MAT 644: Complex Curves and Surfaces
Notes for 04/13/20

Last week: began study of cmpt conn. $\mathbb{C}$-surfaces

Castelnuovo-Enriques Criterion: $\widetilde{S}$ = projective surface, $E\subset\widetilde{S}$ irred. curve. *Then,*
$\widetilde{S}$ is the blowup of a proj. surface S at some $p\in S$ so that $E=E_p$ is the exceptional divisor
iff $E\approx\mathbb{P}^1$ and $E\cdot E=-1$ *iff* $E\cdot K_{\widetilde{S}}<0$ and $E\cdot E<0$

1st $\Longrightarrow$ 2nd $\Longrightarrow$ 3rd: trivial
2nd $\Longrightarrow$ 1st: Monday via Kodaira Vanishing Thm for positive line bundles
3rd $\Longrightarrow$ 2nd: easy from Prp 1

S = cmpt $\mathbb{C}$-surface, $C\subset S$ irred. curve, the arithmetic genus of C is

$$a(C)\equiv 1+\frac{1}{2}\big(C\cdot K_S+C\cdot C\big)$$

Prp 1: $g(\widetilde{C})\leqslant a(C)$, where $\eta\colon\widetilde{C}\longrightarrow C$ is the *normalization of* C; the equality holds *iff* C is smooth

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Prp 2: (started pf on Wed)**:** S = cmpt $\mathbb{C}$-surface, $C\subset S$ irred. curve
(a) $a(C)\in\mathbb{Z}$ (b) $g(\widetilde{C})\leqslant a(C)-\sum_{p\in C}\binom{\mathrm{ord}_pC}{2}$

Wed: Prp 2(b) $\Longrightarrow$ Prp 1

Lemma (proved on Wed)**:** $p\in C$, $U\subset S$ neighborhood of p, $C\cap U=(f)$ with $f\colon U\longrightarrow\mathbb{C}$ holomor.,
$\{B_i\}\equiv$ branches of C at p, $h_i\colon(\mathbb{D},0)\longrightarrow(B_i,p)$ normalization,
$w_i\colon(U,p)\longrightarrow(\mathbb{C},0)$ with $T_pB_i=T_p\{w_i=0\}$. *Then,*

$$\sum_{B_i}\mathrm{ord}_{\tilde{z}=0}\bigg(\frac{\partial f}{\partial w_i}\circ h_i\bigg)\geqslant\big(\mathrm{ord}_pC\big)\big(\mathrm{ord}_pC-1\big)+\sum_{B_i}\big(\mathrm{ord}_pB_i-1\big)\,.$$

Compute $g(\widetilde{C})$ from $-\chi(\widetilde{C})=\deg K_{\widetilde{C}}=\deg(\widetilde{\omega})$, $\widetilde{\omega}$ = nonzero merom. 1-form on $\widetilde{C}$

Start with ω = merom. 2-form on S s.t.

$$(\omega)=-C+D,\qquad\qquad D=\sum_i a_iD_i$$

with $a_i\in\mathbb{Z}-\{0\}$, $D_i\subset S$ irred., $D_i\neq C$, $D_i\cap C_{\mathrm{sing}}=\varnothing$
$\Longleftrightarrow$ ω has simple pole along C, nothing additional on C_{sing}

Note: S projective $\Longrightarrow$ K_S has lots of meromorphic sections

Take $\widetilde{\omega}\equiv\mathrm{PR}_C(\omega)$, the Poincare residue of ω along C, merom. 1-form on C:

$$\mathrm{PR}\colon \mathcal{O}_S\big(K_S(C)\big)\xrightarrow{\text{restr}}\mathcal{O}_S\big(K_S(C)\big)\big|_C\approx\mathcal{O}_C\big(K_S|_C\otimes\mathcal{O}_S(C)|_C\big)\approx\mathcal{O}_C\big(K_S|_C\otimes\mathcal{N}_SC\big)\approx\mathcal{O}_C(K_C).$$

Locally: $C=(f)$, $\omega=g\dfrac{\mathrm{d}z\wedge\mathrm{d}w}{f}\implies\widetilde{\omega}=\frac{g\,\mathrm{d}z}{\partial f/\partial w}\big|_{C^*}$

independent of the coordinates (z,w) b/c $\mathrm{d}z/f_w=-\mathrm{d}w/f_z$ on C^*

$$\big\{\text{zeros/poles of }\widetilde{\omega}\text{ on }C\big\}=\big\{\text{zeros/poles of }f\omega\big\}\cap C^*\quad(\text{or }\cap C)$$
$$\#\;=\;(K_S+C)\cdot C$$

