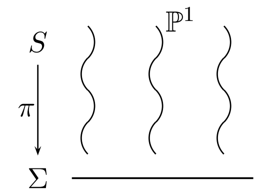
MAT 644: Complex Curves and Surfaces
Notes for 05/04/20

Last time: detecting irrational ruled surfaces



Dfn: $\mathbb{C}$-surface S is ruled
if $\exists$ cmpt conn Riemann surf. Σ and holomor. $\pi\colon S\longrightarrow\Sigma$
s.t. $\pi^{-1}(z)\approx\mathbb{P}^1$ for all $z\in\Sigma$

$\iff$ $S=\mathbb{P}E$ for some holomor. rk 2 v.b. $E\longrightarrow\Sigma$

$\implies$ $\pi^*\colon H^0(K_\Sigma)\longrightarrow H^0(T^*S)$, $\pi^*\eta=\eta\circ\mathrm{d}\pi$, isom.

$\implies$ $g(\Sigma)=h^{1,0}(S)$

Also: $\chi(S)=-4(g-1)$, $K_S^2=-8(g-1)$

Thm 1: $S=$ minimal projective surface, $\chi(S)<0 \implies S$ is irrational ruled ($g>2$)

Lemma 1: $S=$ minimal projective surface.
If $\exists$ cmpt conn Riemann surf. Σ and holomor. $\pi\colon S\longrightarrow\Sigma$
s.t. $\pi^{-1}(z)\approx\mathbb{P}^1$ for *generic* $z\in\Sigma$, then S is ruled.

Lemma 2: $S=$ minimal projective surface.
If $\chi(S)<0$ and $\exists\ \omega_1,\omega_2\in H^0(T^*S)$ lin. indep. s.t. $\omega_1\wedge\omega_2\in H^0(K_S)$,
then S is irrational ruled ($g>2$)

Used map $\Psi\colon U\longrightarrow\mathbb{C}^2$, $\Psi(p')=\left(\int_p^{p'}\omega_1,\int_p^{p'}\omega_2\right)$ $\quad p\in U$, $U\subset S$ small open set; $\int_p^{p'}$ along path in U

Ψ globally depends on the choice of path; well-defined up to

$$\Lambda\equiv\left\{\left(\int_\gamma\omega_1,\int_\gamma\omega_2\right)\colon\gamma\in H_1(S;\mathbb{Z})\right\}\subset\mathbb{C}\qquad\text{subgroup}$$

$\implies$ $\Psi\colon S\longrightarrow\mathbb{C}^2/\Lambda\approx\mathbb{T}^4$ (if Λ is a lattice) $\rightsquigarrow$ analogue of Abel-Jacobi map for S

$\gamma\subset S$ loop $\rightsquigarrow$ $\int_\gamma\cdot\colon H^0(T^*S)\longrightarrow\mathbb{C}$ $\rightsquigarrow$ $\int_\gamma\cdot\in H^0(T^*S)^*$

$$\Lambda_S\equiv\left\{\int_\gamma\cdot\in H^0(T^*S)^*\colon[\gamma]\in H_1(S;\mathbb{Z})/\mathrm{Tor}\right\}\subset H^0(T^*S)^*$$

Claim: $\Lambda_S\subset H^0(T^*S)^*$ is a lattice ($\iff\Lambda_S\otimes_{\mathbb{Z}}\mathbb{R}=H^0(T^*S)^*$); same pf as for curves

$\implies$ (a) $\mathrm{Alb}(S)\equiv H^0(T^*S)^*/\Lambda_S\approx\mathbb{T}^{2h^{1,0}(S)}$

(b) $\forall\,p\in S$, $\Psi_p\colon S\longrightarrow\mathrm{Alb}(S)$, $\Psi_p(p')=\int_p^{p'}\cdot\in H^0(T^*S)^*/\Lambda_S$, is well-defined holomor.

