MAT 545: Complex Geometry
Fall 2008

Problem Set 7 Solutions

Problem 1

(a) Let $X_a \subset \mathbb{P}^3$ *be a smooth hypersurface of degree* $a \ge 1$*. Show that*

$$\dim_{\mathbb{C}} H^0_{\bar\partial}(X_a; \mathcal{K}_{X_a}) = \begin{cases} 0, & \textit{if } a \le 3; \\ 1, & \textit{if } a = 4. \end{cases} \tag{1}$$

Determine the Hodge diamonds for X_a *with* $a \le 4$*.*
(b) Let $Y_a \subset \mathbb{P}^4$ *be a smooth hypersurface of degree* $a \ge 1$*. Determine the Hodge diamonds for* Y_a *with* $a \le 5$*.*
Note: the quartic surface $X_4 \subset \mathbb{P}^3$ *is a K3 surface; the quintic* $Y_5 \subset \mathbb{P}^4$ *is a Calabi-Yau* 3*-fold, popular in string theory.*

(a) By definition, X_a is the zero set of a transverse holomorphic section s of the line bundle $\mathcal{O}_{\mathbb{P}^3}(a) \longrightarrow \mathbb{P}^3$ or equivalently the projectivization of the zero set of a homogeneous degree a polynomial F on $\mathbb{C}^4$ which has no singular values on $\mathbb{C}^4 - 0$. Thus, $[X_a] = \mathcal{O}(a)$ and by Adjunction Formula II (p147),

$$\mathcal{K}_{X_a} = \left(\mathcal{K}_{\mathbb{P}^3} \otimes [X_a]\right)\big|_{X_a} = \mathcal{O}_{\mathbb{P}^3}(a-4)\big|_{X_a} = H^{a-4}|_{X_a},$$

with H denoting the hyperplane line bundle on $\mathbb{P}^3$. In particular, $\mathcal{K}_{X_a} \longrightarrow X_a$ is a negative line bundle if $a < 4$ and thus admits no holomorphic sections by the dual version of the Kodaira Vanishing Theorem (p155); this proves the first case of (1). In the second case of (1), $\mathcal{K}_{X_4} \longrightarrow X_4$ is the trivial line bundle; since X_4 is compact, it follows that $H^0(X_4; \mathcal{K}_{X_4}) \approx \mathbb{C}$. We also can define a nowhere zero holomorphic section Ω of $\mathcal{K}_{X_4} \longrightarrow X_4$ by

$$\Omega_i|_{[Z_0,\ldots,Z_3]} = (-1)^i \frac{dZ_0 \wedge \ldots \widehat{dZ_i} \ldots \wedge dZ_3}{\partial F/\partial Z_i|_{[Z_0,\ldots,Z_3]}} \qquad \forall\, [Z_0,\ldots,Z_3] \in X_4 \text{ s.t. } \frac{\partial F}{\partial Z_i}\Big|_{(Z_0,\ldots,Z_3)} \neq 0.$$

Since $\partial F/\partial Z_i$ is a homogeneous polynomial of degree 3, $\Omega_i|_{[Z_0,\ldots,Z_3]}$ is independent of the choice of the representative $(Z_0,\ldots,Z_3)$ for $[Z_0,\ldots,Z_3]$. The restrictions of the forms Ω_i and Ω_j to the intersection of the domains of their definitions agree, since

$$\frac{\partial F}{\partial Z_0} dZ_0 + \ldots + \frac{\partial F}{\partial Z_3} dZ_3 = 0 \quad \text{on} \quad F^{-1}(0) \subset \mathbb{C}^4.$$

By the Lefschetz Theorem on Hyperplane Sections (p156),

$$H^0(X_a; \mathbb{C}) \approx H^0(\mathbb{P}^3; \mathbb{C}) \approx \mathbb{C}, \quad H^1(X_a; \mathbb{C}) \approx H^1(\mathbb{P}^3; \mathbb{C}) \approx 0 \quad \Longrightarrow \quad H^{1,0}(X_a), H^{0,1}(X_a) = 0. \tag{2}$$

By (1), (2), and Serre duality,

$$h^{0,0}(X_a) = h^{2,2}(X_a) = 1, \qquad h^{1,0}(X_a) = h^{0,1}(X_a) = h^{2,1}(X_a) = h^{1,2}(X_a) = 0,$$

$$h^{2,0}(X_a) = h^{0,2}(X_a) = \begin{cases} 0, & \text{if } a \le 3; \\ 1, & \text{if } a = 4. \end{cases}$$

