

MAT 545: Complex Geometry Fall 2008

Problem Set 5 Solutions

Problem 1 (10 pts)

Suppose (M, J) is an almost complex manifold, g is a J -compatible Riemannian metric on M , and ∇ is the Levi-Civita connection of g (thus, J is a complex structure in the fibers of the vector bundle $TM \rightarrow M$ which preserves g ; ∇ is g -compatible and $[X, Y] = \nabla_X Y - \nabla_Y X$ for any two vector fields X, Y on M). Show that $\nabla J = 0$ if and only if (M, J, g) is Kahler (you can use either of the equivalent conditions in PS2, #1 as the integrability criterion for J).

The nondegenerate 2-form determined by (g, J) is given by

$$\omega(X, Y) = g(JX, Y).$$

Given a fixed point $p \in M$, let X, Y, Z be vector fields on M such that

$$\begin{aligned} & \nabla X|_p = 0, \quad \nabla Y|_p = 0, \quad \nabla Z|_p = 0; \\ \implies & \nabla(JX)|_p = (\nabla J)_p X_p, \quad \nabla(JY)|_p = (\nabla J)_p Y_p, \quad \nabla(JZ)|_p = (\nabla J)_p Z_p, \\ & [X, Y]_p = 0, \quad [X, Z]_p = 0, \quad [Y, Z]_p = 0. \end{aligned}$$

(i) Suppose $\nabla J = 0$. We first show that the Nijenhuis tensor of J ,

$$N_J(X, Y) \equiv \frac{1}{2} \left([X, Y] + J[X, JY] + J[JX, Y] - [JX, JY] \right)$$

vanishes. With X and Y as above,

$$N_J(X, Y)_p = (J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X)_p = 0,$$

since $\nabla J = 0$; thus, (M, J) is a complex manifold. We next show that $d\omega = 0$. With X, Y, Z as above

$$\begin{aligned} & \{d\omega\}_p(X, Y, Z) \\ &= (X\omega(Y, Z) + Y\omega(Z, X) + Z\omega(X, Y) - \omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y))_p \\ &= (Xg(JY, Z) + Yg(JZ, X) + Zg(JX, Y))_p \\ &= (g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y))_p = 0, \end{aligned}$$

since $\nabla J = 0$. Thus, (M, ω) is a symplectic manifold, and (M, g, J) is Kahler.

(ii) Suppose (M, g, J) is Kahler; we show that $\nabla J = 0$.

Solution (a): By Lemma in G&H, p107, there is a complex coordinate system $z = (z_1, \dots, z_m)$, with $z_k = x_k + iy_k$, centered around $p \in M$ such that

$$g = \sum_k (dx_k \otimes dx_k + dy_k \otimes dy_k) + O(z^2).$$

Since ∇ is determined by the values of g and of its first derivatives, $(\nabla T)_p$ agrees with $(DT)_p$ for every tensor T , where D is the connection for the flat metric. In particular, $(\nabla J)_p = 0$.

Solution (b): From the proof in part (i),

$$\begin{aligned} N_J(X, Y)_p &= (J(\nabla_X J)Y - J(\nabla_Y J)X - (\nabla_{JX} J)Y + (\nabla_{JY} J)X)_p; \\ \{d\omega\}_p(X, Y, Z) &= (g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y))_p. \end{aligned}$$

Since $J^2 = -\text{Id}$, $g(J \cdot, \cdot) = -g(\cdot, J \cdot)$, and ∇ is g -compatible,

$$J(\nabla J) = -(\nabla J)J, \quad g((\nabla_X J)Y, Z) = -g(Y, (\nabla_X J)Z).$$

This all gives

$$2g((\nabla_Z J)X, JY) = g(N_J(X, Y), Z) + \omega(JX, Y, Z) + \omega(X, JY, Z).$$

Since M is Kahler, $N_J \omega = 0$ and thus ∇J .

