MAT 545: Complex Geometry
Fall 2008

Problem Set 3

Due on Tuesday, 10/21, at 2:20pm in Math P-131

(or by 2pm on 10/21 in Math 3-111)

Please write up concise solutions to 2 of the 3 problems below.

Problem 1 (10 pts)

Let $\gamma \longrightarrow \mathbb{P}^n$ be the tautological line bundle. Show that
(a) $\gamma^{\otimes a} \longrightarrow \mathbb{P}^n$ admits no nonzero holomorphic section for any $a \in \mathbb{Z}^+$;
(b) every homogeneous polynomial $P = P(X_0, \ldots, X_n)$ on $\mathbb{C}^{n+1}$ of degree a induces a holomorphic section s_P of $\gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$. Furthermore, every holomorphic section of $\gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$ is given by s_P for some homogeneous polynomial P on $\mathbb{C}^{n+1}$ of degree a.

Problem 2 (10 pts)

Show that
(a) every holomorphic line bundle over $\mathbb{C}^n$ is trivial;
(b) every holomorphic line bundle over $\mathbb{P}^n$ is isomorphic to γ^a for some $a \in \mathbb{Z}$;
(c) if $P_0, \ldots, P_n$ are homogeneous polynomials of degree a on $\mathbb{C}^{m+1}$ with no common zeros (other than the origin), then the map

$$f_{P_0 \ldots P_n} \colon \mathbb{P}^m \longrightarrow \mathbb{P}^n, \qquad [X_0, \ldots, X_m] \longrightarrow [P_0(X_0, \ldots, X_m), \ldots, P_n(X_0, \ldots, X_m)],$$

is well-defined and holomorphic and the push-forward of $[\mathbb{P}^m]$ is a^m times the positive generator of $H_{2m}(\mathbb{P}^n; \mathbb{Z})$. Furthermore, every degree a^m holomorphic map $f \colon \mathbb{P}^m \longrightarrow \mathbb{P}^n$ is given by $f = f_{P_0 \ldots P_n}$ for some $P_0, \ldots, P_n$ as above.

Problem 3 (10 pts)

If (X, J_X) and (Y, J_Y) are almost complex manifolds, a smooth map $f \colon X \longrightarrow Y$ is called holomorphic if

$$df \circ J_X = J_Y \circ df.$$

If (X, J_X) is an almost complex manifold and $(V, \mathfrak{i}) \longrightarrow X$ is a smooth complex vector bundle, a $\bar{\partial}$-operator in $(V, \mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar{\partial} \colon \Gamma(X; V) \longrightarrow \Gamma(X; T^*X^{0,1} \otimes_{\mathbb{C}} V) \qquad \text{s.t.}$$
$$\bar{\partial}(f\xi) = (\bar{\partial} f) \otimes \xi + f \bar{\partial} \xi \quad \forall\, f \in C^\infty(M; \mathbb{C}),\, \xi \in \Gamma(M; V).$$

Show that
(a) a connection in V induces a $\bar{\partial}$-operator in V and every $\bar{\partial}$-operator in V arises from a connection in V;
(b) if $\bar{\partial}$ is a $\bar{\partial}$-operator on V, there exists an almost complex structure on J_V on V (the total space of the vector bundle) such that

(i) the bundle projection map $\pi \colon (V, J_V) \longrightarrow (X, J_X)$ is holomorphic,

(ii) for all $v \in V$, the restriction of J_V to $\ker d_v \pi \approx V_{\pi(v)}$ is $\mathfrak{i}|_{V_x}$, and

(iii) if $\xi \in \Gamma(M; V)$, $\bar{\partial} \xi = 0$ if and only if $\xi \colon (X, J_X) \longrightarrow (V, J_V)$ is holomorphic.

Furthermore, every almost complex structure on V satisfying (i)-(iii) arises from a $\bar{\partial}$-operator on V in this way.