(c) if $h^{1,0}(S)\geqslant1$ and $P_1(S)\equiv h^0(K_S)=0$, $\mathrm{Im}\Psi_p\subset\mathrm{Alb}(S)$ is a curve:
if $\mathrm{d}_{p'}\Psi_p$ is injective for $p'\in S$, $\exists\ \omega\in H^0\big(\Lambda^2(T^*\mathrm{Alb}(S))\big)$ s.t. $(\Psi_p^*\omega)_{p'}\equiv\omega_{\Psi_p(p')}\circ\mathrm{d}_{p'}\Psi_p\neq0$

(d) $\Psi_{p*}\colon H_1(S;\mathbb{Z})/\mathrm{Tor}\longrightarrow H_1(\mathrm{Alb}(S);\mathbb{Z})=\pi_1(S)=\Lambda_S$ is isom.

(e) $\Psi_p^*\colon H^0(T^*\mathrm{Alb}(S))\longrightarrow H^0(T^*S)$ is isom.

Crl 1(Wed)**:** $S=$ projective surface. If $\chi(S)<0$, then $h^{1,0}(S)\geqslant 2$.

Crl 2 (Wed)**:** $S=$ (minimal) projective surface with $\chi(S)<0$.
If $\omega_1,\omega_2\in H^0(T^*S)$, then $\omega_1\wedge\omega_2=0\in H^0(K_S)$.
Part of pf of Lemma 3 on Wed; consequence of Lemma 2.

Crl 1,2 $\Longrightarrow$ Thm 1

Thm 2: $S=$ minimal projective surface, $K_S^2<0 \Longrightarrow S$ is irrational ruled ($g>2$)

Prp 1: $S=$ cmpt conn. $\mathbb{C}$-surface.
If $\mathrm{Im}\Psi_p\subset\mathrm{Alb}(S)$ is a curve with normalization $\pi\colon\Sigma\longrightarrow\mathrm{Im}\Psi_p$, then
(i) $h^{1,0}(S)=g(\Sigma)$ (ii) $\exists\ \widetilde{\Psi}_p\colon S\longrightarrow\Sigma$ holomor. s.t. $\Psi_p=\pi\circ\widetilde{\Psi}_p$
(iii) $\widetilde{\Psi}_p^{-1}(z)$ is conn. $\forall\, z\in\Sigma$
(iv) if S is minimal projective, $K_S^2\leqslant 0$, and $\exists$ effective D on S with $K_S{\cdot}D<0$, *then* $\widetilde{\Psi}_p^{-1}(z)\approx\mathbb{P}^1$ for $z\in\Sigma$ generic.

$$\begin{array}{ccccc} & & \Sigma & & \\ & \overset{\widetilde{\Psi}_p}{\nearrow} & \downarrow{\scriptstyle\pi} & & \\ S & \xrightarrow{\Psi_p} & \mathrm{Im}\Psi_p & \hookrightarrow & \mathrm{Alb}(S)\end{array} \qquad\qquad \begin{array}{ccc} & & H^0(K_\Sigma) \\ & \overset{\pi^*}{\nearrow} & \downarrow{\scriptstyle\widetilde{\Psi}_p^*} \\ H^0(T^*\mathrm{Alb}(S)) & \underset{\approx}{\xrightarrow{\Psi_p^*}} & H^0(T^*S)\end{array}$$

Prp 2: $S=$ projective surface with $K_S^2<0$. (i) if S minimal, $P_n(S)\equiv h^0(K_S^{\otimes n})=0\ \forall\, n\in\mathbb{Z}^+$
(ii) $\exists$ effective divisor D on S with $K_S\cdot D<0$

Proof of Thm 2. Prp 2(i) $\Longrightarrow P_2(S)=0$.
If $h^{1,0}(S)=0$, S is rational by Castelnuovo-Enriques $\Longrightarrow K_S^2\in\{8,9\}$ impossible.
$\therefore\ h^{1,0}(S)\geqslant 1$, $P_1(S)=0 \Longrightarrow \mathrm{Im}\Psi_p\subset\mathrm{Alb}(S)$ is a curve
Prp 1,2(ii) $\Longrightarrow$ generic fiber of $\widetilde{\Psi}_p\colon S\longrightarrow\Sigma$ is $\mathbb{P}^1$
S minimal $\Longrightarrow \widetilde{\Psi}_p\colon S\longrightarrow\Sigma$ is a $\mathbb{P}^1$-bundle (by Lemma 1). □

