MAT 644: Complex Curves and Surfaces
Notes for 04/27/20

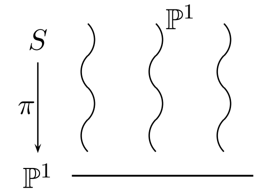
Last week: Rational Surfaces (birational to $\mathbb{P}^2$, blowups/blowdowns of $\mathbb{P}^2$)

Prp 1: $\mathbb{P}^2$ and $\mathbb{F}_k \equiv \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k))$ with $k=0,2,3,\ldots$ are minimal rational surfaces; $\mathbb{F}_1 = \mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$
minimal= no exceptional curves $E \subset S$ ($E \subset S$ irred. $E\cdot K_S, E\cdot E < 0 \implies E \approx \mathbb{P}^1$, $E\cdot E = -1$)

Lemma 1: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S.
If $C_\lambda \cap C_{\lambda'} = \varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and $C_\lambda \approx \mathbb{P}^1$ $\forall\,\lambda\in\mathbb{P}^1$,
then the map

$$\pi\colon S \longrightarrow \mathbb{P}^1, \qquad C_\lambda \in x \longrightarrow \lambda\in\mathbb{P}^1,$$

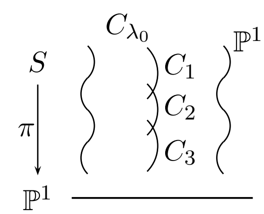
is isomorphic to the projection $\mathbb{F}_k \longrightarrow \mathbb{P}^1$ for some $k\in\mathbb{Z}^{\geqslant 0}$.



Lemma 2: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S.
If $C_\lambda \cap C_{\lambda'} = \varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and $C_\lambda \approx \mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$,
then S is a blowup of some $\mathbb{F}_k$ so that the map

$$\pi\colon S \longrightarrow \mathbb{P}^1, \qquad C_\lambda \in x \longrightarrow \lambda\in\mathbb{P}^1,$$

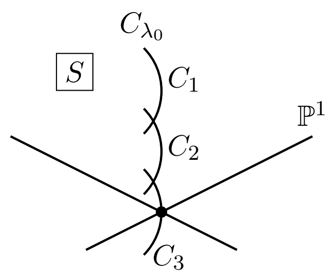
is the composition of the blowdown $S \longrightarrow \mathbb{F}_k$ and projection $\mathbb{F}_k \longrightarrow \mathbb{P}^1$.

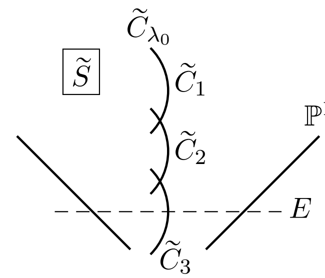


Crl 1: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S so that
$C_\lambda \cap C_{\lambda'} = \varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and $C_\lambda \approx \mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$.
If $\lambda_0\in\mathbb{P}^1$ and $C_{\lambda_0} = \sum_{i=1}^{k} m_i C_i$ with $m_i\in\mathbb{Z}^+$, $C_i \subset S$ irred., then $C_i \approx \mathbb{P}^1$ $\forall\, i$.

Crl 2: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S so that $C_\lambda \approx \mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$.
If $\lambda_0\in\mathbb{P}^1$ and $C_{\lambda_0} = \sum_{i=1}^{k} m_i C_i$ with $m_i\in\mathbb{Z}^+$, $C_i \subset S$ irred., then $C_i \approx \mathbb{P}^1$ $\forall\, i$.

Proof. $\exists$ blowup $\pi\colon \widetilde{S} \longrightarrow S$ and a pencil $\{\widetilde{C}_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on $\widetilde{S}$ so that $\pi(\widetilde{C}_\lambda) = C_\lambda$ $\forall\,\lambda\in\mathbb{P}^1$
and $\widetilde{C}_\lambda \cap \widetilde{C}_{\lambda'} = \varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$ (blowing up at base locus and
taking proper transform of the entire pencil as in pf of Lemma 1 on 04/15/20).
Crl 1 $\implies$ claim for $\widetilde{C}_{\lambda_0}$ $\implies$ claim for C_{λ_0}
b/c pts of the base locus are smooth points of every C_λ. $\square$





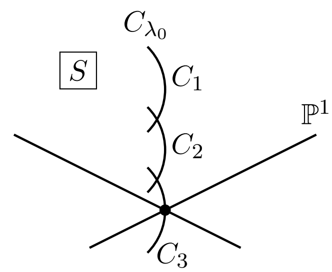
Prp 2: A minimal rational surface S is isomorphic to either $\mathbb{P}^2$ or some $\mathbb{F}_k$
(every rational surface is a blowup of $\mathbb{P}^2$ or some $\mathbb{F}_k$).

