

MAT 545: Complex Geometry

Fall 2008

Problem Set 3

Due on Tuesday, 10/21, at 2:20pm in Math P-131

(or by 2pm on 10/21 in Math 3-111)

Please write up concise solutions to 2 of the 3 problems below.

Problem 1 (10 pts)

Let $\gamma \rightarrow \mathbb{P}^n$ be the tautological line bundle. Show that

- (a) $\gamma^{\otimes a} \rightarrow \mathbb{P}^n$ admits no nonzero holomorphic section for any $a \in \mathbb{Z}^+$;
- (b) every homogeneous polynomial $P = P(X_0, \dots, X_n)$ on \mathbb{C}^{n+1} of degree a induces a holomorphic section s_P of $\gamma^{*\otimes a} \rightarrow \mathbb{P}^n$. Furthermore, every holomorphic section of $\gamma^{*\otimes a} \rightarrow \mathbb{P}^n$ is given by s_P for some homogeneous polynomial P on \mathbb{C}^{n+1} of degree a .

Problem 2 (10 pts)

Show that

- (a) every holomorphic line bundle over \mathbb{C}^n is trivial;
- (b) every holomorphic line bundle over \mathbb{P}^n is isomorphic to γ^a for some $a \in \mathbb{Z}$;
- (c) if P_0, \dots, P_n are homogeneous polynomials of degree a on \mathbb{C}^{m+1} with no common zeros (other than the origin), then the map

$$f_{P_0 \dots P_n}: \mathbb{P}^m \rightarrow \mathbb{P}^n, \quad [X_0, \dots, X_m] \rightarrow [P_0(X_0, \dots, X_m), \dots, P_n(X_0, \dots, X_m)],$$

is well-defined and holomorphic and the push-forward of $[\mathbb{P}^m]$ is a times the positive generator of $H_{2m}(\mathbb{P}^n; \mathbb{Z})$. Furthermore, every degree a holomorphic map $f: \mathbb{P}^m \rightarrow \mathbb{P}^n$ is given by $f = f_{P_0 \dots P_n}$ for some P_0, \dots, P_n as above.

Problem 3 (10 pts)

If (X, J_X) and (Y, J_Y) are almost complex manifolds, a smooth map $f: X \rightarrow Y$ is called holomorphic if

$$df \circ J_X = J_Y \circ df.$$

If (X, J_X) is an almost complex manifold and $(V, i) \rightarrow X$ is a smooth complex vector bundle, a $\bar{\partial}$ -operator in (V, i) is a \mathbb{C} -linear map

$$\bar{\partial}: \Gamma(X; V) \rightarrow \Gamma(X; T^*X^{0,1} \otimes_{\mathbb{C}} V)$$

such that

$$\bar{\partial}(f\xi) = (\bar{\partial}f) \otimes \xi + f\bar{\partial}\xi \quad \forall f \in C^1(M; \mathbb{C}), \xi \in \Gamma(M; V).$$

Show that

(a) any connection in V induces a $\bar{\partial}$ -operator in V and every $\bar{\partial}$ -operator in V arises from a connection in V ;

(b) if $\bar{\partial}$ is a $\bar{\partial}$ -operator on V , there exists an almost complex structure on J_V on V (the total space of the vector bundle) such that

(i) the bundle projection map $\pi: (V, J_V) \rightarrow (X, J_X)$ is holomorphic,

(ii) for all $v \in V$, the restriction of J_V to $\ker d_v\pi \approx V_{\pi(v)}$ is $i|_{V_x}$, and

(iii) if $\xi \in \Gamma(M; V)$, $\bar{\partial}\xi = 0$ if and only if $\xi: (X, J_X) \rightarrow (V, J_V)$ is holomorphic.

Furthermore, every almost complex structure on V satisfying (i)-(iii) arises from a $\bar{\partial}$ -operator on V in this way.