Solution (c): Let D be the metric connection in the holomorphic tangent $(TM, J) \rightarrow M$ with the hermitian inner-product h by g ; in particular, $DJ = 0$. It is sufficient to show that $D = \nabla$, i.e. D is g -compatible and torsion-free. The former is immediate, since D is h -compatible. Since M is Kahler, the ‘‘torsion’’ τ of D defined by (*) in G&H, p76, is zero (G&H, p107). It is thus sufficient to check that

$$\tau(X, Y) = D_X Y - D_Y X - [X, Y]$$

for any two vector fields X, Y on M . Let v_1, \dots, v_m be a local h -orthonormal \mathbb{C} -frame for (TM, J) and $\varphi_1, \dots, \varphi_m$ the dual frame for $(T^*M, J) = (T^*M^{1,0}, i)$. Then, for each i

$$d\varphi_i = \sum_j \psi_{ij} \wedge \varphi_j + \tau_i$$

for unique 1-forms ψ_{ij} and $(2, 0)$ -form τ_i such that $\psi_{ji} = -\bar{\psi}_{ij}$. Furthermore,

$$Dv_i = -\sum_k \psi_{ki} \otimes v_k$$

for all i ; see p77. Thus,

$$\begin{aligned} \varphi_k(Dv_i v_j - Dv_j v_i - [v_i, v_j]) &= -\psi_{kj}(v_i) + \psi_{ki}(v_j) + \{d\varphi_k\}(v_i, v_j) \\ &= -\psi_{kj}(v_i) + \psi_{ki}(v_j) + (\psi_{kj}(v_i) - \psi_{ki}(v_j) + \tau_k(v_i, v_j)) = \tau_k(v_i, v_j). \end{aligned}$$

Thus, τ_k is indeed the v_k -component of the usual torsion and its vanishing implies that D is the Levi-Civita connection for (M, g, J) .

Problem 2 (5 pts)

Let $M_n = (\mathbb{C}^n - 0) / \sim$, where $z \sim 2^k z$ for every $k \in \mathbb{Z}$.

(a) Show that M_n is a complex manifold (with the complex structure inherited from \mathbb{C}^n). What simple smooth manifold is M_n diffeomorphic to?

(b) Give at least two reasons why M_n does not admit a Kahler metric for $n \geq 2$.

(a) Let S^{2n-1} denote the unit sphere in \mathbb{C}^n . The diffeomorphism

$$f: \mathbb{R} \times S^{2n-1} \longrightarrow \mathbb{C}^n - 0, \quad (s, w) \longrightarrow 2^s w,$$

is \mathbb{Z} -equivariant with respect to the above action on $\mathbb{C}^n - 0$ and the action

$$k \cdot (s, z) = (s+k, z)$$

on $\mathbb{R} \times S^{2n-1}$. Thus, M_n is diffeomorphic to $(\mathbb{R}/\mathbb{Z}) \times S^{2n-1}$. Since the standard action of \mathbb{Z} on \mathbb{R} is properly discontinuous, so is the action of \mathbb{Z} on $\mathbb{C}^n - 0$. Thus, the quotient map $q: \mathbb{C}^n - 0 \longrightarrow M_n$ is a covering projection. Since \mathbb{Z} acts by holomorphic transformations on $\mathbb{C}^n - 0$, i.e. the complex structure on $\mathbb{C}^n - 0$ is preserved by the \mathbb{Z} -action, the complex structure on $\mathbb{C}^n - 0$ descends to a complex structure on the quotient M_n .

(b) If $n \geq 2$, by the Kunneth formula

$$H_{deR}^1(M_n; \mathbb{R}) \approx \mathbb{R}, \quad H_{deR}^2(M_n; \mathbb{R}) = \{0\}, \quad H_2(M_n; \mathbb{Z}) = \{0\}.$$

However, the odd betti numbers of a compact Kahler manifolds are even, while the second betti number is non-zero; this contradicts the first statements above. The third statement implies that the complex one-dimensional tori $q(L-0) \subset M_n$, where $L \subset M_n$ is a one-dimensional linear subspace, are trivial in the homology of M ; this is yet another reason that M_n is not Kahler.

