

MAT 545: Complex Geometry Fall 2008

Problem Set 6

Due on Tuesday, 12/2, at 2:20pm in Math P-131

(or by 2pm on 12/2 in Math 3-111)

Please write up **clear** and concise solutions to problems worth 20 pts, including exactly one of the last two problems.

Problem 1 (10 pts)

(a) For each $z \in \mathbb{C}^n$, let \mathcal{O}_z be the ring of germs at z of holomorphic functions on \mathbb{C}^n . If $f, g: \mathbb{C}^n \rightarrow \mathbb{C}$ are holomorphic functions and $p \in \mathbb{C}^n$ are such that $f(p) = 0$, let

$$\text{ord}_{f^{-1}(0), p} g = \max \{ a \in \mathbb{Z} : [g]/[f^a] \in \mathcal{O}_p \}.$$

Show that for any $p \in f^{-1}(0)$ such that $[f] \in \mathcal{O}_p$ is irreducible, there exists a neighborhood $U_p(f, g)$ of p in \mathbb{C}^n with the property such that

$$\text{ord}_{f^{-1}(0), z} g = \text{ord}_{f^{-1}(0), p} g \quad \forall z \in f^{-1}(0) \cap U_p(f, g).$$

(b) Let M be a complex manifold and $V \subset M$ an irreducible analytic hypersurface; thus, $V^* \subset M$ is connected. Suppose $s \equiv \{s_\alpha^+, s_\alpha^-\}_{\alpha \in \mathcal{A}}$ is a meromorphic section of a holomorphic line bundle $L \rightarrow M$. Show that the number

$$\text{ord}_{V, p} s \equiv \text{ord}_{V, p} s_\alpha^+ - \text{ord}_{V, p} s_\alpha^-$$

is independent of the choice of $\alpha \in \mathcal{A}$ and $p \in V^* \cap U_\alpha$.

Problem 2 (5 pts)

Let Σ be a compact connected Riemann surface (complex one-dimensional manifold) and $p, q \in \Sigma$ any two distinct points. Show that

- (a) if $\Sigma = \mathbb{P}^1$, then $[p] = [q]$.
- (b) if $[p] = [q]$, then $\Sigma = \mathbb{P}^1$ (up to bi-holomorphism).

Problem 3 (10 pts)

Let Σ be a compact connected Riemann surface and $V \rightarrow \Sigma$ a holomorphic line bundle.

- (a) Give a necessary and sufficient condition on V so that there exists a holomorphic line bundle $L \rightarrow \Sigma$ such that $L^{\otimes 2} = V$.
- (b) If this condition holds, how many “square roots” does V have?
- (c) If M is a complex surface ($\dim_{\mathbb{C}} M = 2$) and $\Sigma \subset M$ is a smooth canonical divisor with normal bundle N , show that $N^{\otimes 2} = \mathcal{K}_\Sigma$.

Note: a pair (Σ, L) such that $L^{\otimes 2} = \mathcal{K}_\Sigma$ is called a *spin curve*.

Problem 4 (5 pts)

Let M be a complex manifold. Show that

- (a) every C^∞ complex line bundle $L \rightarrow M$ admits at most one holomorphic structure if and only if $H_{\bar{\partial}}^{0,1}(M) = 0$.
 (b) every C^∞ complex line bundle $L \rightarrow M$ admits at least one holomorphic structure if and only if $H_{\bar{\partial}}^{0,2}(M) = 0$.

Problem 5 (10 pts)

Show that

- (a) there exists a short exact sequence of sheaves on \mathbb{P}^n :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow (n+1)\mathcal{O}_{\mathbb{P}^n}(1) \rightarrow \mathcal{O}(T\mathbb{P}^n) \rightarrow 0;$$

- (b) $H_{\bar{\partial}}^q(\mathbb{P}^n; \mathcal{O}(T\mathbb{P}^n)) = 0$ for all $q > 0$;
 (c) $H_{\bar{\partial}}^q(\Sigma; \mathcal{O}(u^*T\mathbb{P}^n)) = 0$ for all $q > 0$ for every compact connected Riemann surface Σ of genus g and holomorphic map $u: \Sigma \rightarrow \mathbb{P}^n$ of degree $d \geq 2g - 1$;
 (d) the homomorphism

$$H_{\bar{\partial}}^0(\Sigma; \mathcal{O}(u^*T\mathbb{P}^n)) \rightarrow \bigoplus_{i=1}^{i=k} a_i T_{u(z_i)} \mathbb{P}^n, \quad s \rightarrow (\nabla_{e_i}^{j-1} s)_{1 \leq j \leq a_i, 1 \leq i \leq k},$$

is surjective for every (Σ, u) as in (c), every choice of distinct points $z_1, \dots, z_k \in \Sigma$, $e_i \in T_{z_i} \Sigma$ with $e_i \neq 0$, and $a_1, \dots, a_k \in \mathbb{Z}^+$ with $d \geq 2g - 1 + \sum_{i=1}^{i=k} a_i$.

Note: $\nabla_{e_i}^j s$ denotes the j -th vertical derivative of the section s with respect to a connection ∇ evaluated at e_j ; thus,

$$\nabla_{e_j}^0(s) = s(z_j), \quad \nabla_{e_j}^1(s) = \{\nabla s\}|_{z_j}(e_j), \quad \nabla_{e_j}^2(s) = \{\nabla(\nabla s)\}|_{z_j}(e_j, e_j).$$

Problem 6 (10 pts)

Let $\gamma \rightarrow \mathbb{P}^1$ be the tautological line bundle, $\mathcal{O}_{\mathbb{P}^1}(-1)$, and $E \rightarrow \mathbb{P}^1$ any holomorphic vector bundle. Show that

- (a) $H_{\bar{\partial}}^1(\mathbb{P}^1; \gamma) = 0$;
 (b) the holomorphic vector bundle $E \otimes \gamma^{*a} \rightarrow \mathbb{P}^1$ admits a non-zero holomorphic section for some $a \in \mathbb{Z}$;
 (c) if $a_E = \min\{a \in \mathbb{Z}: H_{\bar{\partial}}^0(\mathbb{P}^1; E \otimes \gamma^{*a}) \neq 0\}$, then $E \otimes \gamma^{*a}$ admits a nowhere zero holomorphic section and E contains a holomorphic sub-bundle isomorphic to γ^{a_E} ;
 (d) if $F \equiv E/\gamma^{a_E} \approx \gamma^{a_1} \oplus \dots \oplus \gamma^{a_k}$, then $a_E \leq a_i$ for all $i = 1, \dots, k$;
 (e) the exact sequences of holomorphic vector bundles

$$0 \rightarrow F^* \otimes \gamma^{a_E} \rightarrow E^* \otimes \gamma^{a_E} \rightarrow \tau_1 \rightarrow 0, \quad 0 \rightarrow \gamma^{a_E} \rightarrow E \rightarrow F \rightarrow 0$$

split;

- (f) E is isomorphic to a unique vector bundle $\bigoplus_{i=0}^{i=k} \gamma^{*b_i}$ with $b_0 \geq b_1 \geq \dots \geq b_k$.

Note: the last statement is *Grothendieck's theorem*.