

# MAT 615: Complex Curves and Surfaces Spring 2009

## Problem Set 5

Due on Tuesday, 04/28, at 12:40pm

*Please write up concise solutions to 2 problems.*

### Problem 1 (10 pts)

Let  $S$  be a (compact) complex surface,  $\mathcal{C} \subset S$  be a (closed) connected curve, and  $h : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$  a normalization of  $\mathcal{C}$ . Let  $S_0 = S, S_1, \dots$  be a sequence of surfaces and  $\mathcal{C}_0 = \mathcal{C}, \mathcal{C}_1, \dots$  a sequence of curves, with  $\mathcal{C}_k \subset S_k$ , such that  $S_k$  is the blowup of  $S_{k-1}$  at a singular point  $p_k \in \mathcal{C}_{k-1}$  and  $\mathcal{C}_k$  is the proper transform of  $\mathcal{C}_{k-1}$  under the blowup  $S_k \rightarrow S_{k-1}$ . It was shown in class that

$$a(\mathcal{C}_k) = a(\mathcal{C}) - \sum_{l=1}^{k-1} \binom{m_{p_l}(\mathcal{C}_{l-1})}{2}, \quad (1)$$

where  $m_{p_l}(\mathcal{C}_{l-1})$  is the order of the point  $p_l$  in the curve  $\mathcal{C}_{l-1}$ . The above sequences terminate at the  $k$ -th stage if and only if  $\mathcal{C}_k \subset S_k$  is a smooth curve. If this does happen,  $\mathcal{C}_k \approx \tilde{\mathcal{C}}$  and equation (1) implies that

$$g(\tilde{\mathcal{C}}) \leq a(\mathcal{C}) - \sum_{p \in \mathcal{C}} \binom{m_p(\mathcal{C})}{2}, \quad a(\mathcal{C}) \in \mathbb{Z}^{\geq 0}. \quad (2)$$

In particular,  $g(\tilde{\mathcal{C}}) \leq a(\mathcal{C})$  and the equality holds if and only if  $\mathcal{C} \subset S$  is smooth. In class, the first statement in (2) was proved in a different way for  $S$  projective; it implies that  $a(\mathcal{C}) \geq 1 - N(\mathcal{C})$  for any curve, where  $N(\mathcal{C})$  is the number of connected components of  $\mathcal{C}$ . This was used along with (1) to show that the above sequence does terminate. The estimate  $a(\mathcal{C}) \geq 1 - N(\mathcal{C})$  can also be obtained by showing either that  $a(\mathcal{C}) + N(\mathcal{C}) - 1$  is the dimension of the space of holomorphic 1-forms on  $\mathcal{C}$ , properly defined (since  $\mathcal{C}$  is singular), or that  $a(\mathcal{C})$  is the genus of a smooth curve in the same homology class as  $\mathcal{C}$ , which exists at least after deforming the complex structure. The purpose of this exercise is to show that the sequences terminate without an *a priori* lower bound on  $a(\mathcal{C})$ .

(a) Let  $B_{i;0} \subset \mathcal{C}$  be a branch of  $\mathcal{C}$  at a singular point  $p \in \mathcal{C}$  and  $B_{i;k} \subset S_k$  be the proper transform of  $B_{i;0}$  in  $S_k$ . Show that  $\pi_{i;k} : B_{i;k} \rightarrow B_{i;0}$  is a bijection,  $B_{i;k}$  contains at most 1 singular point,  $p_{i;k} \equiv \pi_{i;k}^{-1}(p)$ , and  $p_k \neq p_{i;k}$  if  $k$  is sufficiently large (i.e.  $B_{i;k}$  becomes smooth after enough blowups at  $p_{i;k}$ ).

(b) If  $B_1$  and  $B_2$  are distinct smooth branches at  $p_k \in \mathcal{C}_{k-1}$ , the order of contact of the proper transforms of  $B_1$  and  $B_2$  in  $S_k$  is strictly less than  $\text{mult}_{p_k}(B_1, B_2)$ .

(c) Show that  $\mathcal{C}_k$  is a smooth curve if  $k$  is sufficiently large.

**Problem 2** (10 pts)

(a) If  $S$  is a minimal surface and  $D \subset S$  is an effective divisor such that  $D \cdot K_S < 0$ , then

$$P_n(S) \equiv h^0(K_S^n) = 0 \quad \forall n \in \mathbb{Z}^+.$$

(b) If  $S, S'$  are minimal (projective) surfaces and  $f: S \rightarrow S'$  is a birational map, then either  $f$  is a biholomorphic map or  $P_n(S) = P_n(S') = 0$  for all  $n$ .

*Note:* This says that a surface  $S$  with Kodaira dimension  $\kappa(S) \geq 0$  has a unique minimal model.

**Problem 3** (10 pts)

Let  $S$  be a compact connected surface and  $D$  an effective divisor on  $S$ .

(a) If  $D^2 \equiv D \cdot D < 0$ , show that the base locus of the linear system of curves  $|D|$  contains a curve. Is the converse true?

(b) If the base locus of  $|D|$  does not contain a curve, show that it consists of at most  $D^2$  points.

(c) If  $D$  is irreducible and  $h^0(D) \geq 2$ , then  $D^2 \geq 0$ . Show that the conclusion need not hold if  $D$  is not assumed to be irreducible.

(d) If  $S$  is a minimal rational surface and  $D^2 \geq 0$ , show that  $h^0(D) \geq 2$ . Show that the conclusion need not hold if  $S$  is not assumed to be minimal.

*Hint:* you can achieve arbitrarily high  $D^2$  with  $h^0(D) = 1$  by blowing up  $\mathbb{P}^2$  at sufficiently many general points.

**Problem 4** (5 pts)

Let  $\mathbb{F}_n = S_n$  be the  $n$ -th Hirzebruch surface and  $D \subset \mathbb{F}_n$  an irreducible curve. Compute  $H^q(\mathbb{F}_n, \mathcal{O}(D))$  in terms of the decomposition of  $D$  relative to a standard basis for  $\text{Pic}(\mathbb{F}_n)$ .