearly ordered set and an associated tree of spheres

Proposition 3.2. *Let (M,J) be a compact almost complex manifold, g be a Riemannian metric on M, and $u_i\colon B_1 \longrightarrow M$ be a sequence of J-holomorphic maps converging uniformly in the C^∞-topology on compact subsets of B_1^* to a J-holomorphic map $u\colon B_1 \longrightarrow M$. If the limit*

$$\mathfrak{m} \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g(u_i;B_\delta) \tag{3.9}$$

exists and is nonzero, then there exist

(a) a nodal Riemann surface Σ_∞ consisting of B_1 with a tree of spheres attached at $0\in B_1$,

(b) a continuous map $u_\infty : \Sigma_\infty \longrightarrow M$ which is J-holomorphic map on B_1 and on each of the spheres,

(c) a subsequence of $\{u_i\}$ still denoted by $\{u_i\}$, and

(d) an injective holomorphic map $\psi_i\colon U_i \longrightarrow B_1$, where $U_i\subset\mathbb{C}$ is an open subset,

such that

(1) $E_g(u_\infty;\Sigma_\infty - B_1) = \mathfrak{m}$,

(2) $\mathbb{C} = \bigcup_{i=1}^{\infty} U_i$,

(3) $u_i\circ\psi_i$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty, w_1^*,\ldots,w_k^*$ in the sphere S_0^2 attached at $0\in B_1$,

(4) if $u_\infty|_{S_0^2}$ is constant, S_0^2 contains at least three nodes of Σ_∞;

(5) (d) applies with $(\{u_i\},0)$, B_1, and $\mathfrak{m}$ replaced by $(\{u_i\circ\psi_i\},w_r^*)$, a neighborhood of w_r^* in $\mathbb{C}$, and

$$\mathfrak{m}_r' \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i\circ\psi_i;B_\delta(w_r^*)\big), \tag{3.10}$$

for each $r=1,\ldots,k$.

Proof. Let $\hbar_{J,g}$ be the smallest of the numbers $\hbar_{J,g}$ in Corollaries 1.19 and 1.27 and in Lemma 3.1. In particular, $\mathfrak{m}\ge\hbar_{J,g}$ by Lemma 3.1(1).

For each $i\in\mathbb{Z}^+$ sufficiently large, choose $z_i\in B_1$ so that

$$\big|\mathrm{d}_{z_i}u_i\big| = \sup_{z\in B_1}|\mathrm{d}u_i|. \tag{3.11}$$

Since $z=0$ is the only point in B_1 such that $|\mathrm{d}_z u_i|\longrightarrow\infty$, $z_i\longrightarrow 0$ as $i\longrightarrow\infty$. Thus, there exists $\delta_0\in\mathbb{R}^+$ such that $B_{\delta_0}(z_i)\subset B_1$ for all $i\in\mathbb{Z}^+$ sufficiently large. By Lemma 3.1(4) and (3.9), for all $i\in\mathbb{Z}^+$ sufficiently large there exists $\delta_i\in(0,\delta_0)$ such that

$$E_g\big(u_i;B_{\delta_i}(z_i)\big) = \mathfrak{m} - \frac{\hbar_{J,g}}{2}\,. \tag{3.12}$$

Define

$$\psi_i\colon U_i \equiv B_{\delta_0/\delta_i} \longrightarrow B_1 \qquad\text{by}\qquad \psi_i(w) = z_i + \delta_i w\,.$$

Since $\delta_i\longrightarrow 0$, (2) holds.

For each $i\in\mathbb{Z}^+$ sufficiently large, let

$$v_i = u_i\circ\psi_i : B_{\delta_0/\delta_i} \longrightarrow M.$$

Since u_i is J-holomorphic and ψ_i is biholomorphic onto its image, v_i is J-holomorphic and

$$E_g(v_i) = E_g\big(u_i; B_{\delta_0}(z_i)\big) \le E_g(u_i) \le C \qquad \forall\, i\in\mathbb{Z}^+ .$$

Thus, by the rescaling procedure at the beginning of [5, Section 4.2], there exist a finite collection $w_1^*,\ldots,w_k^*\in\mathbb{C}$ of distinct points, a J-holomorphic map $v: S^2\longrightarrow M$, and a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, such that $u_i\circ\psi_i$ converges to v uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty, w_1^*,\ldots,w_k^*$ in the sphere S_0^2 attached at $0\in B_1$ and the limit (3.10) exists and is at least $\hbar_{J,g}$; see also the proof of Theorem 3.3 below. In particular, (3) holds. Furthermore,

$$\begin{aligned} E_g(v)+\sum_{r=1}^{k}\mathfrak{m}_r' &= \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\Big(v_i, B_R - \bigcup_{r=1}^{k} B_\delta(w_r^*)\Big) + \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E\big(v_i; B_\delta(w_r^*)\big) \\ &= \lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(v_i, B_R\big) = \lim_{R\longrightarrow\infty}\lim_{i\longrightarrow\infty} E_g\big(u_i, B_{R\delta_i}(z_i)\big) = \mathfrak{m}\,; \end{aligned} \tag{3.13}$$

the last equality holds by (3.2).

