# MAT 545: Complex Geometry

## Problem Set 3

**Written Solutions due by Tuesday, 10/08, 1pm**

*Please figure out all of the problems below and discuss them with others.*

**If you have not passed the orals yet**, *please write up concise solutions to 1 of the 3 problems below.*

### Problem 1 (10 pts)

Let $\gamma \longrightarrow \mathbb{P}^n$ be the tautological line bundle. Show that
(a) $\gamma^{\otimes a} \longrightarrow \mathbb{P}^n$ admits no nonzero holomorphic section for any $a \in \mathbb{Z}^+$;
(b) every homogeneous polynomial $P = P(X_0, \ldots, X_n)$ on $\mathbb{C}^{n+1}$ of degree $a$ induces a holomorphic section $s_P$ of $\gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$. Furthermore, every holomorphic section of $\gamma^{*\otimes a} \longrightarrow \mathbb{P}^n$ is given by $s_P$ for some homogeneous polynomial $P$ on $\mathbb{C}^{n+1}$ of degree $a$.

### Problem 2 (10 pts)

Show that
(a) every holomorphic line bundle over $\mathbb{C}^n$ is trivial;
(b) every holomorphic line bundle over $\mathbb{P}^n$ is isomorphic to $\gamma^a$ for some $a \in \mathbb{Z}$;
(c) if $P_0, \ldots, P_n$ are homogeneous polynomials of degree $a$ on $\mathbb{C}^{m+1}$ with no common zeros (other than the origin), then the map

$$f_{P_0 \ldots P_n} : \mathbb{P}^m \longrightarrow \mathbb{P}^n, \qquad [X_0, \ldots, X_m] \longrightarrow [P_0(X_0, \ldots, X_m), \ldots, P_n(X_0, \ldots, X_m)],$$

is well-defined and holomorphic and the push-forward of $[\mathbb{P}^m]$ is $a^m$ times the positive generator of $H_{2m}(\mathbb{P}^n; \mathbb{Z})$. Furthermore, every degree $a^m$ holomorphic map $f : \mathbb{P}^m \longrightarrow \mathbb{P}^n$ is given by $f = f_{P_0 \ldots P_n}$ for some $P_0, \ldots, P_n$ as above.

### Problem 3 (10 pts)

If $(X, J_X)$ and $(Y, J_Y)$ are almost complex manifolds, a smooth map $f : X \longrightarrow Y$ is called holomorphic if

$$df \circ J_X = J_Y \circ df.$$

If $(X, J_X)$ is an almost complex manifold and $(V, \mathfrak{i}) \longrightarrow X$ is a smooth complex vector bundle, a $\bar{\partial}$-operator in $(V, \mathfrak{i})$ is a $\mathbb{C}$-linear map

$$\bar{\partial} : \Gamma(X; V) \longrightarrow \Gamma(X; T^*X^{0,1} \otimes_{\mathbb{C}} V) \qquad \text{s.t.}$$
$$\bar{\partial}(f\xi) = (\bar{\partial} f) \otimes \xi + f \bar{\partial} \xi \quad \forall\, f \in C^\infty(M; \mathbb{C}),\, \xi \in \Gamma(M; V).$$

Show that
(a) a connection in $V$ induces a $\bar{\partial}$-operator in $V$ and every $\bar{\partial}$-operator in $V$ arises from a connection in $V$;
(b) if $\bar{\partial}$ is a $\bar{\partial}$-operator on $V$, there exists an almost complex structure on $J_V$ on $V$ (the total space of the vector bundle) such that

(i) the bundle projection map $\pi\colon (V,J_V) \longrightarrow (X,J_X)$ is holomorphic,

(ii) for all $v\in V$, the restriction of $J_V$ to $\ker d_v\pi \approx V_{\pi(v)}$ is $\mathfrak{i}|_{V_x}$, and

(iii) if $\xi\in\Gamma(M;V)$, $\bar{\partial}\xi = 0$ if and only if $\xi\colon (X,J_X) \longrightarrow (V,J_V)$ is holomorphic.

Furthermore, every almost complex structure on $V$ satisfying (i)-(iii) arises from a $\bar{\partial}$-operator on $V$ in this way.