Problem 3 (10 pts)

Let $M = \mathbb{R}^4 / \sim$, where

$$(s, t, x, y) \sim (s+k, t+l, x+m, y+lx+n) \quad \forall (s, t, x, y) \in \mathbb{R}^4, (k, l, m, n) \in \mathbb{Z}^4.$$

Show that

(a) this is an equivalence relation;

(b) M is a compact symplectic manifold (with the symplectic form, i.e. closed non-degenerate 2-form, inherited from the standard symplectic form on \mathbb{R}^4 , i.e. $ds \wedge dt + dx \wedge dy$).

(c) M does not admit an integrable complex structure compatible with this symplectic form.

Note 1: this is the first known example (due to W. Thurston'76) of a symplectic manifold that admits no Kahler structure.

Note 2: in contrast, every symplectic manifold (M, ω) admits an almost complex structure compatible with ω ; the space of ω -compatible almost complex structures is contractible.

For each $(k, l, m, n) \in \mathbb{Z}^4$, define

$$\varphi_{(k,l,m,n)}: \mathbb{R}^4 \longrightarrow \mathbb{R}^4 \quad \text{by} \quad \varphi_{(k,l,m,n)}(s, t, x, y) = (s+k, t+l, x+m, y+lx+n).$$

Then, $(s, t, x, y) \sim (s', t', x', y')$ if and only if $(s', t', x', y') = \varphi_{(k,l,m,n)}(s, t, x, y)$ for some $(k, l, m, n) \in \mathbb{Z}^4$.

(a) Reflexivity: $(s, t, x, y) = \varphi_{(0,0,0,0)}(s, t, x, y)$.

Symmetry: if $(s', t', x', y') = \varphi_{(k,l,m,n)}(s, t, x, y)$, then

$$(s, t, x, y) = \varphi_{(-k,-l,-m,-n+lm)}(s', t', x', y');$$

Transitivity: if $(s_i, t_i, x_i, y_i) = \varphi_{(k_i,l_i,m_i,n_i)}(s_{i-1}, t_{i-1}, x_{i-1}, y_{i-1})$ for $i=0, 1$, then

$$(s_2, t_2, x_2, y_2) = \varphi_{(k_1+k_2, l_1+l_2, m_1+m_2, n_1+n_2+m_1l_2)}(s_0, t_0, x_0, y_0).$$

It follows that M is the quotient of \mathbb{R}^4 by the group $G = \mathbb{Z}^4$ with the composition law

$$(k_1, l_1, m_1, n_1) \cdot (k_2, l_2, m_2, n_2) = (k_1 + k_2, l_1 + l_2, m_1 + m_2, n_1 + n_2 + m_1l_2),$$

which is acting on the right.

(b) For every $(s, t, x, y) \in \mathbb{R}^4$, there exists $(k, l, m, n) \in \mathbb{Z}^4$ such that

$$\varphi_{(k,l,m,n)}(s, t, x, y) \in [0, 1]^4.$$

Since $[0, 1]^4$ is compact, it follows that so is M . The action group G on \mathbb{R}^4 is properly discontinuous: if p is any point and $B_p(1/2)$ is the open ball of radius $1/2$ around p , in the square or round metric, then

$$B_p(1/2) \cap \varphi_{(k,l,m,n)}(B_p(1/2)) = \emptyset \quad \forall (k, l, m, n) \in \mathbb{Z}^4 - 0.$$

Thus, the quotient map $q: \mathbb{R}^4 \longrightarrow M$ is a covering projection. Since G acts on \mathbb{R}^4 by diffeomorphisms, the smooth structure on \mathbb{R}^4 , descends to M . Since

$$\varphi_{k,l,m,n}^*(ds \wedge dt + dx \wedge dy) = ds \wedge dt + dx \wedge (dy + ldx) = ds \wedge dt + dx \wedge dy,$$

the G acts on \mathbb{R}^4 preserves the symplectic form, which thus descends to a symplectic on M .

(c) Since $\mathbb{R}^4 \longrightarrow M$ is a covering projection, $\pi_1(M) = G$ and $H_1(M; \mathbb{Z}) = G/[G, G]$. Since the projection on the first three coordinates $G \longrightarrow \mathbb{Z}^3$ is a group homomorphism, $[G, G] \subset 0^3 \times \mathbb{Z}$. Since G is not abelian, it follows that

$$H_1(M; \mathbb{R}) = H_1(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \approx \mathbb{R}^3;$$

(in fact $[G, G] = 0^3 \times \mathbb{Z}$, but this is not needed). Thus, the first betti number of M is odd and therefore M does not admit a Kahler metric.

