left-most board

MAT 545

12/09/08 **Kodaira Vanishing Thm**

M cmpt cmpx m-mfld

$V \to M$ positive l.b.

$H^{p,q}_{\bar\partial}(M;V) = 0$ if $p+q>m$

$\Rightarrow \check{H}^1(M;\Omega^m(V)) = 0$

$" H^1_{\bar\partial}(M;\Lambda^m_{\mathbb{C}} T^*M \otimes_{\mathbb{C}} V) = H^1_{\bar\partial}(M;K_M \otimes_{\mathbb{C}} V)$

Lemma 1: $K_{\tilde M_x} = \pi^* K_M + (m-1)E_x$

Lemma 2: if $L \to M$ positive l.b., $L' \to M$ is any l.b.

$\pi^*(L' \otimes L^{\otimes r}) \otimes [-E]^{\otimes k}$ is pos. l.b. $\forall k \in \mathbb{Z}^+$, $r \ge r_x(k, L')$

Lemma 3: if $L \to M$ is holomorphic l.b.,

$$\check{H}^0(M;L) \xrightarrow[\approx]{\pi^*} \check{H}^0(\tilde M_x;\pi^*L), \quad s \to s\circ\pi$$

is an isomorphism

Cor 1: if $L \to M$ is positive l.b., $x \in M$,

$\check{H}^0(M;L^r) \xrightarrow{ev_x} L^r_x$ is onto $\forall r \ge r_x$

Pf: enough to show $\check{H}^0(\tilde M_x;\pi^*L^r) \xrightarrow{r_E} \check{H}^0(E_x;\pi^*L^r)$ is onto

$$0 \to \mathcal{O}_{\tilde M}(\pi^*L^r \otimes [-E_x]) \to \mathcal{O}_{\tilde M_x}(\pi^*L^r) \xrightarrow{r_E} \mathcal{O}_{\tilde M_x}(\pi^*L^r)|_{E_x} \to 0$$

s.e.s of sheaves on $\tilde M$ $\Rightarrow$

$$\check{H}^0(\tilde M;\pi^*L^r) \xrightarrow{r_E} \check{H}^0(E_x;\pi^*L^r) \to \check{H}^1(\tilde M;\pi^*L^r \otimes [-E_x])$$

is exact

do not erase

Last time: Blow up of M at $x \in M$, $\tilde M_x \xrightarrow{\pi} M$

Replace neighborhood $(\mathcal{U}, x) \approx (\mathbb{C}^m, 0)$ with

$\tilde{\mathbb{C}}^m \equiv \{(z,\ell) \in \mathbb{C}^m \times \mathbb{P}^{m-1} : z \in \ell\}$

$\pi^{-1}(x) = 0 \times \mathbb{P}^{m-1} \subset \tilde{\mathbb{C}}^m, \tilde M_x$ exceptional divisor E_x

$\pi: \tilde M - E_x \to M - x$ biholomorphism

$$\begin{tikzcd}
\check{H}^0(M;\mathcal{I}_x(L)) \arrow[r] & \check{H}^0(M;L) \arrow[r, "ev_x"] \arrow[d, "\pi^*", "\approx"'] & L_x \arrow[d, "\pi^*", "\approx"'] \\
\check{H}^0(\tilde M;\pi^*L \otimes [-E_x]) \arrow[r] & \check{H}^0(\tilde M_x,\pi^*L) \arrow[r, "r_E"] & \check{H}^0(E_x,\pi^*L)
\end{tikzcd}$$

$\pi^*L|_{E_x} = E \times L_x$

$\approx$

$\{\tilde s \in \check{H}^0(\tilde M;\pi^*L) : \tilde s|_{E_x} = 0\}$

$\Rightarrow$ enough to show $\check{H}^1(\tilde M; \underbrace{\pi^*L^r \otimes [-E_x]}_{K_{\tilde M} \otimes K^*_{\tilde M} \otimes \pi^*L^r \otimes [-E_x]}) = 0$

$///$

$\check{H}^1(\tilde M;\Omega^m(K^*_{\tilde M} \otimes \pi^*L^r \otimes [-E_x])) =$