Proof. S rational $\implies$ (i) $\chi(\mathcal{O}_S)=h^{0,0}(S)-h^{0,1}(S)+h^{0,2}(S)=1$
(ii) $\exists$ pencil $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on S s.t. $C_\lambda\approx\mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$
(start with a pencil of lines on $\mathbb{P}^2$, pull back under blowups,
push forward under blowdowns)

The base locus

$$B\equiv\bigcap_{\lambda\in\mathbb{P}^1}C_\lambda=C_{\lambda_0}\cap C_{\lambda_1}\quad\text{for any } \lambda_0,\lambda_1\in\mathbb{P}^1,\ \lambda_0\neq\lambda_1,$$

is finite.



$B=\varnothing \implies$ Lemma 2 applies $\qquad\therefore$ assume $C_\lambda\cdot C_\lambda\geqslant|B|>0$

Case 1: C_λ is irred. $\forall\,\lambda\in\mathbb{P}^1\implies\exists$ blowup $\pi\colon\widetilde{S}\longrightarrow S$ and a pencil $\{\widetilde{C}_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on $\widetilde{S}$ so that $\pi(\widetilde{C}_\lambda)=C_\lambda\ \forall\,\lambda\in\mathbb{P}^1$, $\widetilde{C}_\lambda\cap\widetilde{C}_{\lambda'}=\varnothing\ \forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and $\widetilde{C}_\alpha$ is irred. $\forall\,\lambda\in\mathbb{P}^1$
(keep blowing up at base locus and taking proper transform of the entire pencil
as in pf of Lemma 1 on 04/15/20).

Lemma 1 $\implies\widetilde{S}\approx\mathbb{F}_k$ for some $k\in\mathbb{Z}^{\geqslant0}$
$k\neq1\implies$ no exceptional curves in $\widetilde{S}\implies S=\widetilde{S}\approx\mathbb{F}_k$
$k=1\implies\widetilde{S}=\mathrm{Bl}_{\mathrm{pt}}\mathbb{P}^2$, unique exceptional curve in $\widetilde{S}\implies S\approx\mathbb{P}^2$

Case 2: For some $\lambda_0\in\mathbb{P}^1$, $C_{\lambda_0}=\sum_{i=1}^k m_iC_i$ with $k\geqslant2$, $m_i\in\mathbb{Z}^+$, and $C_i\subset S$ irred.

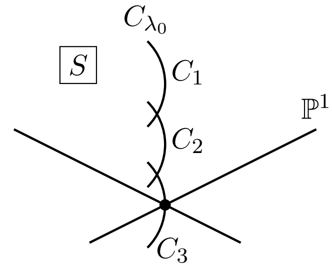
$$0=a(C_\lambda)=a(C_{\lambda_0})\equiv1+\frac{1}{2}(K_S\cdot C_{\lambda_0}+\underbrace{C_{\lambda_0}^2}_{=C_\lambda^2>0})\implies K_S\cdot C_{\lambda_0}<0$$
$$\implies K_S\cdot C_i<0\text{ for some } i$$

if $C_i^2<0$, then can blow down C_i by Castelnuovo-Enriques Criterion II
S minimal $\implies C_i^2\geqslant0$
Riemann-Roch $\implies\chi(C_i)=\chi(\mathcal{O}_S)+\frac{1}{2}(C_i^2-K_S\cdot C_i)>1+0$
Kodaira-Serre $\implies h^2(C_i)\equiv h^{0,2}(C_i)=h^{2,0}(-C_i)=h^0(K_S-C_i)\leqslant h^0(K_S)=0$ (S rational)
$\implies h^0(C_i)=\chi(C_i)+h^1(C_i)-h^2(C_i)\geqslant2$

Crl 2 $\implies C_i\approx\mathbb{P}^1$
$\implies$ can take pencil $\{C'_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on S containing C_i
keep repeating until the base locus of the pencil is empty or C_λ is irred. $\forall\,\lambda\in\mathbb{P}^1$. □

Crl of Case 1: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface $S\not\approx\mathbb{P}^2$
so that $C_\lambda\approx\mathbb{P}^1$ some $\lambda\in\mathbb{P}^1$.
If the base locus of $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ is not empty, then C_{λ_0} is reducible for some $\lambda_0\in\mathbb{P}^1$.

