MAT 541: Algebraic Topology
Suggested Problems for Week 15

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 67.3, 68.5, 68.6, 68.8
From Milnor-Stasheff: 9-C, 11-C

Problem Z-A

Show that
$$H^*(\mathbb{RP}^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[x]/x^{n+1} \qquad \text{and} \qquad H^*(\mathbb{CP}^n;\mathbb{Z}) \approx \mathbb{Z}[u]/u^{n+1}\,,$$
as graded *algebras* over $\mathbb{Z}_2$ and $\mathbb{Z}$, respectively, if the degrees of x and u are 1 and 2, respectively. *Hint:* These algebras have long been shown to be isomorphic as *modules*. Use Poincare Duality and induction to compare the ring structures.

Problem Z-B

Let M be a connected n-manifold. Define
$$\widetilde{M} = \big\{(x,\mu_x)\colon \mu_x \in H_n(M,M-x;\mathbb{Z}) \text{ is a generator, } x\in M\big\}.$$
For each closed ball $B\subset M$ and a generator $\mu_B \in H_n(M,M-B;\mathbb{Z})$, let
$$U(\mu_B) = \big\{(x,\mu_B|_x)\colon\ x\in B\big\},$$
where $\mu_B|_x \in H_n(M,M-x;\mathbb{Z})$ is the image of μ_B under the homomorphism
$$H_n(M,M-B;\mathbb{Z}) \longrightarrow H_n(M,M-x;\mathbb{Z})$$
induced by the inclusion $M-B \longrightarrow M-x$. Show that

(a) $U(\mu_B)\subset\widetilde{M}$ for every closed ball $B\subset M$ and every generator $\mu_B\in H_n(M,M-B;\mathbb{Z})$;

(b) the subsets $U(M_B)\subset\widetilde{M}$ form a basis for a topology on $\widetilde{M}$ so that the map
$$p\colon \widetilde{M} \longrightarrow M, \qquad p(x,\mu_x) = x,$$
is a $2\!:\!1$ covering projection;

(c) $\widetilde{M}$ is an n-manifold with a canonical orientation over $\mathbb{Z}$;

(d) $\widetilde{M}$ is compact if and only if M is compact;

(e) $\widetilde{M}$ is connected if and only if M is non-orientable (over $\mathbb{Z}$).

If M is non-orientable, $\widetilde{M}$ is called orientation double cover of X.

Problem Z-C

Suppose M is a connected n-manifold, which is not orientable over $\mathbb{Z}$, $p\colon \widetilde{M} \longrightarrow M$ is its orientation double cover, and $\sigma\colon \widetilde{M} \longrightarrow \widetilde{M}$ is its deck transformation.

(a) Show that
$$\sigma^* = -\mathrm{id}\colon H_c^n(\widetilde{M};\mathbb{Z}) \longrightarrow H_c^n(\widetilde{M};\mathbb{Z}) \approx \mathbb{Z}.$$

(b) Show that $2\alpha = 0$ for every $\alpha \in H^i(M;\mathbb{Z})$ (resp. $H_c^i(M;\mathbb{Z})$) such that $\sigma^*\alpha = 0$ in $H^i(\widetilde{M};\mathbb{Z})$ (resp. $H_c^i(\widetilde{M};\mathbb{Z})$).

(c) Conclude that $H_c^n(M;\mathbb{Z}) \approx \mathbb{Z}_2$ (a Universal Coefficient Theorem might help).

(d) Assuming M is compact, show that $H_n(M;\mathbb{Z}) = 0$ and the torsion of $H_{n-1}(M;\mathbb{Z})$ is $\mathbb{Z}_2$.

(e) Assuming M is not compact, show that $H^n(M;\mathbb{Z}) = 0$, $H_n(M;\mathbb{Z}) = 0$, and $H_{n-1}(M;\mathbb{Z})$ is torsion-free.

Problem Z-D

Suppose R is a PID and M is a connected n-manifold, which is not orientable over $\mathbb{R}$.

(a) Show that $H_c^n(M;R) \approx R/2R$.

(b) Assuming M is compact, show that $H_n(M;R) = \ker(2\mathrm{id} : R \longrightarrow R)$ and the torsion of $H_{n-1}(M;R)$ over R is $R/2R$.

(c) Assuming M is not compact, show that $H^n(M;R) = 0$, $H_n(M;R) = 0$, and $H_{n-1}(M;R)$ is torsion-free over R.

Problem Z-E

Let $X \subset \mathbb{CP}^3$ be a smooth hypersurface of degree 4. Determine the Hodge diamond of X (this is a K3 surface).