Zeros/Poles of $\widetilde{\omega}$ *on* $\widetilde{C}\supset\eta^{-1}(C^*)$*:*

$p\in C_{\mathrm{sing}}$, $U\subset S$ neighborhood of p, $C\cap U=(f)$ with $f\colon U\longrightarrow\mathbb{C}$ holomor.,
$\{B_i\}\equiv$ branches of C at p, $h_i\colon(\mathbb{D},0)\longrightarrow(B_i,p)$ normalization,
$w_i\colon(U,0)\longrightarrow(\mathbb{C},0)$ with $T_pB_i=T_p\{w_i=0\}$

$$k_i\equiv\mathrm{ord}_pB_i\quad\Longrightarrow\quad\widetilde{h}_i(\widetilde{z})=\big(\widetilde{z}^{k_i},w_i(\widetilde{z})\big)\quad\Longrightarrow\quad\mathrm{ord}_{\widetilde{z}=0}h_i^*\widetilde{\omega}=\underbrace{(k_i-1)}_{\mathrm{d}(\widetilde{z}^{k_i})}-\mathrm{ord}_{\widetilde{z}=0}\bigg(\frac{\partial f}{\partial w_i}\circ h_i\bigg)$$

$$\text{Lemma}\Longrightarrow\sum_{B_i}\mathrm{ord}_{\widetilde{z}=0}h_i^*\widetilde{\omega}\leqslant-\big(\mathrm{ord}_pC\big)\big(\mathrm{ord}_pC-1\big)$$

$$\therefore\underbrace{\deg(\widetilde{\omega})}_{2g(\widetilde{C})-2}\leqslant\underbrace{(K_S+C)\cdot C}_{\deg(\widetilde{\omega}|_{C^*})}-\sum_{p\in C}\big(\mathrm{ord}_pC\big)\big(\mathrm{ord}_pC-1\big)\quad\Longrightarrow\quad g(\widetilde{C})\leqslant\underbrace{1+\frac{1}{2}\big(C\cdot K_S+C\cdot C\big)}_{a(C)}-\sum_{p\in C}\binom{\mathrm{ord}_pC}{2}$$

This completes proof of Prp 2(b).

Normalization of Curve $C\subset S$ via Blowup of Surface S

(1) Pick $p_1\in C_{\mathrm{sing}}$. Take $S_1\equiv\mathrm{Bl}_{p_1}S\xrightarrow{\pi_1}S$; $E_1\equiv\pi_1^{-1}(p_1)$ exceptional divisor for π_1
$C_1\equiv$ proper transform of C in S_1, closure of $C-\{p_1\}$ in S_1

Last week: $E_1\cdot E_1=-1$, $C_1=\pi_1^*C-(\mathrm{ord}_{p_1}C)E_1$

$$\Longrightarrow\qquad C_1\cdot K_{S_1}=C\cdot K_S-(\mathrm{ord}_{p_1}C)\big(\langle K_{E_1},E_1\rangle-E_1\cdot E_1\big)=C\cdot K_S+(\mathrm{ord}_{p_1}C)$$
$$C_1\cdot C_1=C\cdot C+(\mathrm{ord}_{p_1}C)^2(E_1\cdot E_1)=C\cdot C-(\mathrm{ord}_{p_1}C)^2$$
$$a(C_1)\equiv1+\frac{1}{2}\big(C_1\cdot K_{S_1}+C_1\cdot C_1\big)=a(C)-\binom{\mathrm{ord}_{p_1}C}{2}<a(C).\tag{1}$$

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(2) Keep blowing up at singular points. By Prp 2(b) and (1),

$$1-\big|\pi_0(\widetilde{C})\big|=1-\big|\pi_0(\widetilde{C_r})\big|\leqslant g(\widetilde{C_r})\leqslant a(C_r)\leqslant a(C)-r$$

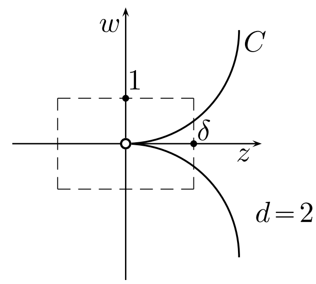
$\Longrightarrow$ process must terminate $\implies$ $C_r\subset S_r$ smooth for some $r\in\mathbb{Z}^{\geqslant0}$

$$g(\widetilde{C_r})=g(C_r)=a(C_r)=a(C)-\sum_{s=1}^{r}\binom{\mathrm{ord}_{p_r}C_{r-1}}{2}\quad\Longrightarrow\quad a(C)\in\mathbb{Z}\quad\Longrightarrow\quad\text{Prp 2(a)}$$

Also, $\pi_1\circ\ldots\circ\pi_r\colon C_r\longrightarrow C$ is finite:1 everywhere, 1:1 over $C^*\implies C_r=\widetilde{C}$ normalization of C