$= \check{H}^1(\tilde M;\Omega^m(\underbrace{\pi^*(K^*_M \otimes L^r) \otimes [-E_x]^{\otimes m}}_{\text{Lemma 2: positive l.b. } \forall r \ge r_x(k)})) = 0$

Kodaira Vanishing

Cor 2: if $L \to M$ is positive l.b., $x, y \in M$, $x \neq y$,

$\check{H}^0(M;L^r) \xrightarrow{ev_{x,y}} L^r_x \oplus L^r_y$ is onto $\forall r \ge r(x,y)$

Pf: $\tilde M \equiv$ blowup of M at x,y = blowup of $\tilde M_x$ at $y \in \tilde M_x$

$E = E_x \sqcup E_y$ exceptional divisor

$\pi: \tilde M \to M$ biholomorphic outside of E_x, E_y

$K_{\tilde M_x} = \pi^* K_M + (m-1)E$; Lemma 2 still ?

$$\begin{array}{ccc} \check{H}^0(M;L^r) & \xrightarrow{ev_{x,y}} & L_x\oplus L_y \\ \pi^*\big\downarrow\approx & & \pi^*\big\downarrow\approx \\ \check{H}^0(\tilde{M};\pi^*L^r) & \xrightarrow{r_E} & \check{H}^0(E;\pi^*L^r) \end{array}$$

$\pi^*L^r|_E = E_x\times L_x \amalg E_y\times L_y \to E_x\amalg E_y$

$\therefore$ enough to r_E is onto or $\check{H}^1(\tilde{M};\pi^*L^r\otimes[-E])=0$

same argument as before

For Kodaira Embedding Thm, also need

$$\begin{array}{ccc} \{s\in\check{H}^0(M;L): s(x)=0\} & \longrightarrow & T^*_xM\otimes L_x,\quad s\to\nabla s|_x, \text{ onto} \\ \pi^*\big\downarrow\approx & & \| \\ \underbrace{\{\tilde{s}\in\check{H}^0(\tilde{M}_x;\pi^*L): \tilde{s}|_{E_x}=0\}}_{\approx} & \longrightarrow & \check{H}^0(E_x;\mathcal{N}^*_E\otimes\pi^*L|_{E_x}),\quad \tilde{s}\to\nabla\tilde{s} \\ & & \| \\ \check{H}^0(\tilde{M}_x;\pi^*L\otimes[-E_x]) & \xrightarrow{r_{E_x}} & \check{H}^0(E_x;\underbrace{\pi^*L|_{E_x}}_{E_x\times L_x}\otimes[-E_x]|_{E_x}) \end{array}$$

do not erase

Notes: (1) $L\to M$ v.b., $s\in\check{H}^0(M;L)$, $X\subset M$ submanifold

$s|_X\equiv 0 \Rightarrow$ get $\nabla s: \mathcal{N}_X\equiv\frac{TM|_X}{TX}\to L_X$

x the vertical differential of s in the normal dir. to X

$\rightarrow \nabla s\in\check{H}^0(E;\mathcal{N}^*_X\otimes L|_X)$

(2) $\mathcal{N}_E=[E]|_E=\gamma=\{(z,l)\in T_xM\times\mathbb{P}(T_xM): z\in l\}$

$$\begin{array}{ccc} \mathcal{N}_E & = & \gamma \\ \downarrow & & \downarrow \\ E_x & = & \mathbb{P}(T_xM) \end{array}$$

$H^0(\mathbb{P}(T_xM);\gamma^*)=\underbrace{T^*_xM}$

homogeneous deg 1 maps $T_xM\to\mathbb{C}$

(3) $\nabla\tilde{s}|_{(z,l)}=\nabla s|_x: l\subset_{T_xM}\longrightarrow\mathbb{C}$ if $\tilde{s}=\pi^*s$

(4) Coordinate charts around $E_x\subset\tilde{\mathbb{C}}^m, M$ are

$\tilde{\mathcal{U}}_i=\{(z,l)\in\mathbb{C}^m\times\mathbb{P}^{m-1}: z\in l, l_i\neq 0\}$