We next show that $u(0)=v(\infty)$, i.e. that the bubble (S_0^2, v) connects to (B_1, u) at $z=0$. Note that

$$\begin{aligned} d_g\big(u(0), v(\infty)\big) &= \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0} d_g\big(u(z_i+\delta), v(R)\big) = \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} d_g\big(u_i(z_i+\delta), v_i(R)\big) \\ &= \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} d_g\big(u_i(z_i+\delta), u_i(z_i+R\delta_i)\big) \\ &\le \lim_{R\longrightarrow\infty}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} \mathrm{diam}_g\big(u_i(B_\delta(z_i) - B_{R\delta_i}(z_i))\big). \end{aligned}$$

Along with (3.2), this implies that $u(0)=v(\infty)$.

Suppose $v: S^2\longrightarrow M$ is a constant map. By (3.13), $k\ge 1$ and so there exists $w^*\in\mathbb{C}$ such that $|\mathrm{d}_{w^*}v_i|\longrightarrow\infty$ as $i\longrightarrow\infty$. By (3.11) and the definition of ψ_i, $|\mathrm{d}_0 v_i|\ge|\mathrm{d}_w v_i|$ for all $w\in\mathbb{C}$ contained in the domain of v_i and so $|\mathrm{d}_0 v_i|\longrightarrow\infty$ as $i\longrightarrow\infty$. By (3.10) and (3.12),

$$\mathfrak{m}_0' \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i\circ\psi_i; B_\delta\big) \le \lim_{i\longrightarrow\infty} E_g\big(u_i\circ\psi_i; B_1\big) = \mathfrak{m} - \frac{\hbar}{2} < \mathfrak{m},$$

and so $k\ge 2$, as claimed in (4). Since the amount of energy of v_i contained in $\mathbb{C}-B_1$ approaches $\hbar_{J,g}/2$, as illustrated in Figure 5, there must be in particular a blowup point w^* with $|w^*|=1$, though this is not material.

The above establishes Proposition 3.2 whenever $k=0$ by taking $u_\infty|_{B^1}=u$ and $u|_{S_0^2}=v$. Since $m_r'\ge\hbar_{J,g}$ for every r, $k=0$ if $\mathfrak{m}<2\hbar_{J,g}$. If $k\ge 1$, $\mathfrak{m}_r'\le\mathfrak{m}-\hbar_{J,g}$ because $E_g(v)\ge\hbar_{J,g}$ if v is not constant and $k\ge 2$ otherwise. Thus, by induction on $[\mathfrak{m}/\hbar_{J,g}]\in\mathbb{Z}^+$, we can assume that Proposition 3.2 holds when applied to $\{v_i\}$ on a small neighborhood of each $w_j^*\in\mathbb{C}$ with $j=1,\ldots,k$. This yields a continuous map $v_j:\Sigma_j\longrightarrow M$ from a tree of spheres Σ_j such that v_j is J-holomorphic on each sphere and $v_j(\infty)=v(w_j^*)$. Identifying ∞ in the base sphere of each Σ_j with $w_j^*\in S_0^2$, which has been already attached to $0\in B_1^*$, we obtain a continuous map $u_\infty:\Sigma_\infty\longrightarrow M$ with the desired properties. □

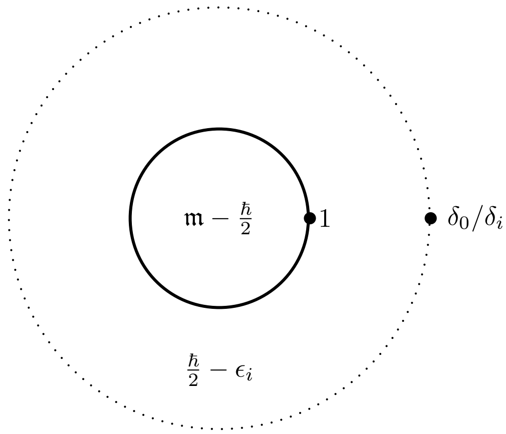

Figure 5: The energy distribution of the rescaled map v_i in the proof of Proposition 3.2