Lemma 3 (Noether's Lemma)**:** Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S. If $C_\lambda\approx\mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$, then S is birational to $\mathbb{P}^2$.



Proof. $\exists$ blowup $\pi\colon \widetilde{S}\longrightarrow S$ and a pencil $\{\widetilde{C}_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on $\widetilde{S}$ so that $\pi(\widetilde{C}_\lambda)=C_\lambda\ \forall\,\lambda\in\mathbb{P}^1$ and $\widetilde{C}_\lambda\cap\widetilde{C}_{\lambda'}=\varnothing\ \forall\,\lambda,\lambda'\in\mathbb{P}^1,\ \lambda\neq\lambda'$ (keep blowing up at base locus and taking proper transform of the entire pencil as in pf of Lemma 1 on 04/15/20).

Lemma 2 $\implies$ $\widetilde{S}$ is a blowup of $\mathbb{F}_k$ for some $k\in\mathbb{Z}^{\geqslant 0}$
$\implies$ S is birational to $\mathbb{P}^2$.

Crl 3: Let S be a a minimal projective surface with $h^1(\mathcal{O}_S), h^0(K_S)=0$. If $\exists$ irred. curve $C\subset S$ with $a(C)=0$ and either $C^2\geqslant 0$ or $C\cdot K_S<0$, then S is rational.

Proof. $C\approx\mathbb{P}^1 \implies$ enough to show $h^0(C)\geqslant 2$

S minimal, $C\cdot K_S<0 \implies C^2\geqslant 0$ (o/w can blow down C)

$h^0(K_S)=0 \implies h^2(C)=h^0(K_S-C)=0$

$\implies h^0(C)\geqslant\chi(C)=\chi(\mathcal{O}_S)+\frac{1}{2}(C^2-K_S\cdot C)=1+\frac{1}{2}(C^2-K_S\cdot C)$

$0=a(C)=1+\frac{1}{2}(C^2+K_S\cdot C),\ C^2\geqslant 0 \implies h^0(C)\geqslant 2+C^2\geqslant 2$ $\square$

Castelnuovo-Enriques Thm: A projective surface S is rational (birational to $\mathbb{P}^2$) *iff* $h^1(\mathcal{O}_S)=0$ and $P_2(S)\equiv h^0(K_S^{\otimes 2})=0$.

$\Longrightarrow$ Trivial b/c $h^{p,q}$ with $(p,q)\neq(1,1)$ and P_n with $n\in\mathbb{Z}^+$ do not change under blowups/downs and $h^{0,1}(\mathbb{P}^2), P_2(\mathbb{P}^2)=0$.

Proof of $\Longleftarrow$***.*** $h^0(K_S^{\otimes 2})=0 \implies h^0(K_S)=0,\ \chi(\mathcal{O}_S)=1$.
Assume S is minimal. Crl 3 $\implies$ enough to find irred. curve $C\subset S$ with $a(C)=0$ and either $C^2\geqslant 0$ or $C\cdot K_S<0$.

By Riemann-Roch and Kodaira-Serre, $C\subset S$ irred. curve $\implies$

$$
\begin{aligned}
0\leqslant a(C) &\equiv 1+\frac{1}{2}\big(C^2+K_S\cdot C\big) = \chi(\mathcal{O}_S)+\frac{1}{2}\big((-C)^2-K_S\cdot(-C)\big)\\
&\overset{\text{RR}}{=} \chi(-C)\equiv \underbrace{h^0(-C)}_{0}-h^1(-C)+h^2(-C) \overset{\text{KS}}{\leqslant} h^0(C+K_S).
\end{aligned}
\tag{1}
$$

Claim 1: If $L\longrightarrow S$ is a positive l.b., $h^0(L+nK_S)=0\ \forall\,n\geqslant n_L$.

Claim 2: $\exists$ positive l.b. $L\longrightarrow S$ s.t. $L\notin\mathbb{Z}K_S$ and $h^0(L)\neq 0$.

Claim 3: If $K_S^2\geqslant 0$, $-K_S$ is effective.

