MAT 615: Complex Curves and Surfaces
Spring 2009

Problem Set 4
Due on Tuesday, 04/14, at 12:40pm

Please write up concise solutions to 2 or 3 problems, worth 15 pts.

Problem 1 (5 pts)

Let $\mathcal{U} \longrightarrow \overline{\mathcal{M}}_{1,1}$ be the universal family of stable genus 1 1-marked curves with section s; see Hain's Section 5.2. Let

$$\mathbb{L}_1 \equiv s^*(T\mathcal{U}^{\mathrm{vert}})^*, \; \mathcal{L}_1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

be the universal cotangent line bundle at the (first) marked point and the line bundle defined in Hain's Section 4, respectively.
(a) Show that $\mathbb{L}_1$ is indeed a line bundle in the orbifold category (i.e. describe equivariant trivializations over the charts and the isomorphism on the overlap).
(b) Show that $\mathbb{L}_1 \approx \mathcal{L}_1$ in the orbifold category.

Problem 2 (10 pts)

Fix any 8 general points, $p_1, \ldots, p_8$, in $\mathbb{P}^2$.
(a) Show that the space of cubics passing through the 8 points is a linearly embedded $\mathbb{P}^1$ in $\mathbb{P}H^0(\mathbb{P}^2; \mathcal{O}(3)) \approx \mathbb{P}^9$.
(b) Show that

$$X \equiv \big\{([f], q) \in \mathbb{P}^1 \times \mathbb{P}^2 \colon \; f(q) = 0\big\}$$

is a smooth submanifold of $\mathbb{P}^1 \times \mathbb{P}^2$. What is its Hodge diamond?
(c) Show that $\pi \colon X \longrightarrow \mathbb{P}^1$ with the holomorphic section

$$s \colon \mathbb{P}^1 \longrightarrow X, \qquad [f] \longrightarrow \big([f], p_1\big),$$

is a family of stable genus 1 1-marked curves (i.e. $(\pi^{-1}(b), s(b))$ is a stable genus 1 1-marked curve for every $b \in \mathbb{P}^1$) and the generic fiber is smooth. Show that the number of singular fibers is 12.
Hint: the set of nodes of the fibers is a subset of X which can be written as the zero set of a holomorphic section of a rank-two vector bundle over X.
(d) Let $L_1 = s^*(TX^{\mathrm{vert}})^* \longrightarrow \mathbb{P}^1$. Show that $L_1 \approx \mathcal{O}(1) \longrightarrow \mathbb{P}^1$ and $L_1 \approx \Phi^*\mathbb{L}_1$, where $\Phi \colon \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$ is the morphism corresponding to the family in (c). Conclude that

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24} \,,$$

where $\psi_1 = c_1(\mathbb{L}_1) \in H^2(\overline{\mathcal{M}}_{1,1})$.

Problem 3 (5 pts)

Let $f: C \longrightarrow \mathbb{P}^2$ be an immersion with only simple normal crossing singularities (thus, $|f^{-1}(p)| \leq 2$ for all $p \in \mathbb{P}^2$; if $f^{-1}(p) = \{z_1, z_2\}$ with $z_1 \neq z_2$, $d_{z_1} f(T_{z_1} C) \neq d_{z_2} f(T_{z_2} C)$). Let S be the blowup of $\mathbb{P}^2$ at the double points of $f(C)$ (the singular values of f).
(a) Show that f lifts to an embedding $\tilde{f}: C \longrightarrow S$.
(b) Use Adjunction Formula in S to show that if $C \subset \mathbb{P}^2$ is of degree d and has δ double points, then

$$g(C) = \binom{d-1}{2} - \delta\,.$$

Problem 4 (5 pts)

Let X be the blowup of $\mathbb{P}^2$ at one point and $E \subset X$ the exceptional divisor. Show that
(a) X is not biholomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$;
(b) the blowup of X at a point of $X - E$ is biholomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point;
(c) $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are minimal surfaces (contain no exceptional curves).

Problem 5 (5 pts)

Let $\pi: S \longrightarrow \mathbb{P}^2$ be the blowup at k general points, $p_1, \ldots, p_k \in \mathbb{P}^2$, with the corresponding exceptional divisors $E_1, \ldots, E_k$. Let

$$D = dH - \sum_{i=1}^{i=k} m_i E_i, \qquad m_i \in \mathbb{Z},$$

and L_{ij} be the proper transform of the line $\overline{p_i p_j}$, with $i \neq j$. Suppose $|D| \neq \emptyset$ (i.e. there is an effective divisor linearly equivalent to D). Show that
(a) E_i is a fixed component of every curve in $|D|$ iff $m_i < 0$;
(b) $L_{i,j}$ is a fixed component of every curve in $|D|$ iff $d < m_i + m_j$.

Problem 6 (10 pts)

Let $\pi: S \longrightarrow \mathbb{P}^2$ be the blowup at k general points, $p_1, \ldots, p_k \in \mathbb{P}^2$.
(a) Determine the number of exceptional curves for $k \leq 8$.
(b) Show that there are infinitely many exceptional curves for $k \geq 9$.