Theorem 3.3 (Gromov's Convergence)**.** *Let (M,J) be a compact almost complex manifold with Riemannian metric g, Σ be a compact Riemann surface, and $u_i : \Sigma \longrightarrow M$ be a sequence of J-holomorphic maps. If $\liminf E_g(u_i) < \infty$, there exist*

(a) a compact nodal Riemann surface Σ_∞ obtained from Σ by identifying a point on each of ℓ trees of spheres, for some $\ell \in \mathbb{Z}^{\geq 0}$, with distinct points $z_1^*, \ldots, z_\ell^* \in \Sigma$,

(b) a continuous map $u_\infty : \Sigma_\infty \longrightarrow M$ which is J-holomorphic map on Σ and on each of the spheres,

(c) a subsequence of $\{u_i\}$ still denoted by $\{u_i\}$, and

(d) for each $z_1^*, \ldots, z_\ell^* \in \Sigma \subset \Sigma_\infty$, a biholomorphic map $\psi_{j;i} : U_{j;i} \longrightarrow U_j$, where $U_{j;i} \subset \mathbb{C}$ is an open subset and $U_j \ni z_j^*$ is an open neighborhood,

such that

(1) $E_g(u_\infty) = \lim_{i \longrightarrow \infty} E_g(u_i)$,

(2) u_i converges to u_∞ uniformly in the C^∞-topology on compact subsets of $\Sigma - \{z_1^*, \ldots, z_\ell^*\}$,

(3) $\mathbb{C} = \bigcup_{i=1}^\infty U_{j,i}$ *for every $j = 1, \ldots, \ell$,*

(4) $u_i \circ \psi_{j;i}$ converges to u_∞ uniformly in the C^∞-topology on compact subsets of the complement of the nodes $\infty, w_{j;1}^*, \ldots, w_{j;k_j}^*$ in the sphere S_j^2 attached at $z_j^* \in \Sigma$,

(5) if $u_\infty|_{S_j^2}$ is constant, S_j^2 contains at least three nodes in total;

(6) (d) applies with $(\{u_i\}, z_1^*, \ldots, z_\ell^*)$ replaced by $(\{u_i \circ \psi_{j;i}\}, w_{j;1}^*, \ldots, w_{j;k_j}^*)$ for each $j = 1, \ldots, \ell$.

Proof. Let $\hbar_{J,g}$ be the smallest of the numbers $\hbar_{J,g}$ in Corollaries 1.19 and 1.27 and in Lemma 3.1.

By the rescaling procedure at the beginning of [5, Section 4.2],

$$\limsup_{i \longrightarrow \infty} \left|\mathrm{d}_{z^*} u\right| = \infty \qquad \Longrightarrow \qquad \lim_{\delta \longrightarrow 0} \limsup_{i \longrightarrow \infty} E_g\big(B_\delta(z^*)\big) \geq \hbar_{J,g},$$

whenever $z^*\in\Sigma$. Since $E_g(u_i)\le C$ for all i, there exist a finite collection $z_1^*,\ldots,z_\ell^*\in\Sigma$ of distinct points and a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$, such that $|\mathrm{d}u_i|$ is uniformly bounded on compact subsets of $\Sigma-\{z_1^*,\ldots,z_\ell^*\}$ and the limit

$$\mathfrak{m}_j \equiv \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E\big(u_i;B_\delta(z_j)\big) \tag{3.14}$$

exists for each $j=1,\ldots,\ell$ and is at least $\hbar_{J,g}$. By the first property and Theorem 2.1, a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$ converges uniformly in the C^∞-topology on compact subsets of $\Sigma-\{z_1,\ldots,z_\ell^*\}$ to a J-holomorphic map u. Furthermore,

$$\begin{split} E_g(u)+\sum_{j=1}^{\ell}\mathfrak{m}_j &= \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u;\Sigma-\bigcup_{j=1}^{\ell}B_\delta(z_j)\big) + \sum_{j=1}^{\ell}\lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g\big(u_i;B_\delta(z_i)\big)\\ &= \lim_{\delta\longrightarrow 0}\lim_{i\longrightarrow\infty} E_g(u_i) = \lim_{i\longrightarrow\infty} E_g(u_i). \end{split} \tag{3.15}$$

Let $U_1,\ldots,U_\ell$ be open neighborhoods of $z_1^*,\ldots,z_\ell^*$, respectively, such that $\overline{U}_{j_1}\cap\overline{U}_{j_2}=\emptyset$ whenever $j_1\neq j_2$.

