MAT 644: Complex Curves and Surfaces
Notes for 04/08/20

Last time: began studying $\mathbb{C}$-surfaces S

Some examples: $\Sigma_{g_1}\times\Sigma_{g_2}$ with $\Sigma_g \equiv$ Riemann surf. of genus g

$\mathbb{P}E\to\Sigma_g$ with $E\to\Sigma_g$ holom. vector bundle of rank 2

$\mathbb{P}(E\otimes L)\approx\mathbb{P}E$ if $L\to\Sigma_g$ holom. l.b.

$\mathbb{P}^2$, $X_a\subset\mathbb{P}^3$ smooth hypersurface of degree a

$X_{a_1,\dots,a_k}\subset\mathbb{P}^{2+k}$ sm. complete intersection of multi-degree $(a_1,\dots,a_k)$

$= X_{a_1}\cap X_{a_2}\cap\dots\cap X_{a_k}\subset\mathbb{P}^{2+k}$

$X_{a_1,\dots,a_k,1}=X_{a_1,\dots,a_k}$, $X_1=\mathbb{P}^2\subset\mathbb{P}^3$

GH, Sect 4.1 $\quad X_2\approx\mathbb{P}^1\times\mathbb{P}^1$, blowup of X_2 at 1pt $\approx$ blowup of $\mathbb{P}^2$ at 2 pts
$\subset\mathbb{P}^3$

HW4: $\mathbb{P}^1\times\mathbb{P}^1\not\approx$ blowup of $\mathbb{P}^2$ at 1pt

$X_3\approx$ blowup of $\mathbb{P}^2$ at 6 general pts
$\subset\mathbb{P}^3$

"general" = (i) no 3 on a line (ii) not all on a conic

Easy: $a\ge 4\implies X_a\subset\mathbb{P}^3$ not a blowup/blowdown of $\mathbb{P}^2$

Reason: $h^{2,0}(X_a)\ge 1\implies \neq h^{2,0}(\mathbb{P}^2)$

Proved Castelnuovo-Enriques Criterion S = projective surface

irred. curve $C\subset S$ can be blown down to a (smooth) point

iff $C\approx\mathbb{P}^1$ and $C\cdot C=-1$ (1)

Stronger version: irred. curve $C\subset S$ can be blown down to a point

iff $C\cdot K_S<0$ and $C\cdot C<0$ (2)

(1) $\Rightarrow$ (2) $\quad C\cdot K_S=-\langle c_1(TS),C\rangle=-\langle c_1(TC)+\underbrace{c_1(N_SC)}_{[C]|_C},C\rangle$

$=-(\underbrace{\chi(C)}_{2}+\underbrace{C\cdot C}_{-1})<0$ ✓

② ⇒ ① $C \subset S$ any curve, define

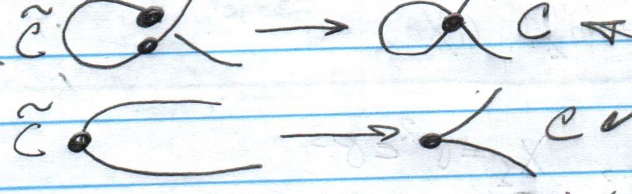
(i) $a(C) \equiv 1 + \frac{1}{2}(C^2 + C \cdot K_S)$ <u>arithmetic genus</u> of C

= genus of any smooth curve C' homologous to C (by adjunction)

(ii) <u>normalization</u> of C is holomorphic $\eta : \tilde{C} \to C$ s.t.

$\tilde{C}$ is smooth curve, η is finite:1 everywhere and 1:1 over $C^* \equiv$ smooth pts of C

<u>Examples:</u>



could be cubic in $\mathbb{P}^2$

$g(\tilde{C}) = 0$, $a(C) = 1$

<u>Prp:</u> (a) $a(C) \in \mathbb{Z}^{\geq 0}$ $\quad a(C) \equiv 1 + \frac{1}{2}(C^2 + C \cdot K_S)$

(b) $g(\tilde{C}) \leq a(C)$ and = holds iff C is smooth, conn.

$\underbrace{C \subset S \text{ irred.}}$, $C^2 < 0$, $C \cdot K_S < 0$

$\Rightarrow \tilde{C}$ conn. $\Rightarrow 0 \leq g(\tilde{C}) \leq a(C) = 1 + \frac{1}{2}(C^2 + C \cdot K_S) \leq 0$

$\Rightarrow C^2, C \cdot K_S = -1$, C smooth, conn., $g(C) = 0 \Rightarrow C \approx \mathbb{P}^1$