Prp 3: $S = \mathbb{C}$-surface, $F \subset S$ discrete, $C \subset S - F$ $\mathbb{C}$-curve $\implies$ $\overline{C} \subset S$ is $\mathbb{C}$-curve

E.g. did this for proper transform under blowup using local coordinates on Monday



In general: enough to consider (i) $C \subset \mathbb{D}^2 - \{0\}$

(ii) $C \not\supset \{(0,w) \in \{0\} \times \mathbb{D}^*\}$ $\implies$ $C \cap \{(0,w)\}$ discrete

(iii) $C \cap \{(0,w) : |w| = 1\} = \varnothing$ $\implies$ $C \cap \{(z,w) : |w| = 1, |z| < \delta\} = \varnothing$

Need to show: $\overline{C} \cap (\mathbb{D}_\delta \times \mathbb{D}) = (g)$ for some $g : \mathbb{D}_\delta \times \mathbb{D} \longrightarrow \mathbb{C}$ holomor.

Proof. (1) de Rham/Dolbeault Thm + $\bar{\partial}$-Poincare Lemma $\implies$ $\check{H}^2(\mathbb{D}_\delta^* \times \mathbb{D}; \underline{\mathbb{Z}}), \check{H}^1(\mathbb{D}_\delta^* \times \mathbb{D}; \mathcal{O}) = 0$

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O} \xrightarrow{\exp(2\pi\mathfrak{i}\cdot)} \mathcal{O}^* \longrightarrow \{1\}$$

$\implies$ $\mathrm{Pic}(\mathbb{D}_\delta^* \times \mathbb{D}) = \check{H}^1(\mathbb{D}_\delta^* \times \mathbb{D}; \mathcal{O}) = 0$

$\implies$ $C \cap (\mathbb{D}_\delta^* \times \mathbb{D}) =$ zero set of holomorphic section of trivial l.b. $\longrightarrow \mathbb{D}_\delta^* \times \mathbb{D}$

$\implies$ $C \cap (\mathbb{D}_\delta^* \times \mathbb{D}) = (f)$ for some $f \in \mathcal{O}(\mathbb{D}_\delta^* \times \mathbb{D})$

(iii) $\implies$ (iv) $f(z,w) \neq 0$ if $|w| = 1$, $0 < |z| < \delta$

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(2) (iv) $\implies$ $\displaystyle \phi_0(z) \equiv \frac{1}{2\pi\mathfrak{i}} \oint_{|w|=1} \frac{\mathrm{d}f}{f}(z,w)$ is well-defined if $0 < |z| < \delta$

(integration over vertical circle with $z = \text{const}$)

$= \#$ of zeros of $w \longrightarrow f(z,w)$ $\equiv d$ independent of z by continuity of ϕ_0

$\therefore$ $C \cap (\mathbb{D}_\delta^* \times \mathbb{D}) \equiv (f) \longrightarrow \mathbb{D}_\delta^*$, $(z,w) \longrightarrow w$, is $d\!:\!1$

$\{w_r(z)\}_{r=1,\ldots,d} \equiv$ roots of $w \longrightarrow f(z,w)$ with $0 < |z| < \delta$

Define $\phi_k : \mathbb{D}_\delta^* \longrightarrow \mathbb{C}$, $\displaystyle \phi_k(z) \equiv \frac{1}{2\pi\mathfrak{i}} \oint_{|w|=1} w^k \frac{\mathrm{d}f}{f}(z,w) = \sum_{r=1}^{d} w_r(z)^k \equiv s_k(w_1(z), \ldots, w_d(k))$

where $\displaystyle s_k(w_1, \ldots, w_d) \equiv \sum_{r=1}^{d} w_r^k$

$\phi_k : \mathbb{D}_\delta^* \longrightarrow \mathbb{C}$ bounded $\implies$ extends to holomor. $\phi_k : \mathbb{D}_\delta \longrightarrow \mathbb{C}$

Define $p_k \in \mathbb{Q}[s_1, \ldots, s_d]$ by $p_k(s_1(w_1, \ldots, w_d), \ldots, s_d(w_1, \ldots, w_d)) = \sigma_k(w_1, \ldots, w_d)$

$\sigma_k(w_1, \ldots, w_d) \equiv$ k-th elementary symm. polyn. in $w_1, \ldots, w_d$

E.g. $p_1(s_1, \ldots, s_d) = s_1$, $\quad p_2(s_1, \ldots, s_d) = \dfrac{1}{2}(s_1^2 - s_2)$