Thus, it remains to find only $h^{1,1}(X_a)$. The euler characteristic of X_a is given by

$$\begin{aligned}\chi(X_a) \equiv \sum_{r=0}^{r=4}(-1)^r h^r(X_a) &= \big\langle e(TX_a), X_a\big\rangle = \big\langle c(T\mathbb{P}^3)/c(\mathcal{O}_{\mathbb{P}^3}(a)), X_a\big\rangle \\ &= \big\langle (1+x)^4(1+ax)^{-1}, X_a\big\rangle = \big\langle (a^2-4a+6)x^2, X_a\big\rangle \\ &= \big\langle (a^2-4a+6)x^2\cdot ax, \mathbb{P}^3\big\rangle = a(a^2-4a+6),\end{aligned}$$

where $x=c_1(H)$ is the first chern class of the hyperplane line bundle. The second, third, and fifth equalities above follow from the multiplicativity of the total chern class and Adjunction Formula I (p146), PS6 #5a, and Poincare duality, respectively. Combining this with the above, we obtain the following Hodge diamonds for X_a:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & a^3-4a^2+6a-2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

a=1,2,3 a=4

The above approach to determining $h^{2,0}(X_a)$ for $a=4$ extends to $a>4$. It shows that $\mathcal{K}_{X_a}=H^{a-4}|_{X_a}$. On the other hand, the short exact sequence of sheaves on $\mathbb{P}^3$

$$0 \longrightarrow \mathcal{O}\big(H^{a-4}\otimes[-X_a]\big) \longrightarrow \mathcal{O}\big(H^{a-4}\big) \longrightarrow \mathcal{O}_{X_a}\big(H^{a-4}\big) \longrightarrow 0,$$

gives rise to the long exact sequence

$$0 \longrightarrow H^0\big(\mathbb{P}^3;\mathcal{O}(-4)\big) \longrightarrow H^0\big(\mathbb{P}^3;\mathcal{O}(a-4)\big) \longrightarrow H^0\big(X_a;\mathcal{O}(a-4)\big) \longrightarrow H^1\big(\mathbb{P}^3;\mathcal{O}(-4)\big) \longrightarrow \ldots$$

By the dual version of Kodaira Vanishing Theorem (p155), the first and the last groups above vanish. Thus, for $a\geq 4$,

$$h^{2,0}(X_a) = \dim H^0(X_a;\mathcal{K}_{X_a}) = \dim H^0\big(X_a;\mathcal{O}(a-4)\big) = \dim H^0\big(\mathbb{P}^3;\mathcal{O}(a-4)\big) = \binom{a-1}{3}.$$

Along with the other computations above (which apply for all a), this determines the Hodge diamond for X_a for any $a\in\mathbb{Z}^+$.

(b) Proceeding as above, we find that

$$\begin{gathered} h^{0,0}(Y_a) = h^{3,3}(Y_a) = 1, \qquad h^{1,0}(Y_a) = h^{0,1}(Y_a) = h^{3,2}(Y_a) = h^{2,3}(Y_a) = 0, \\ h^{2,0}(Y_a) = h^{0,2}(Y_a) = h^{3,1}(Y_a) = h^{1,3}(Y_a) = h^{2,0}(\mathbb{P}^4) = 0, \qquad h^{1,1}(Y_a) = h^{2,2}(Y_a) = h^{1,1}(\mathbb{P}^4) = 1, \\ h^{3,0}(X_a) = h^{0,3}(X_a) = \begin{cases} 0, & \text{if } a\leq 4; \\ 1, & \text{if } a=5. \end{cases} \end{gathered}$$

In order to find $h^{2,1}(Y_a)=h^{1,2}(Y_a)$, we determine the euler characteristic of Y_a:

$$\begin{aligned}\chi(Y_a) \equiv \sum_{r=0}^{r=6}(-1)^r h^r(Y_a) = \big\langle e(TY_a), Y_a\big\rangle &= \big\langle c(T\mathbb{P}^4)/c(\mathcal{O}_{\mathbb{P}^5}(a)), Y_a\big\rangle \\ &= \big\langle (1+x)^5(1+ax)^{-1}, Y_a\big\rangle = \big\langle (10-10a+5a^2-a^3)x^3, Y_a\big\rangle \\ &= \big\langle (10-10a+5a^2-a^3)x^3\cdot ax, \mathbb{P}^4\big\rangle = a(10-10a+5a^2-a^3).\end{aligned}$$

Combining with the above, we obtain the following Hodge diamonds for Y_a:

$$\begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 0&&(a^4-5a^3)/2+5a^2-5a+2&&(a^4-5a^3)/2+5a^2-5a+2&&0 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}$$

a=1,2,3,4

$$\begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&1&&0& \\ 1&&101&&101&&1 \\ &0&&1&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}$$

a=5

Proceeding as at the end of part (a), we can also obtain

$$h^{3,0}(Y_a) = h^{0,3}(Y_a) = \binom{a-1}{4}$$

and thus the Hodge diamond for Y_a for any $a\in\mathbb{Z}^+$.

