Algebraic Topology

Suggested Problems for Week 6

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 14.2, 16.1, 16.3, 19.3, 19.4, 20.2, 20.6

Problem K

Let K and L be simplicial complexes and $g: |K| \longrightarrow |L|$ be a continuous map. For each $v \in \mathrm{Ver}(K)$, let

$$\tau_v = \big\{ w \in \mathrm{Ver}(L) \colon g\big(\mathrm{St}(v,K)\big) \subset \mathrm{St}(w,L) \big\}.$$

For each simplex $\sigma = \{v_0, \ldots, v_p\} \in K$, let

$$\tau_\sigma = \tau_{v_0} \cup \ldots \cup \tau_{v_p} \subset \mathrm{Ver}(L).$$

(1) Suppose $x \in \mathrm{Int}\,\sigma$ for some $\sigma \in K$ and $g(x) \in \mathrm{Int}\,\tau$ for some $\tau \in L$. Show that $\tau_\sigma \subset \tau$ and that this inclusion may be strict even if σ is a vertex and $\tau_v \neq \emptyset$ for every $v \in \mathrm{Ver}(K)$.

(2) Let $\sigma \in K$ and σ' be a face of σ. Show that $\tau_\sigma \in L$ and $\tau_{\sigma'}$ is a face of τ_σ.

(3) Let $\widehat{g} \colon \mathrm{Ver}(K) \longrightarrow \mathrm{Ver}(L)$ be a simplicial approximation to g, i.e.

$$g\big(\mathrm{St}(v,K)\big) \subset \mathrm{St}\big(\widehat{g}(v), L\big) \qquad \forall\ v \in \mathrm{Ver}(K).$$

Show that $\widehat{g}$ is carried by $\sigma \longrightarrow \tau_\sigma$, i.e. $\widehat{g}(\sigma) \subset \tau_\sigma$ for every $\sigma \in K$.

(4) By definition, g admits a simplicial approximation if and only if $\tau_v \neq \emptyset$ for every $v \in \mathrm{Ver}(K)$. Conclude that the homomorphism

$$g_* \equiv \widehat{g} \colon H_*(K) \longrightarrow H_*(L)$$

is independent of the choice of the simplicial approximation $\widehat{g}$ to g, if one exists.

This gives a more systematic perspective on the proof of Lemma 14.1 in Munkres.