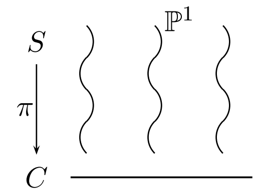
MAT 644: Complex Curves and Surfaces
Notes for 04/22/20

Last time: Ruled Surfaces
Dfn: $\mathbb{C}$-mfld S is ruled if $\exists$ holomor. $\pi\colon S \longrightarrow C$ s.t. $\pi^{-1}(p)\approx\mathbb{P}^1\ \forall\, p\in C$



$$\Longrightarrow\quad \forall\, p\in S,\ \langle c_1(\mathcal{N}_S\pi^{-1}(p)),\pi^{-1}(p)\rangle=0 \ \Longrightarrow\ K_S\cdot\pi^{-1}(p)=-2+0<0$$
$$\Longrightarrow\quad p_n(S)\equiv\dim H^0(S;K_S)=0,\ \kappa(S)=-\infty$$

Prp: if $\pi\colon S\longrightarrow\mathbb{P}^1$ is a projective ruled surface (and π is submersion),
then $\exists\ k\in\mathbb{Z}^{\geqslant 0}$ s.t. $S\approx\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))\equiv\mathbb{F}_k$ (k-th Hirzebruch surface) as $\mathbb{P}^1$-bundles over $\mathbb{P}^1$.

Examples: $\mathbb{F}_0\approx\mathbb{P}^1\times\mathbb{P}^1$ (trivial), $\quad\mathbb{F}_1\approx\mathrm{Bl}_p\mathbb{P}^2$ (later)

Properties of Hirzebruch Surfaces $\pi\colon\mathbb{F}_k\equiv\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))\longrightarrow\mathbb{P}^1$ **(last time)**

(1) $\pi_1(\mathbb{F}_k)=\{0\}$ $\qquad$ (2) Hodge diamond of $\mathbb{F}_k$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(2′) tautological line bundle $\gamma\equiv\big\{(\ell,v)\in\mathbb{P}^1\times\mathbb{C}^2\colon v\in\ell\big\}\longrightarrow\mathbb{P}^1,\qquad \omega\equiv c_1(\gamma^*)\in H^2(\mathbb{P}^1;\mathbb{Z})$
tautological line bundle $\widetilde\gamma\equiv\big\{(\widetilde\ell,\widetilde v)\in\mathbb{F}_k\times(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))\colon\widetilde v\in\widetilde\ell\big\}\longrightarrow\mathbb{F}_k,\ \ \widetilde\omega\equiv c_1(\widetilde\gamma^*)\in H^2(\mathbb{F}_k;\mathbb{Z})$

$$H^*(\mathbb{P}^1;\mathbb{Z})[\widetilde\omega]/(\widetilde\omega^2+k\widetilde\omega)\xrightarrow{\ \approx\ }H^*(\mathbb{F}_k;\mathbb{Z}),\quad \alpha\widetilde\omega^i\longrightarrow(\pi^*\alpha)\cup\widetilde\omega^i,$$
isomorphism of graded algebras over $H^*(\mathbb{P}^1;\mathbb{Z})$

$$\omega=\mathrm{PD}_{\mathbb{P}^1}\mathrm{pt}\in H^2(\mathbb{P}^1;\mathbb{Z}),\ \pi\text{ submersion}\Longrightarrow\pi^*\omega=\mathrm{PD}_{\mathbb{F}_k}F\in H^2(\mathbb{F}_k;\mathbb{Z}),\ F\equiv[\pi^{-1}(p)]\in H_2(\mathbb{F}_k;\mathbb{Z})$$
fiber class

Properties of Hirzebruch Surfaces $\pi\colon\mathbb{F}_k\equiv\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))\longrightarrow\mathbb{P}^1$ **(more)**

(3) $\mathrm{Pic}(\mathbb{F}_k)\xrightarrow{c_1}H^2(\mathbb{F}_k;\mathbb{Z})$ is isom.: short exact sequence

$$\{0\}\longrightarrow\mathbb{Z}\longrightarrow\mathbb{C}\xrightarrow{\exp(2\pi\mathrm{i}\cdot)}\mathbb{C}^*\longrightarrow\{1\}$$

of abelian groups gives exact sequence in $\check{H}^*$:

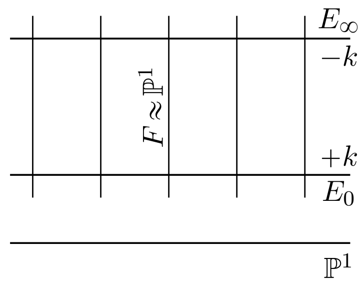
$$\underbrace{\check{H}^1(\mathbb{F}_k;\mathcal{O})}_{H^{0,1}(\mathbb{F}_k)=\{0\}}\longrightarrow\check{H}^1(\mathbb{F}_k;\mathcal{O}^*)=\mathrm{Pic}(\mathbb{F}_k)\xrightarrow{c_1}\check{H}^2(\mathbb{F}_k;\mathbb{Z})=H^2(\mathbb{F}_k;\mathbb{Z})\longrightarrow\underbrace{\check{H}^2(\mathbb{F}_k;\mathcal{O})}_{H^{0,2}(\mathbb{F}_k)=\{0\}}$$

(4) $\mathrm{Pic}(\mathbb{F}_k) \overset{c_1}{\approx} H^2(\mathbb{F}_k;\mathbb{Z}) \overset{\mathrm{PD}}{\approx} H_2(\mathbb{F}_k;\mathbb{Z})$ generated by fiber class $F=[\pi^{-1}(p)]$
and class of either section of $\pi\colon \mathbb{F}_k \longrightarrow \mathbb{P}^1$,

$$E_0 \equiv \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\oplus 0\big) \subset \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)\big) \quad \text{or} \quad E_\infty = \mathbb{P}\big(0\oplus\mathcal{O}_{\mathbb{P}^1}(k)\big) \subset \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)\big).$$

$F\cdot E_0, F\cdot E_\infty = 1 \quad\Longrightarrow\quad \mathrm{Pic}(\mathbb{F}_k), H^2(\mathbb{F}_k;\mathbb{Z}), H_2(\mathbb{F}_k;\mathbb{Z}) \approx \mathbb{Z}F\oplus\mathbb{Z}E_0 \approx \mathbb{Z}F\oplus\mathbb{Z}E_\infty$

$$\begin{array}{c} \mathbb{F}_k \equiv \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)\big) \\ \big\downarrow \pi \\ \mathbb{P}^1 \end{array}$$



$E_0\cap E_\infty = \varnothing \quad\Longrightarrow\quad E_0\cdot E_\infty = 0$
$\pi^{-1}(p)\cap\pi^{-1}(p') = \varnothing$ if $p,p'\in\mathbb{P}^1$ distinct $\quad\Longrightarrow\quad F\cdot F = 0$
$\mathcal{N}_{\mathbb{F}_k}E_0 \approx \widetilde{\gamma}^*|_{E_0}\otimes\mathcal{O}_{\mathbb{P}^1}(k) \approx \mathcal{O}_{\mathbb{P}^1}(k) \quad\Longrightarrow\quad E_0\cdot E_0 = \langle e(\mathcal{N}_{\mathbb{F}_k}E_0), E_0\rangle = k$
$\mathcal{N}_{\mathbb{F}_k}E_\infty \approx \widetilde{\gamma}^*|_{E_\infty}\otimes\mathcal{O}_{\mathbb{P}^1} \approx \mathcal{O}_{\mathbb{P}^1}(-k) \quad\Longrightarrow\quad E_\infty\cdot E_\infty = \langle e(\mathcal{N}_{\mathbb{F}_k}E_\infty), E_\infty\rangle = -k$

$E_\infty = aE_0+bF,\ E_\infty\cdot F = 1,\ E_\infty\cdot E_0 = 0 \quad\Longrightarrow E_\infty = E_0-kF$
$K_{\mathbb{F}_k} = aE_0+bF,\ K_{\mathbb{F}_k}\cdot F = K_F\cdot F - F\cdot F = -2,\ K_{\mathbb{F}_k}\cdot E_0 = K_{E_0}\cdot E_0 - E_0\cdot E_0 = -2-k \Longrightarrow K_{\mathbb{F}_k} = -2E_0+(k-2)F$
$\langle c_1(\widetilde{\gamma}^*), F\rangle = \langle c_1(\gamma^*), \mathbb{P}^1\rangle = 1,\ \langle c_1(\widetilde{\gamma}^*), E_0\rangle = \langle c_1(\mathcal{O}_{\mathbb{P}^1}), \mathbb{P}^1\rangle = 0 \quad\Longrightarrow c_1(\widetilde{\gamma}^*) = E_\infty$