Problem 2

Let $u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ *be a holomorphic map of degree* d *(thus,* $u_*[\mathbb{P}^1]=d[\mathbb{P}^1]\in H_2(\mathbb{P}^n)$*). If* $d\le n$*, show that* $u(\mathbb{P}^1)$ *is contained in some linearly embedded* $\mathbb{P}^d$ *in* $\mathbb{P}^n$*.*
Note: this is a special case of the Castelnuovo bound. It implies for example that every degree 2 (rational) curve in $\mathbb{P}^3$ *is in fact contained in some hyperplane* $\mathbb{P}^2\subset\mathbb{P}^3$*. This makes it possible to use classical Schubert calculus (homology intersections on* $G(k,n)$*) to determine the number of such conics in* $\mathbb{P}^3$ *that pass through* a *points and* $8-2a$ *lines in general position.*

The assumptions and PS3, #2b imply that $u^*\mathcal{O}_{\mathbb{P}^n}(1)=\mathcal{O}_{\mathbb{P}^1}(d)$. If the image of u is not contained in any hyperplane of $\mathbb{P}^n$, then u corresponds to a subspace of $H^0(\mathbb{P}^1;u^*\mathcal{O}_{\mathbb{P}^n}(1))\approx\mathbb{C}^{d+1}$ by p177 and thus $n\le d$. This implies the claim.

Problem 3

Let Σ be a compact connected Riemann surface (complex one-dimensional manifold). Show that Σ can be holomorphically embedded into $\mathbb{P}^N$ for some N.

By the Kodaira Embedding Theorem (p191), it is sufficient to show that Σ admits a positive rational $(1,1)$-form. Any volume form on Σ scaled so that the volume of Σ is 1 is such a form.

Problem 4

Let M be a complex manifold of dimension at least 2 and $x\in M$. Show that the sheaf $\mathfrak{I}_x$ of $\mathcal{O}$-modules is not isomorphic to the sheaf of holomorphic sections of any line bundle $L\longrightarrow M$.
Note: Recall that for any open subset $U\subset M$,

$$\mathfrak{I}_x(U)=\big\{f\in\mathcal{O}(U):\ f(x)=0 \text{ if } x\in U\big\};$$

this is a module over the ring $\mathcal{O}(U)$.

For any line bundle L and any sufficiently small open subset $U\neq\emptyset$ of M, there exists $e_U\in\{\mathcal{O}(L)\}(U)$ such that

$$\{\mathcal{O}(L)\}(U)=\big\{f\cdot e_U:\ f\in\mathcal{O}(U)\big\}.$$

On the other hand, if U is a sufficiently small neighborhood of x and $e_U\in\mathfrak{I}_x(U)$, then

$$\mathfrak{I}_x(U)\neq\big\{f\cdot e_U:\ f\in\mathcal{O}(U)\big\}.$$

The reason is that the homomorphism,

$$\mathfrak{I}_x(U)\longrightarrow T_x^*M,\qquad s\longrightarrow d_xs,$$

is well-defined and surjective. Since $e_U(x)=0$, $d(f\cdot e_U)=f(x)\cdot(d_xe_U)$; thus, the image of the restriction of this homomorphism to $\mathcal{O}(U)e_U$ is a linear subspace of T_x^*M of dimension at most one.

Problem 5

Let Γ be a complete lattice in $\mathbb{C}^2$ (i.e. the $\mathbb{Z}$-span of 4 $\mathbb{R}$-linearly independent vectors $v_1,\ldots,v_4\in\mathbb{C}^2$). Thus, $M\equiv\mathbb{C}^2/\Gamma$ is diffeomorphic to $(S^1)^4$.
(a) Show that the Kahler structure (complex structure and symplectic form) on $\mathbb{C}^4$ induce a Kahler structure on M. Describe a basis for $H_2(M;\mathbb{Z})$.
(b) Find a lattice Γ so that $H^{1,1}(M;\mathbb{Z})=\{0\}$ and thus M is not projective (cannot be embedded into $\mathbb{P}^N$ for any N).