$(z,l)\to\left(z_i,\frac{l_1}{l_i},\ldots,\widehat{\frac{l_i}{l_i}},\ldots\frac{l_m}{l_i}\right)$

$E_x\cap\tilde{\mathcal{U}}_i=\{z_i=0\}\Rightarrow\{z_i\}_i\sim$ section $s_E\neq 0$ of $[E_x]\to\tilde{M}_x$

$\{\tilde{s}\in\check{H}^0(\tilde{M}_x;\pi^*L): \tilde{s}|_{E_x}=0\}\xrightarrow{\approx}\check{H}^0(\tilde{M}_x;\pi^*L\otimes[-E_x])$

$\tilde{s}\longrightarrow\tilde{s}/s_E\sim(\tilde{s}/z_i)$

do not erase

do not erase

$\therefore$ the diagram commutes

<u>Crl 3:</u> If $L\to M$ positive line bundle, $x\in M$.

$\{s\in\check{H}^0(M;L^r): s(x)=0\}\xrightarrow{\nabla}T^*_xM\otimes L_x$ is onto $\forall r\geq r_2$

<u>Pf:</u> to show $\check{H}^0(\tilde{M}_x;\pi^*L^r\otimes[-E_x])\xrightarrow[\text{onto}]{r_{E_x}}\check{H}^0(E_x;\pi^*L^r\otimes[-E_x]|_{E_x})$

$0\to\mathcal{O}_{\tilde{M}_x}(\pi^*L^r\otimes[-2E_x])\to\mathcal{O}_{\tilde{M}_x}(\pi^*L^r\otimes[-E_x])\to\mathcal{O}_{\tilde{M}_x}(\pi^*L^r\otimes[-E_x]|_{E_x})\to 0$

s.e.s. $\Rightarrow$ enough to show $\check{H}^1(\tilde{M}_x;\pi^*L^r\otimes[-2E_x])=0$

Follows from Lemmas 1,2 + Kodaira Vanishing Thm.

Crl 4: If $L \to M$ positive, $\exists$ $\forall r \geq r_0(L)$,

(1) $\check{H}^0(M;L^r) \xrightarrow{ev_{x,y}} L_x^r \oplus L_y^r$ is $\forall\, x,y \in M,\ x \neq y$

(2) $\{s \in \check{H}^0(M;L^r) : s(x)=0\} \xrightarrow{\nabla} T_x^*M \otimes L_x^r$ $\quad\forall\, x \in M.$

Pf: Crl 2 $\Rightarrow$ $\forall\, x,y \in M$, $x \neq y$, $\exists$ s.t. $r_{x,y}(L)$

$ev_{x,y}$ onto $\Rightarrow$ $ev_{x',y'}$ onto $\forall (x',y')$ close to (x,y)

Crl: If M_1, M_2 admit embedding $\iota_j : M_j \to \mathbb{P}^{N_j}$, then so does $M_1 \times M_2$.

Pf 1: $M_1 \times M_2 \xrightarrow{\iota_1 \times \iota_2} \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \hookrightarrow \mathbb{P}$

Pf 2: If $L_j \to M_j$ positive line bundle, so is $L_1 \times L_2 \to M_1 \times M_2$ $\quad \Theta_{L_1 \times L_2} = \pi_1^* \Theta_{L_1} + \pi_2^* \Theta_{L_2}$

Crl: If M admits an embedding, so does $\tilde{M}_x$.

Pf: $L \to \tilde{M}_x$ positive $\Rightarrow$ $L^r \otimes [-E] \to \tilde{M}_x$ is

Same for finite unbranched covers

Crl (Kodaira Embedding Thm)

If M is cmpt complex mfld that admits a positive line bundle, then $\exists$ embedding $\iota : M \to \mathbb{P}^N$ for some N

Pf: $\iota_{L^r} : M \to \mathbb{P}(\check{H}^0(M;L^r)^*)$, $x \to \{s \in \check{H}^0(M;L^r) : s(x)=0\}$

well-defined injective immersion by Crl 2

(& last time)

Crl