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(5) If curve $C\subset\mathbb{F}_k$ is irred., $C\neq E_\infty$, and $C\sim aE_0+bF$, *then* $a,b\in\mathbb{Z}^{\geqslant 0}$:
C, E_∞ irred., $C\neq E_\infty \quad\Longrightarrow\quad C\cdot E_\infty\geqslant 0 \quad\Longleftrightarrow\quad b\in\mathbb{Z}^{\geqslant 0}$
if $C=F$, $(a,b)=(0,1)$ ✓
C, F irred., $C\neq F \quad\Longrightarrow\quad C\cdot F\geqslant 0 \quad\Longleftrightarrow\quad a\in\mathbb{Z}^{\geqslant 0}$

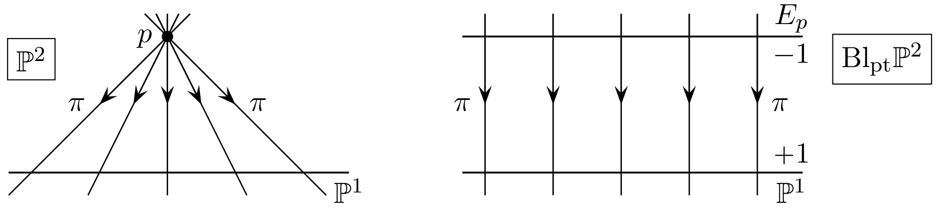
Crl 1: For $k\in\mathbb{Z}^+$, $\mathbb{F}_k$ is *the* projective surface s.t.
$\exists$ holomor. (submersion) $\pi\colon S\longrightarrow\mathbb{P}^1$ with $\pi^{-1}(p)\approx\mathbb{P}^1\ \forall\, p\in\mathbb{P}^1$ and
irred. curve $C\subset\mathbb{F}_k$ with $C^2=-k$.

Proof. $C=E_\infty\subset\mathbb{F}_k \implies C^2=-k$ ✓
$C\subset\mathbb{F}_k$ irred., $C\neq E_\infty \implies C \sim aE_0+bF$ with $a,b\in\mathbb{Z}^{\geqslant 0}$
$\implies C^2 = a^2k+2ab\geqslant 0$
$\implies E_\infty\subset\mathbb{F}_k$ is the unique irred. curve with negative self-intersection
$\implies \mathbb{F}_{k'}$ with $k'\neq k$ does not contain irred. curve C with $C^2=-k$. □

Crl 2: $\mathbb{F}_1 \approx \mathrm{Bl}_p\mathbb{P}^2$.

Proof 1. $E_\infty \subset \mathbb{F}_1$ exceptional curve ($E_\infty \approx \mathbb{P}^1$, $E_\infty^2 = -1$)
Castelnuovo-Enriques Criterion $\Longrightarrow$ $\mathbb{F}_1 =$ blowup of projective surface S at some $p \in S$
so that $E_\infty = E_p$ exceptional divisor
$E_0 \cap E_\infty = \varnothing \;\Longrightarrow\; K_S \cdot \pi(E_0) = K_{\mathbb{F}_1} \cdot E_0 = -2-1 = -3 < 0$
$h^{p,q}(S) = h^{p,q}(\mathbb{P}^2)\ \forall\,(p,q)$, K_S not positive $\Longrightarrow$ $S \approx \mathbb{P}^2$ □