Proof of Prp 1. (ii) Let $A\subset\mathrm{Im}\Psi_p$ be the set of singular pts $\Longrightarrow \Psi_p^{-1}(A)\subset S$ divisor
$\exists\ \widetilde{\Psi}_p\colon S-\Psi_p^{-1}(A)\longrightarrow\Sigma$ holomor. s.t. $\Psi_p=\pi\circ\widetilde{\Psi}_p$ on $S-\Psi_p^{-1}(A)$
$\widetilde{\Psi}_p$ is bounded around each pt of $\Psi_p^{-1}(A)$ (b/c Ψ_p maps to $\mathrm{Alb}(S)$) $\Longrightarrow \widetilde{\Psi}_p$ extends

(i) $\Psi_p^*\colon H^0(T^*\mathrm{Alb}(S))\longrightarrow H^0(T^*S)$ isom $\Longrightarrow \widetilde{\Psi}_p^*$ onto
$\widetilde{\Psi}_p$ onto $\Longrightarrow \widetilde{\Psi}_p^*$ injective $\Longrightarrow h^{1,0}(S)=g(\Sigma)$
(iii) Lemma 0 (Wed) $\Longrightarrow \exists$ branch cover $\sigma\colon\widetilde{\Sigma}\longrightarrow\Sigma$ and holomor. $\widetilde{\Psi}'_p\colon S\longrightarrow\widetilde{\Sigma}$
s.t. $\widetilde{\Psi}_p=\widetilde{\Psi}'_p\circ\sigma$ and $\widetilde{\Psi}'^{-1}_p(\tilde{z})$ is conn. $\forall\,\tilde{z}\in\widetilde{\Sigma}$.
$\widetilde{\Psi}'_p$ onto $\Longrightarrow \widetilde{\Psi}'^*_p\colon H^0(K_{\widetilde{\Sigma}})\longrightarrow H^0(T^*S)$ injective
$\Longrightarrow g(\Sigma)\leqslant g(\widetilde{\Sigma})\leqslant h^{1,0}(S)=g(\Sigma) \Longrightarrow \sigma$ is bijection or $h^{1,0}(S)=1$ and

$$\begin{array}{ccc}\widetilde{\Sigma} & \overset{\sigma}{\dashrightarrow} & \Sigma \\ \uparrow{\scriptstyle\widetilde{\Psi}'_p} & \overset{\widetilde{\Psi}_p}{\nearrow} & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{\Psi_p} & \mathrm{Alb}(S)\end{array}$$

$$H_1(S;\mathbb{Z})/\mathrm{Tor}\xrightarrow{\widetilde{\Psi}'_{p*}}H_1(\widetilde{\Sigma}=\mathbb{T}^2;\mathbb{Z})\xrightarrow{\sigma_*}H_1(\Sigma=\mathbb{T}^2;\mathbb{Z})\xrightarrow{\pi_*}H_1(\mathrm{Alb}(S);\mathbb{Z}),\qquad \Psi_{p*}\colon H_1(S;\mathbb{Z})/\mathrm{Tor}\xrightarrow{\approx}H_1(\mathrm{Alb}(S);\mathbb{Z})\qquad\Longrightarrow \sigma \text{ is bijection}$$

Lemma: $S =$ minimal projective surface s.t. $\Psi_p(S) \subset \mathrm{Alb}(S)$ is a curve

$\Sigma =$ normalization of $\Psi_p(S)$, $\widetilde{\Psi}_p \colon S \longrightarrow \Sigma$ lift of Ψ_p

If $C \subset S$ is an irred. curve with $K_S \cdot C < 0$ and $|\widetilde{\Psi}_p(C)| = 1$, then $\widetilde{\Psi}_p^{-1}(z) \approx \mathbb{P}^1$ for $z \in \Sigma$ generic.

Proof. *Suppose* $C \subset F_z \equiv \widetilde{\Psi}_p^{-1}(z)$ and $F_z = \sum_i m_i F_i$ with $m_i \in \mathbb{Z}^+$, $F_i \subset S$ irred.

