MAT 545: Complex Geometry

Problem Set 5

Written Solutions due by Tuesday, 11/12, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

Suppose (M,J) is an almost complex manifold, g is a J-compatible Riemannian metric on M, and ∇ is the Levi-Civita connection of g (thus, J is a complex structure in the fibers of the vector bundle $TM \longrightarrow M$ which preserves g; ∇ is g-compatible and $[X,Y] = \nabla_X Y - \nabla_Y X$ for any two vector fields X,Y on M). Show that $\nabla J = 0$ if and only if (M,J,g) is Kahler (you can use either of the equivalent conditions in PS2,#1 as the integrability criterion for J).

Problem 2 (15 pts)

Let (M,J) be a compact complex surface ($\dim_{\mathbb{C}} M = 2$). Show that
(a) if $f\colon M \longrightarrow \mathbb{C}$ is a smooth function s.t. $\overline{\partial}\partial f = 0$, then f is constant;
(b) there are natural injections $\overline{H^{1,0}(M)} \longrightarrow H^{0,1}(M)$ and $H^{1,0}(M) \oplus \overline{H^{1,0}(M)} \longrightarrow H^1(M;\mathbb{C})$;
(c) $b_1(M) \leqslant h^{1,0}(M) + h^{0,1}(M)$;
(d) there are natural injections $\overline{H^{2,0}(M)} \longrightarrow H^{0,2}(M)$ and $H^{2,0}(M) \oplus \overline{H^{2,0}(M)} \longrightarrow H^2(M;\mathbb{C})$;
(e) $\big(b_2^+(M) - 2h^{0,2}(M)\big) + \big(2h^{0,1}(M) - b_1(M)\big) = 1$;
(f) $b_1(M) = h^{1,0}(M) + h^{0,1}(M)$ and either $h^{1,0}(M) = h^{0,1}(M)$ and $b_2^+(M) = 2h^{0,2}(M) + 1$
or $h^{1,0}(M) = h^{0,1}(M) - 1$ and $b_2^+(M) = 2h^{0,2}(M)$;
(g) $h^{1,0}(M), h^{0,1}(M), h^{2,1}(M), h^{1,2}(M)$ (resp. $h^{2,0}(M), h^{0,2}(M)$) are topological invariants of the unoriented (resp. oriented) manifold M.
Hints: (a) show that f is harmonic in each variable in each coordinate chart;
(b) for $\alpha \in H^{1,0}(M)$, show that $\int_M (\mathrm{d}\alpha) \wedge (\mathrm{d}\overline{\alpha}) = 0$;
(c) let $\mathcal{Z}$ denote the sheaf of closed holomorphic 1-forms on M; use the short exact sequence

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \mathcal{O}_M \stackrel{\mathrm{d}}{\longrightarrow} \mathcal{Z} \longrightarrow 0$$

of sheaves on M;
(d) for $\beta \in H^{2,0}(M) - \{0\}$, show that $\int_M \beta \wedge \overline{\beta} \neq 0$;
(e) use Noether's and Hirzerbruch's formulas:

$$\chi(\mathcal{O}_M) = \frac{1}{12}\big(K_M^2 + \chi(M)\big) \qquad \text{and} \qquad b_2^+(M) - b_2^-(M) = \frac{1}{2}\big(K_M^2 - 2\chi(M)\big).$$

Note: From Serre Duality and Riemann-Roch for $T^*M \longrightarrow M$,

$$2h^{1,0}(M) - h^{1,1}(M) = \chi(T^*M) = \frac{1}{6}\big(K_M^2 - 5\chi(M)\big),$$

we find that $h^{1,1}(M)$ is also a topological invariant of the oriented manifold M and that

$$b_2(M) = h^{2,0}(M) + h^{1,1}(M) + h^{0,2}(M).$$

Problem 3 (10 pts)

Let $M_n = (\mathbb{C}^n - 0)/\sim$, where $z \sim 2^k z$ for every $k \in \mathbb{Z}$.
(a) Show that M_n is a complex manifold (with the complex structure inherited from $\mathbb{C}^n$). What simple smooth manifold is M diffeomorphic to?
(b) Give at least two reasons why M_n does not admit a Kahler metric for $n \geqslant 2$.
(c) Determine the groups $\check{H}^1(M_n; \mathcal{O}^*)$ of holomorphic line bundles and $H^{0,1}(M_n; \mathcal{O}^*)$.
(d) Determine $h^{p,q}(M_2)$ with $(p,q) \neq (1,1)$.

Problem 4 (10 pts)

Let $M = \mathbb{R}^4/\sim$, where

$$(s,t,x,y) \sim \big(s+k, t+l, x+m, y+lx+n\big) \qquad \forall\ (s,t,x,y) \in \mathbb{R}^4,\ (k,l,m,n) \in \mathbb{Z}^4 .$$

Show that
(a) this is an equivalence relation;
(b) M is a compact symplectic manifold (with the symplectic form, i.e. closed non-degenerate 2-form, inherited from the standard symplectic form on $\mathbb{R}^4$, i.e. $ds \wedge dt + dx \wedge dy$).
(c) M does not admit an integrable complex structure compatible with this symplectic form.

Note 1: M in (c) is the first known example (due to W. Thurston'76) of a symplectic manifold that admits no Kahler structure.
Note 2: in contrast, every symplectic manifold (M, ω) admits an *almost* complex structure compatible with ω; the space of ω-compatible almost complex structures is contractible.

Problem 5 (5 pts)

Let (X, J, g) be a Kahler manifold and ω its symplectic form. Show that ω is harmonic with respect to g.

Problem 6 (10 pts)

Let M be a compact complex manifold that admits a Kahler metric.
(a) Let α be a (p,q)-form on M such that $\mathrm{d}\alpha = 0$. Show that the following are equivalent:

(i) $\alpha = \mathrm{d}\beta$ for some $(p+q-1)$-form β;

(ii) $\alpha = \partial\beta$ for some $(p-1,q)$-form β;

(ii) $\alpha = \bar{\partial}\beta$ for some $(p,q-1)$-form β;

(ii) $\alpha = \partial\bar{\partial}\beta$ for some $(p-1,q-1)$-form β.

(b) Let ω and ω' be symplectic forms compatible with the complex structure on M (thus ω and ω' arise from Kahler metrics on M). If $[\omega] = [\omega'] \in H^2_{deR}(M)$, show that $\omega' = \omega + \mathrm{i}\partial\bar{\partial} f$ for some $f \in C^\infty(M; \mathbb{R})$.