

MAT 545: Complex Geometry

Fall 2008

Problem Set 4

Due on Tuesday, 11/4, at 2:20pm in Math P-131
(or by 2pm on 11/4 in Math 3-111)

Please write up concise solutions to problems worth 20 pts.

Problem 1 (5 pts)

Let (V, J) be a complex vector space of complex dimension m and g a Riemannian metric on V compatible with J . Show that

$$*: \Lambda^{p,q}V \longrightarrow \Lambda^{m-p,m-q}V \subset \Lambda_{\mathbb{C}}^{2m-p-q}(V \otimes_{\mathbb{R}} \mathbb{C}),$$

where $*$ is the Hodge operator on (V, g) with the orientation induced by J and extended \mathbb{C} -anti-linearly to $V \otimes_{\mathbb{R}} \mathbb{C}$.

Problem 2 (5 pts)

Let M and N be complex manifolds with hermitian metrics h_M and h_N . Show that

$$\Delta_{M \times N} = \Delta_M \otimes 1 + 1 \otimes \Delta_N,$$

where Δ_M , Δ_N , and $\Delta_{M \times N}$ are the Laplacians on M , N , and $M \times N$ with respect to the metrics h_M , h_N , and $h_M \otimes 1 + 1 \otimes h_N$.

Problem 3 (5 pts)

Let $E \rightarrow M$ be a holomorphic vector bundle with a hermitian inner-product over a compact complex manifold with a hermitian metric and Δ_E the corresponding Laplacian on E . Show that

- all eigenvalues of Δ_E are non-negative;
- eigenfunctions corresponding to distinct eigenvalues of Δ_E are orthogonal;
- eigenspaces of Δ_E are finite-dimensional;
- the set of eigenvalues of Δ_E has no limit point.

Problem 4 (10 pts)

With notation as in Problem 3, show that

- (a) Δ_E has a positive eigenvalue;
- (b) Δ_E has infinitely many positive eigenvalues;
- (c) the linear span of eigenfunctions of Δ_E is L^2 -dense in $\Gamma(M; \Lambda_{\mathbb{C}}^*(T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E)$;
- (d) the linear span of eigenfunctions of Δ_E is L^∞ -dense in $\Gamma(M; \Lambda_{\mathbb{C}}^*(T^*M \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} E)$.

This problems supplies the missing ingredient for the proof of Kunneth formula in the textbook: the existence of a complete set of eigenfunctions of a laplacian.

Hint for (a): Let G be the inverse of Δ_E on $\mathcal{H}^{*,*}(M, E)^\perp$. Show that $\lambda_1 \in \mathbb{R}^+$ given by

$$1/\lambda_1 = \sup_{\alpha \in \mathcal{H}^{*,*}(M, E)^\perp, \|\alpha\|=1} \|G\alpha\|$$

is the smallest positive eigenvalue of Δ_E .