Proof of Claim 3. Use Riemann-Roch for $-K_S$:

$$
h^0(-K_S)+\underbrace{h^2(-K_S)}_{h^0(K_S+K_S)=0}\geqslant\chi(-K_S)=\chi(\mathcal{O}_S)+\frac{1}{2}(K_S^2+K_S^2)\geqslant 1
$$

Proof of Thm for $K_S^2\!=\!0$. Claim 3 $\implies$ $-K_S$ effective
$\implies$ $K_S\cdot L<0$ for any positive l.b. $L\longrightarrow S$
Claim 1 $\implies$ $\exists$ positive l.b. $L\longrightarrow S$ and $n\in\mathbb{Z}^+$ s.t.

$$h^0(L+nK_S)\geqslant 1 \qquad\text{and}\qquad h^0(L+(n+1)K_S)=0.$$

$\implies$ $\exists$ effective divisor $C=\sum m_iC_i\sim L+nK_S$
$K_S\cdot C=K_S\cdot L<0 \implies K_S\cdot C_i<0$ for some i
(1) $\implies$ $0\leqslant a(C_i)\leqslant h^0(C_i+K_S)\leqslant h^0((L+nK_S)+K_S)=0$ $\implies$ done. $\square$

Proof of Thm for $K_S^2\!>\!0$. $\implies$ $h^0(-K_S)+\underbrace{h^2(-K_S)}_{h^0(K_S+K_S)}\geqslant\chi(-K_S)\overset{\text{RR}}{=}1+\dfrac{1}{2}(K_S^2+K_S^2)\geqslant 2.$

$\implies$ $\exists$ effective divisor F and pencil $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on S s.t.

$$-K_S\sim F+C_\lambda \;\;\forall\,\lambda\in\mathbb{P}^1 \qquad\text{and}\qquad \bigcap_{\lambda\in\mathbb{P}^1}C_\lambda\subset S \;\text{ is finite}$$

($F=$ curve part of base locus of a pencil in $|-K_S|$).
$C_\lambda=\sum\limits_{i=1}^{k}m_iC_{\lambda,i}$ for $\lambda\in\mathbb{P}^1$ generic, $m_i\in\mathbb{Z}^+$, $C_{\lambda,i}$ irred. $\implies$ $C_{\lambda,i}^2\geqslant 0\;\forall\, i$
(1) $\implies$ $0\leqslant a(C_{\lambda,i})\leqslant h^0(C_{\lambda,i}+K_S)\leqslant h^0(-F)\leqslant 1$ if $a(C_{\lambda,i})=0$, then done.

$a(C_{\lambda,i})=1$ $\implies$ $-K_S\sim C_{\lambda,1}$
$C_{\lambda,1}$ irred., $(-K_S)^2>0$ $\implies$ $(-K_S)\cdot C\geqslant 0$ $\forall$ effective divisors C on S

Claims 2,1 $\implies$ $\exists$ positive l.b. $L\longrightarrow S$ and $n\in\mathbb{Z}^+$ s.t.

$$L\notin\mathbb{Z}K_S, \qquad h^0(L+nK_S)\geqslant 1, \qquad\text{and}\qquad h^0(L+(n+1)K_S)=0.$$

$\implies$ $\exists$ effective divisor $C=\sum m_iC_i\sim L+nK_S$
$K_S\cdot C_i\leqslant 0$ (1) $\implies 0\leqslant a(C_i)\leqslant h^0(C_i+K_S)\leqslant h^0((L+nK_S)+K_S)=0$
if $K_S\cdot C_i<0$, then done.

$K_S\cdot C_i=0$, $a(C_i)=0$ $\implies$ $C_i^2=-2$

$$h^0(-K_S-C_i)+\underbrace{h^2(-K_S-C_i)}_{h^0((K_S+C_i)+K_S)}\geqslant\chi(-K_S-C_i)\overset{\text{RR}}{=}1+\frac{1}{2}\big((-K_S-C_i)^2-K_S\cdot(-K_S-C_i)\big)$$

$$=1+\frac{1}{2}(2K_S^2+C_i^2+3K_S\cdot C_i)=K_S^2\geqslant 1$$

$$h^0(C_i+2K_S)\leqslant h^0\big((L+nK_S)+2K_S\big)\leqslant h^0\big(L+(n+1)K_S\big)=0$$

$\implies$ $\exists$ $D=\sum\limits_{j=1}^{k}n_jD_j\sim -K_S-C_i$ with $n_j\in\mathbb{Z}^+$, $D_j\subset S$ irred.
(1) $\implies$ $0\leqslant a(D_j)\leqslant h^0(D_j+K_S)\leqslant h^0(-C_i)=0$
$-K_S\cdot D=K_S^2>0$ $\implies$ $K_S\cdot D_j<0$ for some j $\implies$ done $\square$

Proof of Thm for $K_S^2<0$. Claim 1 $\implies$ $\exists$ positive l.b. $L\longrightarrow S$ and $n\in\mathbb{Z}^+$ s.t.

$$h^0(L+nK_S)\geqslant 2 \qquad \text{and} \qquad h^0(L+(n+1)K_S)\leqslant 1.$$

$\implies$ $\exists$ effective divisor F and pencil $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on S s.t.

$$L+nK_S \sim F+C_\lambda \;\; \forall\,\lambda\in\mathbb{P}^1 \qquad \text{and} \qquad \bigcap_{\lambda\in\mathbb{P}^1} C_\lambda\subset S \;\text{ is finite}$$

($F=$ curve part of base locus of a pencil in $|L+nK_S|$).

