MAT 541: Algebraic Topology
Suggested Problems for Week 1

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 1.5, 2.3, 2.5, 2.8, 2.9, 3.2, 4.2, 38.1

Problem A

(a) Let V be a vector space and $K \subset V$ be a geometric simplicial complex as in §2. Show that the simplicial topology on $|K|$ is the quotient topology with respect to the projection

$$\bigsqcup_{\sigma\in K}\sigma \longrightarrow |K| \equiv \bigcup_{\sigma\in K}\sigma \subset V$$

and the subspace topology on each σ on the left-hand side above.

(b) A finite-dimensional vector space V has a unique T1 topology with respect to which the vector space operations are continuous. If J is an infinite set, there are a number of such topologies on $\mathbb{R}^J$: product, uniform, box (all described in Munkres's point-set topology book) and "coherent". A set $U \subset \mathbb{R}^J$ is defined to be open in the last topology if $U\cap V$ is open in V for every finite-dimensional linear subspace $V \subset \mathbb{R}^J$ (this is equivalent to the same definition with open replaced by closed). Show that the vector space operations

$$\mathbb{R}^J\times\mathbb{R}^J \longrightarrow \mathbb{R}^J, \quad (v,w)\longrightarrow v+w, \qquad \mathbb{R}\times\mathbb{R}^J \longrightarrow \mathbb{R}^J, \quad (r,v)\longrightarrow rw,$$

are continuous with respect to all four topologies.

(c) Let $\mathcal{S}$ be an abstract simplicial complex as in §3, $\mathrm{Ver}(\mathcal{S})$ be its vertex set, and $|\mathcal{S}| \subset \mathbb{R}^{\mathrm{Ver}(\mathcal{S})}$ be its canonical geometric realization. Show that the simplicial topology on $|S|$ is the subspace topology with respect to the coherent topology on $\mathbb{R}^{\mathrm{Ver}(\mathcal{S})}$.

(d) Suppose in addition that the set $\{S\in\mathcal{S} : v\in S\}$ is (at most) countable for every $v\in\mathrm{Ver}(\mathcal{S})$. Show that the simplicial topology on $|S|$ is the subspace topology with respect to the box topology on $\mathbb{R}^{\mathrm{Ver}(\mathcal{S})}$.

Problem B

(a) Describe simplicial and CW decompositions of S^1 and S^2 with as few cells as possible.

(b) Describe CW decompositions of $\mathbb{RP}^n$ and $\mathbb{CP}^n$ with precisely $n+1$ cells each.

(c) Use them to show that

$$H_p(\mathbb{RP}^n;\mathbb{Z}_2)=\begin{cases}\mathbb{Z}_2, & \text{if } p\le n;\\ 0, & \text{if } p>n;\end{cases} \qquad H_p(\mathbb{CP}^n;\mathbb{Z})=\begin{cases}\mathbb{Z}, & \text{if } p\le 2n,\ p\in 2\mathbb{Z};\\ 0, & \text{otherwise.}\end{cases}$$

Since H_* does not depend on the CW decomposition, the above conclusions imply that $\mathbb{RP}^n$ and $\mathbb{CP}^n$ do not admit CW decompositions with fewer than $n+1$ cells.