$\Longrightarrow$ $K_S \cdot F_i < 0$ for some i, $0 = F_z \cdot F_i = \sum_{j \neq i} m_j (F_j \cdot F_i) + F_i^2$

if $k \geqslant 2$, then $\sum > 0$ (b/c F_z conn.) $\Longrightarrow$ $F_i^2 < 0$ $\rightsquigarrow$ impossible (S is minimal)

if $k = 1$, then $0 > K_S \cdot F_z = 2(a(F_z) - 1) \geqslant 0$ $\Longrightarrow$ $a(F_z) = 0$

$\Longrightarrow$ generic fiber of $\widetilde{\Psi}_p$ is $\mathbb{P}^1$ $\square$

Prp 1: $S =$ cmpt conn. $\mathbb{C}$-surface.

If $\mathrm{Im}\Psi_p \subset \mathrm{Alb}(S)$ is a curve with normalization $\pi \colon \Sigma \longrightarrow \mathrm{Im}\Psi_p$, then

(i) $h^{1,0}(S) = g(\Sigma)$ (ii) $\exists\ \widetilde{\Psi}_p \colon S \longrightarrow \Sigma$ holomor. s.t. $\Psi_p = \pi \circ \widetilde{\Psi}_p$

(iii) $\widetilde{\Psi}_p^{-1}(z)$ is conn. $\forall\, z \in \Sigma$

(iv) if S is minimal projective, $K_S^2 \leqslant 0$, and $\exists$ effective D on S with $K_S \cdot D < 0$, *then* $\widetilde{\Psi}_p^{-1}(z) \approx \mathbb{P}^1$ for $z \in \Sigma$ generic.

$$\begin{array}{ccccc} & & \Sigma & & \\ & \overset{\widetilde{\Psi}_p}{\nearrow} & \downarrow \pi & & \\ S & \xrightarrow{\Psi_p} & \mathrm{Im}\Psi_p & \hookrightarrow & \mathrm{Alb}(S) \end{array}$$

Proof of (iv). Can assume $D \subset S$ irred. $D \cdot K_S < 0$ $\Longrightarrow$ $D^2 \geqslant 0$ (b/c S is minimal)

$\Longrightarrow$ $D \cdot D' \geqslant 0$ for all effective D' on S

$\Longrightarrow$ $\exists\, n \in \mathbb{Z}^{\geqslant 0}$ s.t. $h^0(D + nK_S) \geqslant 1$, $h^0(D + (n+1)K_S) = 0$

$\Longrightarrow$ $D + nK_S \sim \sum_i m_i C_i$ with $m_i \in \mathbb{Z}^+$, $C_i \subset S$ irred. curve

$K_S \cdot D < 0$, $K_S^2 \leqslant 0$ $\Longrightarrow$ $K_S \cdot (D + nK_S) < 0$ $\Longrightarrow$ $K_S \cdot C_i < 0$ for some i

$h^0(C_i + K_S) \leqslant h^0(D + nK_S + K_S) = 0$

$\therefore$ Can assume $\exists\ C \subset S$ irred. curve s.t. $K_S \cdot C < 0$, $C^2 \geqslant 0$, $h^0(C + K_S) = 0$

$C^2 - 2(a(C) - 1) = -K_S \cdot C > 0$, $C \cdot D' \geqslant 0$ for all effective D' on S

Lemma $\Longrightarrow$ can assume $\widetilde{\Psi}_p \colon C \longrightarrow \Sigma$ is a $k\!:\!1$ branch cover, $k \in \mathbb{Z}^+$

$\Longrightarrow$ $a(C) \geqslant g(\widetilde{C}) \geqslant g(\Sigma) = h^{1,0}(S) \geqslant 1$, $\widetilde{C} \longrightarrow C$ is normalization of C

$C^2 = 2(a(C) - 1) - K_S \cdot C > 0$

Exact sequence $0 \longrightarrow K_S \longrightarrow K_S(C) \longrightarrow K_S(C)|_C \longrightarrow 0$ of sheaves over S gives exact sequence

$$\{0\} = H^0\big(S; K_S(C)\big) \longrightarrow H^0(C; K_S(C)|_C) = H^0(C; K_C) \longrightarrow H^1(S; K_S) = H^{2,1}(S)$$

$\Longrightarrow$ $a(C) = h^0(K_C) \leqslant h^{2,1}(C) = h^{1,0}(C)$ $\Longrightarrow$ $a(C) = g(\widetilde{C}) = g(\Sigma) = h^{1,0}(C) \geqslant 1$, $C = \widetilde{C}$