For each $j=1,\ldots,\ell$, Proposition 3.2 provides a continuous map $v_j:\Sigma_j\longrightarrow M$ from a tree of spheres Σ_j such that v_j is J-holomorphic on each sphere and $v_j(\infty)=u(z_j^*)$. Identifying ∞ in the base sphere of each Σ_j with $z_j^*\in\Sigma$, we obtain a continuous map $u_\infty:\Sigma_\infty\longrightarrow M$ with the desired properties. □

4 An example

We now give an example illustrating Gromov's convergence in a classical setting.

Let $n\in\mathbb{Z}^+$, with $n\ge 2$, and $\mathbb{P}^{n-1}=\mathbb{CP}^{n-1}$. Denote by ℓ the positive generator of $H_2(\mathbb{P}^{n-1};\mathbb{Z})\approx\mathbb{Z}$, i.e. the homology class represented by the standard $\mathbb{P}^1\subset\mathbb{P}^{n-1}$. A degree d map $f:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a continuous map such that $f_*[\mathbb{P}^1]=d\ell$. A holomorphic degree d map $f:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is given by

$$[u,v]\longrightarrow\big[R_1(u,v),\ldots,R_n(u,v)\big]$$

for some degree d homogeneous polynomials $R_1,\ldots,R_d$ on $\mathbb{C}^2$ without a common linear factor. Since the tuple $(\lambda R_1,\ldots,\lambda R_n)$ determines the same map as $(R_1,\ldots,R_n)$ for any $\lambda\in\mathbb{C}^*$, the space of degree d holomorphic maps $f:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a dense open subset of

$$\mathfrak{X}_{n,d}\equiv\big((\mathrm{Sym}^d\mathbb{C}^2)^n-\{0\}\big)/\mathbb{C}^*\approx\mathbb{P}^{(d+1)n-1}.$$

Suppose $f_k:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ is a sequence of holomorphic degree d maps determining the equivalence classes of n-tuples of homogeneous polynomials

$$\mathbf{R}_k=\big[R_{k;1},\ldots,R_{k;n}\big]\in\mathfrak{X}_{n,d}$$

without a common linear factor. Passing to a subsequence, we can assume that $[\mathbf{R}_k]$ converges to some

$$\mathbf{R}\equiv\big[(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_1,\ldots,(v_1u-u_1v)^{d_1}\ldots(v_mu-u_mv)^{d_m}S_n\big]\in\mathfrak{X}_{n,d}\,, \tag{4.1}$$

with $d_1,\ldots,d_m\in\mathbb{Z}^+$ and homogeneous polynomials

$$\mathbf{S}\equiv[S_1,\ldots,S_n]\in\mathfrak{X}_{n,d_0}$$

without a common linear factor and with $d_0\in\mathbb{Z}^{\geq 0}$. By (4.1),

$$d_0+d_1+\ldots+d_m=d.$$

Suppose $z_0\in\mathbb{C}-\{u_1/v_1,\ldots,u_m/v_m\}$ and $S_{i_0}(z_0,1)\neq 0$ for some $i_0=1,\ldots,n$ (such i_0 exists, since $S_1,\ldots,S_n$ do not have a common linear factor). This implies that $R_{k;i_0}(z_0,1)\neq 0$ for all k large enough and so

$$\lim_{k\longrightarrow\infty}\frac{R_{k;i}(z,1)}{R_{k;i_0}(z,1)}=\frac{\lim\limits_{k\longrightarrow\infty}R_{k;i}(z,1)}{\lim\limits_{k\longrightarrow\infty}R_{k;i_0}(z,1)}=\frac{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_i(z,1)}{(v_1z-u_1)^{d_1}\ldots(v_mz-u_m)^{d_m}S_{i_0}(z,1)}=\frac{S_i(z,1)}{S_{i_0}(z,1)}$$

for all $i=1,\ldots,n$ and z close to z_0. Furthermore, the convergence is uniform on a neighborhood of z_0. Thus, the sequence f_k C^∞-converges on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$ to the holomorphic degree d_0 map $g\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ determined by $\mathbf{S}$.