Let $z=(z_1,z_2)$, with $z_1=x_1+\mathfrak{i}y_1$ and $z_2=x_2+\mathfrak{i}y_2$, be the standard coordinates on $\mathbb{C}^2$.
(1) The action of the group Γ on $\mathbb{C}^2$ (by addition) is properly discontinuous and thus $\mathbb{C}^2\longrightarrow M$ is

a covering projection. Since this action is holomorphic (and thus preserves the standard complex structure on $\mathbb{C}^2$) and symplectic (i.e. preserves the standard symplectic form on $\mathbb{C}^2$, $dx_1\wedge dy_1+dx_2\wedge dy_2$), the standard complex structure and symplectic form on $\mathbb{C}^2$ descend to M. Since the natural map

$$\big(\mathbb{R}v_1/\mathbb{Z}v_1\big)\times\big(\mathbb{R}v_2/\mathbb{Z}v_2\big)\times\big(\mathbb{R}v_3/\mathbb{Z}v_3\big)\times\big(\mathbb{R}v_4/\mathbb{Z}v_4\big)\longrightarrow M=\mathbb{C}^2/\Gamma$$

is a diffeomorphism, by the Kunneth formula a basis for $H_2(M;\mathbb{Z})$ is formed by the six 2-tori

$$T_{ij}\equiv\big(\mathbb{R}v_i/\mathbb{Z}v_i\big)\times\big(\mathbb{R}v_j/\mathbb{Z}v_j\big)\subset M,\qquad 1\le i<j\le 4.$$

(2) By the Kunneth formula, the rank of $H^2(M;\mathbb{C})$ is 6. Thus, the set of two-forms

$$dz_1\wedge dz_2,\quad dz_1\wedge d\bar{z}_1,\ dz_1\wedge d\bar{z}_2,\ dz_2\wedge d\bar{z}_1,\ dz_2\wedge d\bar{z}_2,\quad d\bar{z}_1\wedge d\bar{z}_2,$$

is a basis for $H^2(M;\mathbb{C})$. These closed 2-forms, originally defined on $\mathbb{C}^2$, descend to M, since they are preserved by the Γ-action. They are linearly independent in $H^2(M;\mathbb{C})$, since the pairing

$$H^2(M;\mathbb{C})\times H^2(M;\mathbb{C})\longrightarrow\mathbb{C},\qquad \int_M\alpha\wedge\beta\,,$$

does not vanish on non-trivial linear combinations of these forms. Taking into consideration the types of the forms, it follows that the middle four forms above form a basis for $H^{1,1}(M)$, and so do

$$dx_1\wedge dy_1,\quad dx_2\wedge dy_2,\quad dx_1\wedge dx_2+dy_1\wedge dy_2,\quad dx_1\wedge dy_2-dy_1\wedge dx_2.$$

Thus, by part (a), $H^{1,1}(M;\mathbb{Z})=0$ if and only if no fixed non-trivial linear combination of the last four forms integrates to an integer on all 6 of the tori T_{ii}. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ in the usual way, let

$$\big(v_1\ v_2\ v_3\ v_4\big)=\begin{pmatrix}1&0&0&0\\0&1&\sqrt{3}&0\\0&\sqrt{2}&1&0\\0&0&0&\sqrt{5}\end{pmatrix}.$$

Then, the integrals of the elements of the last basis for $H^{1,1}(M)$ over the 6 tori are given by

	T_{12}	T_{13}	T_{14}	T_{23}	T_{24}	T_{34}
$dx_1\wedge dy_1$	1	$\sqrt{3}$	0	0	0	0
$dx_2\wedge dy_2$	0	0	0	0	$\sqrt{10}$	$\sqrt{5}$
$dx_1\wedge dx_2+dy_1\wedge dy_2$	$\sqrt{2}$	1	0	0	$\sqrt{5}$	$\sqrt{15}$
$dx_1\wedge dy_2-dy_1\wedge dx_2$	0	0	$\sqrt{5}$	$\sqrt{6}-1$	0	0