Take $g(z,w) = w^d - p_1(\phi_1(z), \ldots, \phi_d(z)) w^{d-1} + \ldots + (-1)^d p_d(\phi_1(z), \ldots, \phi_d(z))$

$= w^d - \sigma_1(w_1(z), \ldots, w_d(z)) w^{d-1} + \ldots + (-1)^d \sigma_d(w_1(z), \ldots, w_d(z))$

$\big\{$zeros of $w \longrightarrow g(z,w)\big\} = \{w_r(z)\}_{r=1,\ldots,d} = \big\{$zeros of $w \longrightarrow f(z,w)\big\}$ if $0 < |z| < \delta$

$\implies$ $C \cap (\mathbb{D}_\delta^* \times \mathbb{D}) = (f) \cap (\mathbb{D}_\delta^* \times \mathbb{D}) = (g) \cap (\mathbb{D}_\delta^* \times \mathbb{D})$

$C \cap \{(0,w)\}$ discrete $\implies$ $\overline{C} \cap (\mathbb{D}_\delta \times \mathbb{D}) = (g)$ $\square$

Crl: $S=\mathbb{C}$-surface, $F\subset S$ discrete, $f: S-F\longrightarrow\mathbb{P}^n$ holomor.
(1) $f^*\mathcal{O}_{\mathbb{P}^n}(1)\longrightarrow S-F$ extends to a holomor. l.b. $L\longrightarrow S$
(2) f induces homom. $f^*: H^0(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(1))\longrightarrow H^0(S;L)$

Proof. $H\subset\mathbb{P}^n$ hyperplane s.t. $\mathrm{Im}(f)\not\subset H \implies f^{-1}(H)\subset S-F$ is a curve
$f^*\mathcal{O}_{\mathbb{P}^n}(1)=[f^{-1}(H)]\longrightarrow S-F$
Prp 3 $\implies \overline{f^{-1}(H)}\subset S$ is a curve $\implies f^*\mathcal{O}_{\mathbb{P}^n}(1)$ extends to l.b. $L\equiv[\overline{f^{-1}(H)}]\longrightarrow S$

Hartog's $\implies$ every $s\in H^0(S-F;f^*\mathcal{O}_{\mathbb{P}^n}(1))$ extends to $\tilde{s}\in H^0(S;L)$
$\implies$ get $f^*: H^0(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(1))\longrightarrow H^0(S;L)$
$\implies L\longrightarrow S$ independent of $H\subset\mathbb{P}^n$ hyperplane s.t. $\mathrm{Im}(f)\not\subset H$

Note: $f^*s\equiv 0$ for $s\in H^0(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(1)) \iff \mathrm{Im}(f)\subset H\equiv s^{-1}(0)\subset\mathbb{P}^n$

Other Tools to Study Cmpt $\mathbb{C}$-Surface S

(1) Adjunction Formula: If $C\subset S$ is a smooth curve,

$$g(C)=1+\frac{1}{2}\big(C\cdot K_S+C\cdot C\big)\equiv a(C).$$

Follows from $2g(C)-2\equiv-\chi(C)=-\langle c_1(TC),C\rangle$ and $\mathcal{O}_S(C)|_C\approx NC\equiv TS|_C/TC$.

(2) Noether's Formula: $\chi(\mathcal{O}_S)\equiv h^0_{\bar\partial}(S)-h^1_{\bar\partial}(S)+h^2_{\bar\partial}(S)=\frac{1}{12}\big(\chi(S)+K_S\cdot K_S\big)$.
Proved directly for S projective in Section 4.6 of G&H.

(3) Riemann-Roch for Line Bundle on a Surface: If $L\longrightarrow S$ is a holomor. l.b.,

$$\chi(L)\equiv h^0_{\bar\partial}(S;L)-h^1_{\bar\partial}(S;L)+h^2_{\bar\partial}(S;L)=\chi(\mathcal{O}_S)+\frac{1}{2}\big(L\cdot L-L\cdot K_S\big).$$

Obtained from RR for Line Bundle on a Curve in Section 4.1 of G&H.

(4) If S is a (projective) surface with the same Hodge $\diamond$ as $\mathbb{P}^2$ and K_S is not positive, then $S\approx\mathbb{P}^2$.
Obtained from (2), (3), and Kodaira Vanishing Theorem in Section 4.1 of G&H.

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(3) is tautology for $L=$ trivial holomor. l.b.
(2) and (3) are special cases of Hirzebruch-Riemann-Roch:
$\chi(E)=\ldots$ for holomor. v.b. E over projective X
HRR is special case of Atiyah-Singer Index Thm:
$\mathrm{ind}\,D=\ldots$ for elliptic operator D on smooth cmpt X
discussed at the beginning of the semester

(3) and (4) might be discussed at the end of some lecture, time-permitting