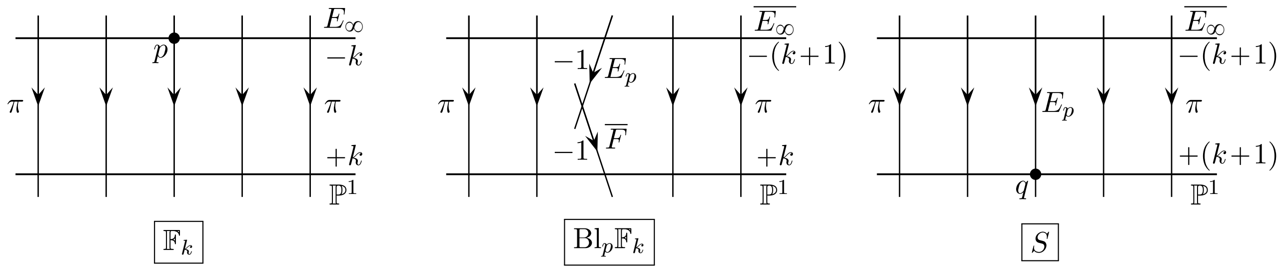
Proof 2. Take $p \in \mathbb{P}^2$ and $\mathbb{P}^1 \subset \mathbb{P}^2 - \{p\}$.
Projection $\mathbb{P}^2 - \{p\} \longrightarrow \mathbb{P}^1$ from p extends to holomor. submersion $\pi\colon \mathrm{Bl}_p\mathbb{P}^2 \longrightarrow \mathbb{P}^1$
with $\pi^{-1}(x) \approx \mathbb{P}^1\ \forall\, x \in \mathbb{P}^1$
b/c proper transforms of lines thr. p are disjoint in $\mathrm{Bl}_p\mathbb{P}^2$



$E_p^2 = -1 \quad\Longrightarrow\quad \mathrm{Bl}_p\mathbb{P}^2 \approx \mathbb{F}_1$ (by Crl. 1)

Crl 3: $\mathrm{Bl}_p\mathbb{F}_k \approx \mathrm{Bl}_q\mathbb{F}_{k+1}$ if
$p \in E_\infty(\mathbb{F}_k) \qquad q \notin E_\infty(\mathbb{F}_{k+1})$

Proof. $E_p \subset \mathrm{Bl}_p\mathbb{F}_k$ exceptional divisor
$\overline{E_\infty} \subset \mathrm{Bl}_p\mathbb{F}_k$ proper transform of $E_\infty(\mathbb{F}_k)$
$\overline{F} \subset \mathrm{Bl}_p\mathbb{F}_k$ proper transform of fiber thr. p
$\overline{E_\infty} \cap \overline{F} = \varnothing$, $\overline{E_\infty}^2 = E_\infty(\mathbb{F}_k)^2 + \mathrm{ord}_p(E_\infty(\mathbb{F}_k))^2 E_p^2 = -(k+1)$
$\overline{F} \approx F \approx \mathbb{P}^1$, $\overline{F}^2 = F^2 + \mathrm{ord}_p(F)^2 E_p^2 = -1$
Castelnuovo-Enriques Criterion $\Longrightarrow$ $\mathrm{Bl}_p\mathbb{F}_k =$ blowup of projective surface S at some $q \in S$
so that $\overline{F} = E_q$ exceptional divisor



S contains irred. curve $C \equiv \pi(\overline{E_\infty})$ with $C^2 = -(k+1)$
$\pi\colon \mathrm{Bl}_p\mathbb{F}_k \longrightarrow \mathbb{P}^1$ induces holomor. submersion $\pi\colon S \longrightarrow \mathbb{P}^1$ with $\pi^{-1}(x) \approx \mathbb{P}^1\ \forall\, x \in \mathbb{P}^1$
Crl. 1 $\Longrightarrow$ $S \approx \mathbb{F}_{k+1}$ □

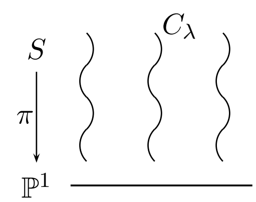
Crl 4: $\mathbb{F}_k$ is rational (birational to $\mathbb{P}^2$) $\forall\, k\in\mathbb{Z}^{\geqslant 0}$
Proof. $\mathbb{F}_1=\mathrm{Bl}_p\mathbb{P}^2$, $\mathbb{F}_{k+1}=$ blowdown of a blowup of $\mathbb{F}_k$

Thm: The minimal rational surfaces are $\mathbb{P}^2$ and $\mathbb{F}_k$ with $k=0,2,3,\ldots$
Crl 1,4 $\implies$ $\mathbb{P}^2$ and $\mathbb{F}_k$ with $k=0,2,3,\ldots$ are minimal rational surfaces
Need to show: every projective surface S birational to $\mathbb{P}^2$
can be obtained by a sequence of blowups from $\mathbb{P}^2$ or some $\mathbb{F}_k$.

Lemma 1: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a surface S.
If $C_\lambda\cap C_{\lambda'}=\varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and C_λ is smooth $\forall\,\lambda\in\mathbb{P}^1$, then

$$\pi\colon S\longrightarrow\mathbb{P}^1,\qquad C_\lambda\ni x\longrightarrow\lambda,$$

is a well-defined holomor. submersion.