$\Rightarrow$ can blow down C to a (smooth) pt by Castelnuovo-Enriques □

Prp to be proved; also need existence and uniqueness of normalization

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<u>Normalization is unique:</u>

$$\begin{tikzcd} \tilde{C}_1 \arrow[rr, dashed, "\phi ?", "\approx"'] \arrow[dr, "\eta_1"'] & & \tilde{C}_2 \arrow[dl, "\eta_2"] \\ & C & \end{tikzcd} \supset \begin{tikzcd} \eta_1^{-1}(C^*) \equiv \tilde{C}_1^* \arrow[rr, "\phi"] \arrow[dr, "\eta_1"'] & & \tilde{C}_2^* \equiv \eta_2^{-1}(C^*) \arrow[dl, "\eta_2"] \\ & C^* = \text{smooth pts of } C & \end{tikzcd}$$

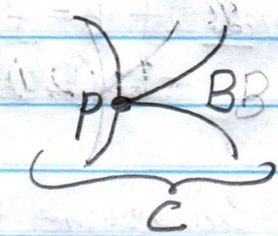
$\eta_i : \tilde{C}_i^* \to C^*$ biholom. $\Rightarrow$ so is $\phi \equiv \eta_2^{-1} \circ \eta_1 : \tilde{C}_1^* \to \tilde{C}_2^*$

ϕ is bounded around each $p \in \tilde{C}_1 - \tilde{C}_1^*$

$\Rightarrow$ extends to $\tilde{C}_1 \to \tilde{C}_2$

Construction of Normalization $\eta: \tilde{C} \to C \subset S$

Take $p \in C_{sing}$, B = branch of C at p



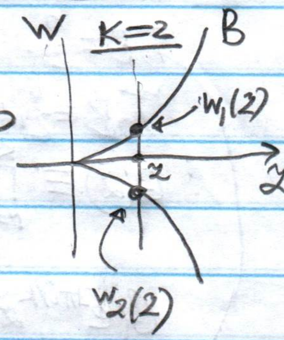
$B = (f)$, $f = f(z,w)$ Weierstrass polyn.

$= w^k + a_1(z) w^{k-1} + \dots + a_k(z)$

a_ℓ = holom., $a_\ell(0) = 0$

$w \to f(z,w)$ has only simple roots $\{w_r(z)\}_{r=1,\dots,k}$ if $z \neq 0$

(o/w get a non-smooth pt $\in B - \{p\}$)



$\Rightarrow$ $\pi_1: \underbrace{B - \{p\}}_{\text{connected b/c } B \text{ is irred.}} \to \mathbb{D}^*$, $(z,w) \to z$, is $k:1$ covering map, con

$\Rightarrow$ $\exists$ biholom. $h: \mathbb{D}^* \to B - \{p\} \subset \mathbb{C}^2$ s.t. $\pi_1(h(\tilde{z})) = \tilde{z}^k$

$$\begin{tikzcd} \mathbb{D}^* \arrow[r, "h", "\approx"'] \arrow[dr, "\tilde{z}^k"'] & B - \{p\} \arrow[d, "\pi_1"] \\ & \mathbb{D}^* \end{tikzcd}$$

$\Rightarrow h(\tilde{z}) = (\tilde{z}^k, \tilde{w}_1(\tilde{z}^k))$, $\tilde{w}_1: \mathbb{D}^* \to \mathbb{C}$

(1) $\{\underbrace{\tilde{w}_1(e^{2\pi i \cdot r/k} \tilde{z})}_{\equiv \tilde{w}_{r+1}(\tilde{z})}\}_{r=0,\dots,k-1} = \{w_r(\tilde{z}^k)\}_{r=1,\dots,k} \equiv \{\text{roots of } w \to f(\tilde{z}^k, w)\}$

b/c $B = (f)$

(2) $\tilde{w}_{r+1}: \mathbb{D}^* \to \mathbb{C}$ bounded $\Rightarrow$ extends holom. to $\mathbb{D} \to \mathbb{C}$

$\therefore$ get $h: \mathbb{D} \to B \subset S$, $\tilde{z} \to (\tilde{z}^k, \tilde{w}_1(\tilde{z}))$ biholom. except at $0 \in \mathbb{D}$

<u>Define</u>

$$\begin{array}{ccc} \tilde{C} \equiv \Big(C^* \amalg \coprod_{p \in C_{sing}} \coprod_{B = \text{branch at } p} \mathbb{D}\Big)\Big/\tilde{\sim} & & \text{via each } h: \mathbb{D}^* \to B - \{p\} \subset C^* \\ \downarrow \eta \quad \downarrow id \qquad \downarrow h & & \\ C \equiv \Big(C^* \amalg \coprod_{p \in C_{sing}} \coprod_{B = \text{branch at } p} B\Big)\Big/\sim & & \text{via } B - \{p\} \hookrightarrow C^* \end{array}$$

<u>Crl:</u> each branch B of C at p has a well-defined tangent line

$T_p B = \mathbb{C} \cdot$ (first nonzero Taylor coefficient of $h: \mathbb{D} \to B \subset \mathbb{C}^2$ at $\tilde{z} = 0$)