Let ω be the Fubini-Study symplectic form on $\mathbb{P}^{n-1}$ normalized so that $\langle\omega,\ell\rangle=1$. For each $\delta>0$ and $j=1,\ldots,m$, denote by $B_\delta([u_j,v_j])$ the ball of radius δ around $[u_j,v_j]$ in $\mathbb{P}^1$ and let

$$\mathbb{P}^1_\delta=\mathbb{P}^1-\bigcup_{j=1}^m B_\delta([u_j,v_j])\,.$$

For each $j=1,\ldots,m$, let

$$\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)=\lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}E\big(f_k|_{B_\delta([u_j,v_j])}\big)\in\mathbb{R}^{\geq 0}$$

be the energy sinking into the bubble point $[u_j,v_j]$. By Gromov's Compactness Theorem, the number $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})$ is the value of ω on some element of $H_2(\mathbb{P}^{n-1};\mathbb{Z})$, i.e. an integer. Below we show that $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$.

Since the sequence f_k C^∞-converges to the degree d_0 map $g:\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ on compact subsets of $\mathbb{P}^1-\{[u_1,v_1],\ldots,[u_m,v_m]\}$,

$$d_0=\langle\omega,d_0\ell\rangle=E(g)=\lim_{\delta\longrightarrow 0}E\big(g|_{\mathbb{P}^1_\delta}\big)=\lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}E\big(f_k|_{\mathbb{P}^1_\delta}\big).$$

Thus,

$$\begin{aligned}\sum_{j=1}^m\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)&=\sum_{j=1}^m\lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}E\big(f_k|_{B_\delta([u_j,v_j])}\big)=\lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}E\big(f_k|_{\bigcup\limits_{j=1}^m B_\delta([u_j,v_j])}\big)\\&=\lim_{\delta\longrightarrow 0}\lim_{k\longrightarrow\infty}\big(E(f_k)-E\big(f_k|_{\mathbb{P}^1_\delta}\big)\big)=d-d_0=d_1+\ldots+d_m\,.\end{aligned}$$

In particular, $\mathfrak{m}_{[u_j,v_j]}(\{f_k\})=d_j$ if $m=1$, no matter what the "residual" tuple of polynomials $\mathbf{S}$ is. In the next paragraph, we show that this mass identity holds for $m>1$ as well.

By the assumption on $\mathbf{R}_k$, there exist $\lambda_{k;i;j;p}\in\mathbb{C}$ with $k\in\mathbb{Z}^+$ large, $i=1,\ldots,n$, $j=1,\ldots,m$, and $p=1,\ldots,d_j$ and tuples

$$\mathbf{S}_k \equiv \big[S_{k;1},\ldots,S_{k;n}\big] \in \mathfrak{X}_{n;d_0}$$

of polynomials without a common linear factor such that

$$\lim_{k\longrightarrow\infty}\mathbf{S}_k=\mathbf{S}, \qquad \lim_{k\longrightarrow\infty}\lambda_{k;i;j;p}=1 \quad \forall\, i,j,p,$$
$$R_{k;i}(u,v)=\prod_{j=1}^{m}\prod_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v) \quad \forall\; k,i\,.$$

For each $j_0=1,\ldots,m$, let

$$\mathbf{T}_{j_0} \equiv \big[T_{j_0;1},\ldots,T_{j_0;n}\big] \in \mathfrak{X}_{n;d-d_{j_0}}$$

be a tuple of polynomials without a common linear factor. If in addition, $i=1,\ldots,n$, $\epsilon\in\mathbb{R}$, and $k\in\mathbb{Z}^+$, let

$$S_{i;j_0;\epsilon}(u,v)\equiv\prod_{j\neq j_0}^{m}(v_ju-u_jv)^{d_j}\cdot S_i(u,v)+\epsilon T_{j_0;i}(u,v)\,, \qquad i=1,\ldots,n,$$
$$R_{k;i;j_0;\epsilon}(u,v)\equiv R_{k;i}(u,v)+\epsilon\prod_{p=1}^{d_{j_0}}(v_{j_0}u-\lambda_{k;i;j_0;p}u_{j_0}v)\cdot T_{j_0;i}(u,v), \qquad i=1,\ldots,n.$$

The polynomials in each of the above two sets have no common linear factor for all $i=1,\ldots,n$, $\epsilon\in\mathbb{R}^+$ sufficiently small, and k sufficiently large (with the conditions on ϵ and k independent of each other). We denote by $f_{k;j_0;\epsilon}\colon\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$ the holomorphic degree d map determined by the tuple

$$\mathbf{R}_{k;j_0;\epsilon}\equiv\big[R_{k;1;j_0;\epsilon},\ldots,R_{k;n;j_0;\epsilon}\big]\,.$$