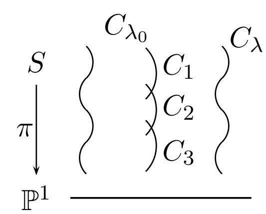


Proof. $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ pencil of curves on S $\iff$ $\exists$ holomor.l.b. $L\longrightarrow S$ and
2-dim lin. subspace $V\subset H^0(L)$
s.t. $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}=\{s^{-1}(0)\}_{[s]\in\mathbb{P}V}$

$C_\lambda\cap C_{\lambda'}=\varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$ $\iff$ $V\longrightarrow L_x$, $s\longrightarrow s(x)$, onto $\forall\,x\in S$
$\implies$ $\pi\colon S\longrightarrow\mathbb{P}V^*\approx\mathbb{P}^1$, $x\longrightarrow\{s\in V\colon s(x)=0\}$,
is well-defined, holomor., $\pi^{-1}(\lambda)=C_\lambda$ $\forall\,\lambda\in\mathbb{P}^1$

$[C_\lambda]=[C_{\lambda'}]\in H_2(S;\mathbb{Z})$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$ $\implies$ $\langle c_1(\mathcal{N}_SC_\lambda),C_\lambda\rangle=C_\lambda^2=0$ $\forall\,\lambda\in\mathbb{P}^1$
$k_x\equiv$ smallest order of nonvanishing derivative $D^i\pi$ of π at $x\in S$
$\lambda\in\mathbb{P}^1$ any, $k\equiv\min\{k_x\colon x\in C_\lambda\}\implies D^k\pi\colon(\mathcal{N}_SC_\lambda)^{\otimes k}\longrightarrow T_\lambda\mathbb{P}^1$ isom. $\implies[C_\lambda]=k\times$ generic $C_{\lambda'}$
$[C_\lambda]=[C_{\lambda'}]\in H_2(S;\mathbb{Z})$ $\implies$ $k=1$ $\implies$ π submersion $\square$

Lemma 2: Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a pencil of curves on a projective surface S.
If $C_\lambda\cap C_{\lambda'}=\varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$, and $C_\lambda\approx\mathbb{P}^1$ for some $\lambda\in\mathbb{P}^1$,
then S is a blowup of some $\mathbb{F}_k$.



Proof. If C_λ irred. $\forall\,\lambda$, then $C_\lambda\approx\mathbb{P}^1$ $\forall\lambda$ and done by Lemma 1 and Prp

Suppose $C_{\lambda_0}=\sum\limits_{i=1}^{m}a_iC_i$ with $m\geqslant 2$, $a_i\in\mathbb{Z}^+$, $C_i\subset S$ irred. curve

$C_\lambda\cap C_i=\varnothing$ $\forall\,\lambda\in\mathbb{P}^1-\{\lambda_0\}$ $\implies$ $0=C_\lambda\cdot C_i=C_{\lambda_0}\cdot C_i=\sum\limits_{j=1}^{m}a_jC_j\cdot C_i$ $\implies$ $C_i^2\leqslant 0$

C_{λ_0} connected $\implies$ $C_j\cdot C_i>0$ for some $j\neq i$ $\implies$ $C_i^2<0$ $\forall\,i$

$C_\lambda\cap C_{\lambda'}=\varnothing$ $\forall\,\lambda,\lambda'\in\mathbb{P}^1$, $\lambda\neq\lambda'$ $\implies$ $K_S\cdot C_\lambda=\langle K_{C_\lambda},C_\lambda\rangle-\langle c_1(\mathcal{N}_SC_\lambda),C_\lambda\rangle=-2-0$ if $C_\lambda\approx\mathbb{P}^1$
$\implies$ $-2=K_S\cdot C_\lambda=K_S\cdot C_{\lambda_0}=\sum\limits_{i=1}^{m}a_iK_S\cdot C_i$
$\implies$ $K_S\cdot C_i<0$ for some i

Castelnuovo-Enriques $\implies$ can blow down C_i to get projective S' with pencil $\{C'_\lambda\}_{\lambda\in\mathbb{P}^1}$
$\rightsquigarrow$ keeping blowing down to get $\{C'_\lambda\}_{\lambda\in\mathbb{P}^1}$ as in Lemma 1