$h(\tilde{z}) = A \tilde{z}^m + O(\tilde{z}^{m+1}) \in B \subset \mathbb{C}^2$, $A \neq 0 \Rightarrow T_p B = \mathbb{C} A \subset T_p S = \mathbb{C}^2$

Lemma 1 If $B=(f)$ is a branch of C at $0\in C_{sing}$ with $f=w^{\kappa}+a_1(z)w^{\kappa-1}+\ldots+a_{\kappa}(z)$ and $T_0B=\{w=0\}\subset T_0\mathbb{C}^2=\mathbb{C}^2$, then

$$\kappa=\mathrm{ord}_0 B \quad\text{and}\quad \mathrm{ord}_{z=0}\, a_\ell\geq \ell+1 \text{ for all } \ell.$$

Pf: Let $f(z,w)=\underbrace{f_m(z,w)}_{\text{homogeneous of degree } m}+r(z,w)$ ← higher order terms

$$\equiv \prod_{i=1}^{m}(a_iw-b_iz) \text{ with } (a_i,b_i)\neq 0$$

$T_pB=\{w=0\}\Rightarrow h(\tilde{z})=\tilde{z}^n(1,g(\tilde{z}))$ with $g(0)=0$

↖ normalization of B as before

$$\Rightarrow 0=\tilde{z}^{-mn}f(\tilde{z}^n,\tilde{z}^ng(\tilde{z}))=\prod_{i=1}^{m}(a_ig(\tilde{z})-b_i)+\underbrace{\tilde{z}^{-mn}r(\tilde{z}^n,\tilde{z}^ng(\tilde{z}))}_{\text{vanishes to order}\geq n}$$

$\Rightarrow$ at least one $b_i=0$

B irred. at 0 (only 1 tangent line) $\Rightarrow$ all $[a_i,b_i]\in\mathbb{P}^1$ are the same $\Rightarrow$ ✓

Lemma 2: If $B=(f)$ is a branch of C at $p\in C_{sing}$, $T_0B=\{w=0\}\subset T_0\mathbb{C}^2=\mathbb{C}^2$,
$h:(\mathbb{D},0)\to(B,p)$ is normalization, then

$$\mathrm{ord}_{\tilde{z}=0}\left(\frac{\partial f}{\partial w}\circ h\right)\geq(\mathrm{ord}_pB)^2-1$$

Pf: Can assume f = Weierstrass polynomial w (if $f|_{w\text{-axis}}\not\equiv 0$)

$$\hookrightarrow w^k+a_1(z)w^{k-1}+\dots+a_k(z)$$

$$T_0B=\{w=0\}\xRightarrow{\text{Lemma 1}}\mathrm{ord}_{z=0}\,a_l\geq l+1\quad\forall l=1,\dots,k \qquad (*)$$

$$G(\tilde{z})\equiv\prod_{r=0}^{k-1}\frac{\partial f}{\partial w}\Big(\underbrace{h(e^{2\pi i r/k}\tilde{z})}_{(\tilde{z}^k,\,\tilde{w}_{r+1}(\tilde{z}^k))}\Big)$$

$$G(e^{2\pi i/k}\tilde{z})=G(\tilde{z})\Longrightarrow G(\tilde{z})=g(\tilde{z}^k)=g(z)$$

$\hookrightarrow g(z)$ = symmetric polynomial in $\{w_r(z)\}_{r=1,\dots,k}$ of deg. $(k-1)\cdot k$

$$=\sum_{\sum l\cdot q_l=k(k-1)}C_q\,a_1(z)^{q_1}\cdots a_k(z)^{q_k}$$

$$\overset{(*)}{\Longrightarrow}\mathrm{ord}_{z=0}\,g(z)\geq\sum(l+1)q_l\geq k(k-1)+k-1=k^2-1$$

$$\Longrightarrow\underbrace{\mathrm{ord}_{\tilde{z}=0}\,G(\tilde{z})}_{k\cdot\mathrm{ord}_{\tilde{z}=0}\frac{\partial f}{\partial w}(h(\tilde{z}))}\geq k\cdot(k^2-1)\Longrightarrow\checkmark$$

Cor: if $C=(f)$ near 0, $f:\mathbb{C}^2\to\mathbb{C}$, $\{B_i\}$ = branches of C at 0,
$h_i:(\mathbb{D},0)\to(B_i,0)$ is normalization, $T_0B_i=\{w_i=0\}$,

$$\text{then}\sum_{\substack{\text{branches } B_i\\ \text{at } 0}}\mathrm{ord}_{\tilde{z}=0}\left(\frac{\partial f}{\partial w_i}\circ h_i\right)\geq(\mathrm{ord}_0C)(\mathrm{ord}_0C-1)+\sum_{B_i}(\mathrm{ord}_0B_i-1)$$

Pf: Write $f=f_1\cdots f_m$, product of irred., $B_i=(f_i)\Rightarrow\mathrm{ord}_0C=\sum_{B_i}\mathrm{ord}_0B_i$
Apply Lemma 2 to each B_i