For $\delta\in\mathbb{R}^+$ sufficiently small, the ratios

$$\frac{R_{k;i;j_0;\epsilon}(u,v)}{R_{k;i}(u,v)}=1+\epsilon\frac{T_{j_0;i}(u,v)}{\prod\limits_{j\neq j_0}^{m}\prod\limits_{p=1}^{d_j}(v_ju-\lambda_{k;i;j;p}u_jv)\cdot S_{k;i}(u,v)}$$

converge uniformly to 1 on $B_\delta([u_{j_0},v_{j_0}])$ as $\epsilon\longrightarrow0$, since the denominator in the last fraction does not vanish on $B_\delta([u_{j_0},v_{j_0}])$. Thus, there exists $k^*\in\mathbb{Z}^+$ such that

$$\lim_{\epsilon\longrightarrow0}\sup_{k\geq k^*}\sup_{z\in B_\delta([u_{j_0},v_{j_0}])}\left|\frac{|\mathrm{d}_zf_{k;j_0;\epsilon}|}{|\mathrm{d}_zf_k|}-1\right|=0.$$

Thus, for any $j=1,\ldots,m$,

$$\begin{aligned}\mathfrak{m}_{[u_j,v_j]}\big(\{f_k\}\big)&\equiv\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_k|_{B_\delta([u_j,v_j])}\big)=\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}\lim_{\epsilon\longrightarrow0}E\big(f_{k;j;\epsilon}|_{B_\delta([u_j,v_j])}\big)\\&=\lim_{\epsilon\longrightarrow0}\lim_{\delta\longrightarrow0}\lim_{k\longrightarrow\infty}E\big(f_{k;j;\epsilon}|_{B_\delta([u_j,v_j])}\big)=\lim_{\epsilon\longrightarrow0}\mathfrak{m}_{[u_j,v_j]}\big(\{f_{k;j;\epsilon}\}\big)=\lim_{\epsilon\longrightarrow0}d_j=d_j;\end{aligned}$$

the second-to-last inequality above holds by the $m=1$ case considered above, since

$$\lim_{k\longrightarrow\infty}\mathbf{R}_{k;j;\epsilon}=\big[(v_1u-u_1v)^{d_1}S_{1;j;\epsilon},\ldots,(v_1u-u_1v)^{d_1}S_{n;j;\epsilon}\big]\in\mathfrak{X}_{n;d}$$

and the polynomials $S_{1;j;\epsilon},\ldots,S_{n;j;\epsilon}$ have no linear factor in common.

By Gromov's Compactness Theorem, a subsequence of $\{f_k\}$ converges to the equivalence class of a holomorphic degree d_0 map $f:\Sigma\longrightarrow\mathbb{P}^{n-1}$, where Σ is a nodal Riemann surface consisting of the component $\Sigma_0=\mathbb{P}^1$ corresponding to the original $\mathbb{P}^1$ and finitely many trees of $\mathbb{P}^1$'s coming off from Σ_0; the maps on the components in the trees are defined only up reparametrization of the domain. By the above, $f|_{\Sigma_0}$ is the map g determined by the "relatively prime part" $\mathbf{S}$ of the limit $\mathbf{R}$ of the tuples of polynomials. The trees are attached at the roots $[u_j,v_j]$ of the common linear factors v_ju-u_jv of the polynomials in $\mathbf{R}$; the degree of the restriction of f to each tree is the power of the multiplicity d_j of the corresponding common linear factor.

This example shows that there is a continuous surjective map

$$\overline{\mathfrak{M}}_{0,0}\big(\mathbb{P}^1\times\mathbb{P}^{n-1},(1,d)\big)\longrightarrow\mathfrak{X}_{n,d} \tag{4.2}$$

which restricts to $[f,g]\longrightarrow[g\circ f^{-1}]$ on $\mathfrak{M}_{0,0}(\mathbb{P}^1\times\mathbb{P}^{n-1},(1,d))$. In particular, Gromov's moduli spaces refine classical compactifications of spaces of holomorphic maps $\mathbb{P}^1\longrightarrow\mathbb{P}^{n-1}$. On the other hand, the former are defined for arbitrary almost Kahler manifolds, which makes them naturally suited for applying topological methods. The right-hand side of (4.2) is known as the linear sigma model in the Mirror Symmetry literature. The morphism (4.2) plays a prominent role in the proof of mirror symmetry for the genus 0 Gromov-Witten invariants in [2] and [3]; see [4, Section 30.2].

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