$\Longrightarrow$ *either* $k = 1$ or $g(\widetilde{C}) = h^{1,0}(S) = 1$

$$\text{Fix } z_0\in\Sigma. \text{ For } z\in\Sigma, \text{ take } D_z\equiv C+F_z-F_{z_0} \implies C\cdot(K_S-D_z)=C\cdot(K_S-C)<0 \implies h^0(K_S-D_z)=0$$

$$\implies h^0(D_z)\geqslant\chi(D_z)=\chi(C)\geqslant 1-h^{0,1}(S)+\frac{1}{2}(C^2-K_S\cdot C)=C^2-2\big(g(C)-1\big)>0$$

$$[C_z]=[D_z]=[C]\in H_2(S;\mathbb{Z})\ \forall\ C_z\sim D_z \implies C\not\subset C_z\ \forall \text{ effective } C_z\sim D_z \text{ with } z\neq z_0$$

Suppose $k=1 \implies C=\Sigma$, $C_z\cap F$ is $k=1$ pt $\forall$ effective $C_z\sim D_z$ and fibers F of $\widetilde{\Psi}_p$
$C_z\cap F\neq C\cap F$ $\forall$ effective $C_z\sim D_z$, $z\neq z_0$, and generic fibers F of $\widetilde{\Psi}_p$
$C_z\cap F\sim D_z\cap F\sim C\cap F$ as divisors on F (b/c $F_z\cap F, F_{z_0}\cap F=\varnothing$)
$\implies F\approx\mathbb{P}^1$ (F contains two equivalent points)

Suppose $g(C)=h^{1,0}(S)=1 \implies (\Sigma,z_0)\approx(\mathbb{C}/\Lambda,0+\Lambda)$, $d\equiv C^2>0$, $h^0(D_z)\geqslant d\ \forall\, z\in\Sigma$
$C\not\subset C_z$ $\forall$ effective $C_z\sim D_z$ with $z\neq z_0 \implies H^0(S;D_z)\longrightarrow H^0(C;D_z|_C)$ injective
$\deg D_z|_C=D_z\cdot C=C^2=d$, $g(C)>0 \implies h^0(D_z|_C)\leqslant d$
$\implies \dim H^0(S;D_z)=d$

Let $p_1,\ldots,p_{d-1}\in S$ be generic $\implies$ $\forall\ z\in\Sigma$ generic, $\exists!$ effective $C_z\sim D_z$ s.t. $C_z\ni p_1,\ldots,p_{d-1}$
$\Sigma^*\subset\Sigma$ subset of these generic z
$\forall\ z,z'\in\Sigma^*$ generic, $C_z\cap C_{z'}-\{p_1,\ldots,p_{d-1}\}$ consists of 1 pt $p_d(z,z')$
$U\subset(\Sigma^*)^2$ of these generic (z,z')
For $z\in\Sigma^*$ and $(z,z'),(z,z'')\in U$ generic, $C_z\cap F_{z'}\not\sim C_z\cap F_{z''} \implies p_d(z,z')\neq p_d(z,z'')$
$\implies$ for $z\in\Sigma^*$ generic, $\{p_d(z,z')\colon (z,z')\in U\}\subset C_z$ is dense
$\implies$ ⊛ $\{p_d(z,z')\colon (z,z')\in U\}\subset S$ is dense

For $z\in\Sigma^*$, define $\Sigma^*(z)=\{z'\in\Sigma^*\colon z-z'\in\Sigma^*,\ (z',z-z')\in U\}$
$f_z\colon \Sigma^*(z)/\mathbb{Z}_2\longrightarrow S$, $f(z')=p_d(z',z-z')$
S projective $\implies$ f_z extends to $f_z\colon\mathbb{P}^1=\Sigma/\mathbb{Z}_2\longrightarrow S$
$g(\Sigma)>0 \implies$ (a) $\mathrm{Im} f_z$ is contained in a fiber of $\widetilde{\Psi}_p$
⊛ $\implies$ (b) $\bigcup_{z\in\Sigma^*}\mathrm{Im} f_z\subset S$ is dense
generic fiber of $\widetilde{\Psi}_p$ is smooth + (a) + (b) $\implies$ generic fiber of $\widetilde{\Psi}_p$ is $\mathbb{P}^1$ □