MAT 541: Algebraic Topology

Suggested Problems for Week 4

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 12.4, 12.5, 13.6, 11.1

Problem G

Suppose R is a commutative ring with 1 and $(\mathcal{C}_*, \partial)$ and $(\mathcal{C}'_*, \partial')$ are chain complexes over R such that $(\mathcal{C}_*, \partial)$ is free, $\mathcal{C}_p = 0$ for $p < 0$, and $H_p(\mathcal{C}', \partial') = 0$ for all $p > 0$. Let

$$g\colon H_0(\mathcal{C}_*, \partial) \longrightarrow H_0(\mathcal{C}'_*, \partial')$$

be a homomorphism of R-modules. Show that there exists a chain map

$$f_*\colon (\mathcal{C}_*, \partial) \longrightarrow (\mathcal{C}'_*, \partial') \qquad \text{s.t.} \quad f_{0*} = g\,.$$

This problem is from the midterm in MIT's 18.905 in Fall 1996 taught by F. Peterson.

Problem H

(a) State and prove excision for relative ordered simplicial homology. Show that the excision isomorphisms in the ordered and oriented homologies commute with the canonical isomorphisms between the two homologies.

(b) State and prove Mayer-Vietoris for ordered simplicial homology. Show that the Mayer-Vietoris long exact sequences in the ordered and oriented homologies commute with the canonical isomorphisms between the two homologies.

Problem I

The (topological) mapping cylinder of a map $f\colon X \longrightarrow Y$ between two sets is the quotient

$$M_f \equiv \Big(([0,1]\times X)\sqcup Y\Big)\Big/\sim, \quad [0,1]\times X \ni (1,x) \sim f(x) \in Y \;\; \forall\, x \in X;$$

sometimes $(0,x)$ is identified with $f(x)$, instead of $(1,x)$. The maps

$$i\colon X \longrightarrow M_f, \quad i(x) = [0,x], \qquad \text{and} \qquad j\colon Y \longrightarrow M_f, \quad j(y) = [y],$$

are then inclusions. If f is a continuous map between topological spaces, M_f is topologized by the quotient topology; the maps i and j are then continuous. If f is a simplicial/CW map between simplicial/CW complexes, then M_f is a simplicial/CW complex so that the maps i and j are simplicial/CW maps.

(a) Suppose f is a continuous map between topological spaces. Show that the map j is an embedding and a homotopy equivalence, $j(Y) \subset M_f$ is a strong deformation retract, and the maps $i, j\circ f\colon X \longrightarrow M_f$ are homotopic.

(b) Suppose f is a simplicial/CW map between simplicial/CW complexes. Relate the singular/CW chain complex of M_f with the algebraic mapping cylinder of the homomorphism $f_\#$ between the chain complexes of X and Y induced by f.