MAT 541: Algebraic Topology

Suggested Problems for Week 8

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 31.1, 32.1-3, 33.2-5, 34.1

Problem M

Let X, Z be smooth manifolds, $f\colon Z \longrightarrow X$ be a smooth map, and $W \subset X$ be a neighborhood of $f(Z) \subset X$. Show that there exists a neighborhood $U \subset W$ of $f(Z) \subset X$ such that

$$H_p(U;R) = 0 \qquad \forall\ p > \dim Z\,.$$

Hint. By Sard's theorem, there exists a smooth triangulation of X such that $\operatorname{Int}\sigma$ is transverse to f for every simplex σ.

Note. Let $N \in \mathbb{Z}^+$ with $N \geq 3$. Let

$$X = \bigcup_{i=1}^{\infty} \big\{ x \in \mathbb{R}^N \colon \big| x - (1/i, 0, \ldots, 0) \big| = 1/i \big\}\,.$$

Thus, X is a wedge of countably many $(N$–$1)$-spheres with one point (the origin) in common and with collapsing radii. It is the image of a smooth map

$$f\colon \mathbb{Z}^+ \times S^{N-1} \longrightarrow \mathbb{R}^N\,.$$

It is shown in the 1962 paper of Barratt-Milnor that $H_p(X) \neq 0$ for infinitely many values of p. If you can find a direct proof of this, you can submit it as Problem M$'$.

Problem N

Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere and define

$$r_1\colon S^n \longrightarrow S^n, \qquad r_1\big(x_1, \ldots, x_{n+1}\big) = \big(-x_1, x_2, \ldots, x_{n+1}\big)\,.$$

Use Mayer-Vietoris to show that there is a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_p(S^n) & \xrightarrow{\ r_{1*}\ } & \widetilde{H}_p(S^n) \\ \approx\downarrow & & \downarrow\approx \\ \widetilde{H}_{p-1}(S^{n-1}) & \xrightarrow[\ r_{1*}\ ]{} & \widetilde{H}_{p-1}(S^{n-1}) \end{array}$$

Conclude that the degree of r_1 is -1 and the degree of the antipodal map on S^n is $(-1)^{n+1}$.