

# MAT 566: Differential Topology Spring 2018

## Problem Set 5

Due on Monday, 04/09, by 5pm in Math 3-111

Please do two out of 12-A, 14-B, 14-E, (vi), and (vii).

**Problem (vi):** Suppose  $f : M \rightarrow M$  is a diffeomorphism. A fixed point  $x$  of  $f$  (i.e.  $f(x) = x$ ) is called *nondegenerate* if the isomorphism  $df|_x : T_x M \rightarrow T_x M$  has no eigenvalues equal to 1.

- (a) Show that a nondegenerate fixed point is necessarily isolated (in the set of fixed points).
- (b) Suppose  $M$  is compact and  $f$  is homotopic to the identity.
  - (b-i) If  $M$  is oriented and the Euler characteristic  $\chi(M)$  is non-zero, show that  $f$  has at least one fixed point. If in addition all fixed points are nondegenerate, show that their number is at least  $|\chi(M)|$ .
  - (b-ii) If  $\chi(M)$  is odd, show that  $f$  has at least one fixed point.

*Hint:* This is a sequel to Problem (v).

**Problem (vii):** A homogeneous cubic polynomial in four variables,

$$f(X_0, X_1, X_2, X_3) = \sum_{i+j+k+l=3} a_{ijkl} X_0^i X_1^j X_2^k X_3^l,$$

naturally corresponds to an element

$$\tilde{f} \in (\text{Sym}^3((\mathbb{C}^4))^* \approx \text{Sym}^3((\mathbb{C}^4)^*) \equiv (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^* \otimes (\mathbb{C}^4)^* / \sim, \quad \alpha \otimes \beta \otimes \gamma \sim \beta \otimes \alpha \otimes \gamma \sim \alpha \otimes \gamma \otimes \beta.$$

This is because each  $X_i$  defines a linear map  $X_i : \mathbb{C}^4 \rightarrow \mathbb{C}$ , i.e. the projection onto the  $i$ th coordinate. Since every fiber of the tautological 2-plane bundle  $\gamma_2 \rightarrow \text{Gr}_2 \mathbb{C}^4$  is a linear subspace of  $\mathbb{C}^4$ , by restriction  $f$  induces a section

$$s_f \in \Gamma(\text{Gr}_2 \mathbb{C}^4; \text{Sym}^3(\gamma_2^*)).$$

Note that the complex rank of the bundle  $\text{Sym}^3(\gamma_2^*) \rightarrow \text{Gr}_2 \mathbb{C}^4$  equals to the dimension of  $\text{Gr}_2 \mathbb{C}^4$ . Thus, if  $s_f$  is transverse to the zero set (as is the case for a generic  $f$ ), the set  $s_f^{-1}(0)$  is finite and its signed cardinality is

$$\pm |s_f^{-1}(0)| = \langle e(\text{Sym}^3(\gamma_2^*)), \text{Gr}_2 \mathbb{C}^4 \rangle \in \mathbb{Z}.$$

In fact,  $s_f$  is a holomorphic section of a holomorphic vector bundle and thus all its zeros are positive.

A cubic surface  $Y$  in  $\mathbb{C}P^3$  is the zero set of a homogeneous cubic polynomial in four variables, i.e.

$$Y \equiv Y_f = \{[X_0, X_1, X_2, X_3] \in \mathbb{C}P^3 : \sum_{i+j+k+l=3} a_{ijkl} X_0^i X_1^j X_2^k X_3^l = 0\} = (f^{-1}(0) - 0) / \mathbb{C}^*.$$

A projective line  $\ell$  in  $\mathbb{C}P^3$  corresponds to a 2-plane  $P$  in  $\mathbb{C}^4$ , i.e. an element of  $\text{Gr}_2 \mathbb{C}^4$ . Such a line  $\ell$  lies in  $Y_f$  if and only if  $f|_P$  vanishes identically, i.e.  $P \in s_f^{-1}(0)$ . Thus, if  $f$  is generic, the number of lines in  $Y_f$  is finite and is given by the euler class of  $\text{Sym}^3(\gamma_2^*)$ .

- (a) Formulate and prove a splitting principle for Chern classes.
- (b) Use it to determine the number of lines that lie on a generic cubic surface in  $\mathbb{C}P^3$ .
- (c) Determine the number of lines that lie on a generic quintic threefold in  $\mathbb{C}P^4$ .