

MAT 545: Complex Geometry Fall 2008

Problem Set 7

Due on Monday, 12/15, at 2:20pm in Math P-131
(or by 2pm on 12/15 in Math 3-111)

Please write up **clear** and **concise** solutions to problems worth 20 pts.

Problem 1 (10 pts)

(a) Let $X_a \subset \mathbb{P}^3$ be a smooth hypersurface of degree $a \geq 1$. Show that

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^0(X_a; \mathcal{K}_{X_a}) = \begin{cases} 0, & \text{if } a \leq 3; \\ 1, & \text{if } a = 4. \end{cases}$$

Determine the Hodge diamonds for X_a with $a \leq 4$.

(b) Let $Y_a \subset \mathbb{P}^4$ be a smooth hypersurface of degree $a \geq 1$. Determine the Hodge diamonds for Y_a with $a \leq 5$.

Note: the quartic surface $X_4 \subset \mathbb{P}^3$ is a K3 surface; the quintic $Y_5 \subset \mathbb{P}^4$ is a Calabi-Yau 3-fold, popular in string theory.

Problem 2 (5 pts)

Let $u: \mathbb{P}^1 \rightarrow \mathbb{P}^n$ be a holomorphic map of degree d (thus, $u_*[\mathbb{P}^1] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^n)$). If $d \leq n$, show that $u(\mathbb{P}^1)$ is contained in some linearly embedded \mathbb{P}^d in \mathbb{P}^n .

Note: this is a special case of the Castelnuovo bound. It implies for example that every degree 2 (rational) curve in \mathbb{P}^3 is in fact contained in some hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$. This makes it possible to use classical Schubert calculus (homology intersections on $G(k, n)$) to determine the number of such conics in \mathbb{P}^3 that pass through a points and $8 - 2a$ lines in general position.

Problem 3 (5 pts)

Let Σ be a compact connected Riemann surface (complex one-dimensional manifold). Show that Σ can be holomorphically embedded into \mathbb{P}^N for some N .

Problem 4 (5 pts)

Let M be a complex manifold of dimension at least 2 and $x \in M$. Show that the sheaf \mathfrak{J}_x of \mathcal{O} -modules is not isomorphic to the sheaf of holomorphic sections of any line bundle $L \rightarrow M$.

Note: Recall that for any open subset $U \subset M$,

$$\mathfrak{J}_x(U) = \{f \in \mathcal{O}(U) : f(x) = 0 \text{ if } x \in U\};$$

this is a module over the ring $\mathcal{O}(U)$.

Problem 5 (10 pts)

Let Γ be a complete lattice in \mathbb{C}^2 (i.e. the \mathbb{Z} -span of 4 \mathbb{R} -linearly independent vectors $v_1, \dots, v_4 \in \mathbb{C}^2$). Thus, $M \equiv \mathbb{C}^2/\Gamma$ is diffeomorphic to $(S^1)^4$.

(a) Show that the Kahler structure (complex structure and symplectic form) on \mathbb{C}^4 induce a Kahler structure on M . Describe a basis for $H_2(M; \mathbb{Z})$.

(b) Find a lattice Γ so that $H^{1,1}(M; \mathbb{Z}) = \{0\}$ and thus M is not projective (cannot be embedded into \mathbb{P}^N for any N).

Problem 6 (10 pts)

(a) Let $C \subset \mathbb{P}^3$ be a complete intersection of bi-degree (a, b) (so $C = s^{-1}(0)$, where s is a holomorphic section of the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b) \rightarrow \mathbb{P}^3$ which is transverse to the zero set). Determine the degree of C in \mathbb{P}^3 and the genus of C .

(b) If $C \subset \mathbb{P}^3$ is a smooth rational curve of degree 3 (thus, $C \approx \mathbb{P}^1$ and $[C] = 3[\mathbb{P}^1] \in H_2(\mathbb{P}^3)$) and C is not contained in any hyperplane \mathbb{P}^2 of \mathbb{P}^3 , then C is not a complete intersection in \mathbb{P}^3 . Show that such a curve C actually exists.