If the two-form

$$\omega\equiv a\,dx_1\wedge dy_1+b\,dx_2\wedge dy_2+c(dx_1\wedge dx_2+dy_1\wedge dy_2)+f(dx_1\wedge dy_2-dy_1\wedge dx_2)$$

is integral on the 6 toris, then $f=0$ by the last line in the table. Furthermore, for some $\alpha,\beta,\gamma,\delta\in\mathbb{Z}$,

$$\begin{cases}a+\sqrt{2}c=\alpha\\ \sqrt{3}a+c=\beta\\ \sqrt{10}b+\sqrt{5}c=\gamma\\ \sqrt{5}b+\sqrt{15}c=\delta\end{cases}\quad\Longrightarrow\quad\begin{cases}(\sqrt{6}-1)c=\sqrt{3}\alpha-\beta\\ \sqrt{5}(\sqrt{6}-1)c=\sqrt{2}\delta-\gamma\end{cases}\quad\Longrightarrow\quad\sqrt{5}\big(\sqrt{3}\alpha-\beta\big)=\sqrt{2}\delta-\gamma.$$

Since $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, the last equation implies that $\alpha, \beta, \gamma, \delta = 0$ and thus $a, b, c = 0$. We conclude that $H^{1,1}(M) = 0$ and M can't be embedded into $\mathbb{P}^N$ for any N by Kodaira Embedding Theorem (p191).

Problem 6

(a) Let $C \subset \mathbb{P}^3$ be a complete intersection of bi-degree (a, b) (so $C = s^{-1}(0)$, where s is a holomorphic section of the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b) \longrightarrow \mathbb{P}^3$ which is transverse to the zero set). Determine the degree of C in $\mathbb{P}^3$ and the genus of C.
(b) If $C \subset \mathbb{P}^3$ is a smooth rational curve of degree 3 (thus, $C \approx \mathbb{P}^1$ and $[C] = 3[\mathbb{P}^1] \in H_2(\mathbb{P}^3)$) and C is not contained in any hyperplane $\mathbb{P}^2$ of $\mathbb{P}^3$, then C is not a complete intersection in $\mathbb{P}^3$. Show that such a curve C actually exists.

(a) The homology class of C is Poincare dual to

$$e\big(\mathcal{O}(a) \oplus \mathcal{O}(b)\big) = ab\, x^2 \,,$$

if $x \in H^2(\mathbb{P}^3)$ is the first chern of the hyperplane line bundle. Thus, the degree of C in $\mathbb{P}^3$ is ab. The euler characteristic of C is given by

$$\begin{aligned}\chi(C) &= \big\langle e(TC), C \big\rangle = \big\langle c(T\mathbb{P}^3)/c(\mathcal{O}(a) \oplus \mathcal{O}(b)), C \big\rangle = \big\langle (1+x)^4/((1+ax)(1+bx)), C \big\rangle \\ &= \big\langle (4-a-b)x, C \big\rangle = \big\langle (4-a-b)x \cdot abx^2, \mathbb{P}^3 \big\rangle = (4-a-b) \cdot ab;\end{aligned}$$

see the analogous computation in Problem 1a for comments. Thus, the genus of C is

$$g(C) = \frac{1}{2}\big(2 - \chi(C)\big) = \frac{1}{2}ab(a+b-4) \cdot ab + 1.$$

(b) The first statement is immediate from (a), since there exist no $a, b \in \mathbb{Z}^+$ such that

$$ab = 3, \qquad \frac{1}{2}ab(a+b-4) \cdot ab + 1 = 0.$$

For the second statement, let

$$\iota \colon C \longrightarrow \mathbb{P}\big(H^0(C; \mathcal{O}(3))^*\big) \approx \mathbb{P}^3, \qquad x \longrightarrow \big\{ s \in H^0(C; \mathcal{O}(3)) \colon\, s(x) = 0 \big\}.$$

This map is a well-defined injective immersion, since

$$\begin{aligned} H^1(\mathbb{P}^1; \mathcal{O}(3) \otimes [-x]) &= H^1(\mathbb{P}^1; \mathcal{O}(2)) \approx H^0(\mathbb{P}^1; \mathcal{O}(-2) \otimes \mathcal{K}_{\mathbb{P}^1})^* \approx H^0(\mathbb{P}^1; \mathcal{O}(-4))^* = 0, \\ H^1(\mathbb{P}^1; \mathcal{O}(3) \otimes [-x-y]) &= H^1(\mathbb{P}^1; \mathcal{O}(1)) \approx H^0(\mathbb{P}^1; \mathcal{O}(-1) \otimes \mathcal{K}_{\mathbb{P}^1})^* \approx H^0(\mathbb{P}^1; \mathcal{O}(-3))^* = 0 \end{aligned}$$

for all $x, y \in \mathbb{P}^1$; see p181.