$C_\lambda=\sum_{i=1}^{k} m_iC_{\lambda,i}$ for $\lambda\in\mathbb{P}^1$ generic, $m_i\in\mathbb{Z}^+$, $C_{\lambda,i}$ irred. $\implies$ $C_{\lambda,i}^2\geqslant 0\;\forall\,i$

(1) $\implies$ $0\leqslant a(C_{\lambda,i})\leqslant h^0(C_{\lambda,i}+K_S)\leqslant h^0((L+nK_S)+K_S)\leqslant 1$ if $a(C_{\lambda,i})=0$, then done.

$a(C_{\lambda,i})=1$ $\implies$ $\exists\, D=\sum n_jD_j\sim C_{\lambda,i}+K_S$ with $n_j\in\mathbb{Z}^+$, $D_j\subset S$ irred.

$\implies$ $C_{\lambda,i}\cdot D=2(a(C_{\lambda,i})-1)=0$ $\implies$ (a) $C_{\lambda,i}\cdot D_j=0\;\forall\,j$ b/c $C_{\lambda,i}^2\geqslant 0$

$a(C_{\lambda,i})=1$, $C_{\lambda,i}^2\geqslant 0$ $\implies$ $K_S\cdot C_{\lambda,i}\leqslant 0$ $\implies$ $K_S\cdot D<0$ (b/c $K_S^2<0$)

$\implies$ (b) $K_S\cdot D_j<0$ for some j

(a)+(b) $\implies$ $0>(C_{\lambda,i}+K_S)\cdot D_j=\sum_\ell n_\ell(D_\ell\cdot D_j)\geqslant n_jD_j^2$

$\therefore$ $D_j^2, K_S\cdot D_j<0$ $\implies$ $D_j\subset S$ exceptional; impossible. □

Claim 1: If $L\longrightarrow S$ is a positive l.b., $h^0(L+nK_S)=0\;\forall\,n\geqslant n_L$.

Proof for $K_S^2\geqslant 0$. Claim 3 $\implies$ $-K_S$ is effective

$\implies$ $L\cdot(-K_S)>0$ $\implies$ $L\cdot(L+nK_S)<0\;\forall\,n\geqslant n_L$

$\implies$ $h^0(L+nK)=0$ b/c L is positive. □

Proof for $K_S^2<0$, L any. $K_S\cdot(L+nK_S)<0\;\forall\,n\geqslant n'_L$

Suppose $n\geqslant n'_L$ and $D=\sum m_iC_i\sim L+nK_S$ with $m_i\in\mathbb{Z}^+$ and $C_i\subset S$ irred.

$\implies$ $K_S\cdot C_i<0$ for some i $\implies$ $C_i^2\geqslant 0$ (o/w could blow down C_i) $\implies$ (a) $C_i\cdot D'\geqslant 0\;\forall\; D'$ effective

$\implies$ (b) $C_i\cdot(L+nK_S)<0\;\;\forall\,n\geqslant n_L\geqslant n'_L$

(a)+(b) $\implies$ $h^0(L+nK)=0\;\forall\,n\geqslant n_L$. □

Claim 2: $\exists$ positive l.b. $L\longrightarrow S$ s.t. $L\notin\mathbb{Z}K_S$ and $h^0(L)\neq 0$.

Proof. Suppose not: $L\longrightarrow S$ positive l.b. with $h^0(L)\neq 0$ $\implies$ $L\in\mathbb{Z}K_S$.

For any l.b. $L'\longrightarrow S$, $\exists\,n\in\mathbb{Z}^+$ s.t. $L'+nL$ is positive, $h^0(L'+nL)\neq 0$

$\implies L'\in\mathbb{Z}K_S$ $\implies$ $H^2(S;\mathbb{Z})\approx\mathrm{Pic}(S)=\mathbb{Z}K_S$, $K_S^2=1$ (by PD and $K_S^2>0$), $b_2(S)=1$

Noether's Formula $\implies$ $1=\chi(\mathcal{O}_S)=\frac{1}{12}(\chi(S)+K_S^2)=\frac{1}{12}(3+1)$ impossible □