Problem 4 (5 pts)

Let (X, J, g) be a Kahler manifold and ω its symplectic form. Show that ω is harmonic with respect to g .

Let L be the wedging with ω operator and Δ either of the three laplacians. By one of the main Hodge relations, $\Delta L = L\Delta$ (this is used in the proof of the Lefschetz theorem). Thus,

$$\Delta\omega = \Delta(L1) = L(\Delta 1) = L0 = 0.$$

Problem 5 (10 pts)

Let M be a compact complex manifold that admits a Kahler metric.

(a) Let α be a (p, q) -form on M such that $d\alpha=0$. Show that the following are equivalent

(i) $\alpha = d\beta$ for some $(p+q-1)$ -form β ;

(ii) $\alpha = \partial\beta$ for some $(p-1, q)$ -form β ;

(iii) $\alpha = \bar{\partial}\beta$ for some $(p, q-1)$ -form β ;

(iv) $\alpha = \partial\bar{\partial}\beta$ for some $(p-1, q-1)$ -form β .

(b) Let ω and ω' be symplectic forms compatible with the complex structure on M (thus ω and ω' arise from Kahler metrics on M). If $[\omega] = [\omega'] \in H_{deR}^2(M)$, show that $\omega' = \omega + i\partial\bar{\partial}f$ for some $f \in C^\infty(M; \mathbb{R})$.

(a) (iv) \implies (i),(ii),(iii), since $d = \partial + \bar{\partial}$, $\partial^2 = 0$, $\bar{\partial}^2 = 0$, and $\partial\bar{\partial} = -\bar{\partial}\partial$ on any complex manifold.

(i) \implies (ii),(iii): Since $\alpha = d\beta$, α is orthogonal to the harmonic forms (since M is Kahler, harmonic forms for d , ∂ , and $\bar{\partial}$ are the same). Thus, by the Hodge decomposition for ∂ and $\bar{\partial}$,

$$\alpha = \partial\gamma_- + \partial^*\gamma_+ = \bar{\partial}\gamma'_- + \bar{\partial}^*\gamma'_+.$$

Since $\partial\alpha=0$ and $\bar{\partial}\alpha=0$ (because α is of pure bi-degree), $\partial^*\gamma_+, \bar{\partial}^*\gamma'_+ = 0$.

(ii) \implies (iv): By Hodge decomposition, $\beta = \beta_0 + \bar{\partial}\beta_- + \bar{\partial}^*\beta_+$, where β_0 is harmonic. Thus,

$$\alpha = \partial\beta = \partial\bar{\partial}\beta_- + \partial\bar{\partial}^*\beta_+ \implies 0 = \bar{\partial}\beta = \bar{\partial}\partial\bar{\partial}\beta_- + \bar{\partial}\partial\bar{\partial}^*\beta_+.$$

Since $\bar{\partial}\partial = -\partial\bar{\partial}$ and $\bar{\partial}^2 = 0$ on all complex manifolds, while $\partial\bar{\partial}^* = -\bar{\partial}^*\partial$ on Kahler manifolds, it follows that

$$\bar{\partial}\bar{\partial}^*\partial\beta_+ = 0 \implies \bar{\partial}^*\partial\beta_+ = 0 \implies \partial\bar{\partial}^*\beta_+ = 0 \implies \alpha = \partial\bar{\partial}\beta_-,$$

as required.

(b) Let $\alpha = \omega - \omega'$. By our assumptions, $\alpha \in A^{1,1}(M)$, $d\alpha=0$, and $\alpha = d\beta$. Thus, $\alpha = i\partial\bar{\partial}g$ for some $g \in C^\infty(M; \mathbb{C})$ by part (a). Since α is a real $(1, 1)$ -form, it follows that

$$\alpha = \bar{\alpha} = (-i)\bar{\partial}\partial\bar{g} = i\bar{\partial}\partial\bar{g} \implies \alpha = i\bar{\partial}\partial(g + \bar{g})/2;$$

so we can take $f = \text{Re } g$.