# MAT 541: Algebraic Topology
## Suggested Problems for Week 14

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Milnor-Stasheff: A.1, A.2
From Munkres: 65.1

### Problem W

Let $M$ be an $n$-manifold, possibly with boundary.

(a) Suppose $x \in M - \partial M$ and $\mu_x \in H_n(M, M-x)$ is an orientation for $M$ at $x \in M$. Describe $\mu_x$ and its dual generator $u_x \in H_n(M, M-x)$ using a chart centered at $x$.

(b) Suppose $M$ is an oriented manifold, $\partial M \subset M$ is its boundary with the induced orientation, $K \subset M$ is a compact subset. Show that the image of the fundamental class

$$\mu_{M;K} \in H_n\big(M; \partial M \cup (M-K)\big)$$

for $M$ under an appropriate boundary homomorphism $\partial$ is the fundamental class

$$\mu_{\partial M; \partial M \cap K} \in H_{n-1}\big(\partial M; \partial M - K\big).$$

### Problem X

Suppose $M$ is a smooth $n$-manifold.

(a) Let $x \in M$. Show that a local orientation $\mu_x$ of $M$ at $x$ in the sense of p273 in Milnor-Stasheff with $\mathbb{Z}$-coefficients corresponds to an orientation of $T_x M$.

(b) Show that an orientation for $M$ in the sense of p273 in Milnor-Stasheff with $\mathbb{Z}$-coefficients corresponds to an orientation of $TM$ as a vector bundle (or some other standard notion of orientation for a smooth manifold).

### Problem Y

Let $X$ be a topological space and $R$ be a ring. Show that

(a) $\mathrm{Tor}(H^1(X;R)), \mathrm{Tor}(H^1_c(X;R)) = \{0\}$;

(b) $H^0_c(X;R) = \{0\}$ if $X$ is connected and not compact.

### Problem Z

Suppose $X$ is a topological space.

(a) Let $\{K_j\}_{j \in \mathcal{A}}$ be a collection of compact subsets of $X$ such that every compact subset $K \subset X$ is contained in some $K_j$. Show that the $R$-modules $H^*(X, X-K_j)$ form a direct system and

$$\varinjlim H^*(X, X-K_j) \approx H^*_c(X).$$

(b) Let $\{U_j\}_{j \in \mathcal{A}}$ be a collection of open subsets of $X$ such that their union is $X$ and the union of every pair of these subsets is contained in some $U_j$. Show that the $R$-modules $H^*_c(U_j)$ form a direct system and

$$\varinjlim H^*_c(U_j) \approx H^*_c(X).$$