MAT 531 Geometry/Topology Homework 5

1. Prove that the distribution $dx + zdy + ydz = 0$ in $\mathbb{R}^3$ is integrable. Find the integral surfaces and a rectifying diffeomorphism (which turns out to be defined globally).

2. Prove that the distribution $\alpha = 0$ in $\mathbb{R}^3$ (here α is a smooth 1-form) is integrable if and only if $\alpha \wedge d\alpha = 0$.

3. Consider a distribution $adx + bdy + cdz = 0$ in $\mathbb{R}^3$, where a, b and c are smooth functions. Prove that this distribution is integrable if and only if the following symbolic determinant vanishes:

$$\det \begin{pmatrix} a & b & c \\ \partial_x & \partial_y & \partial_z \\ a & b & c \end{pmatrix}$$

(to compute this determinant, one needs to expand it in the first row).

4. A differential 1-form α on a smooth manifold of dimension $2n+1$ is called a *contact form* (or a *maximally non-integrable form*) if $\alpha \wedge d\alpha^n \neq 0$ (here $d\alpha^n$ denotes the n-th wedge power of the 2-form $d\alpha$).

Consider the following differential 1-form on the space $\mathbb{R}^{2n+1}$ with coordinates $(H, p_1, \dots, p_n, q_1, \dots, q_n)$:

$$\alpha = dH - p_1 dq_1 - p_2 dq_2 - \cdots - p_n dq_n.$$

(the notation for coordinates comes from classical mechanics). Prove that α is a contact form.

5. For the hyperplane distribution $\alpha = 0$ from the last exercise, give examples of partial integral submanifolds of dimension n. Prove that for any smooth function h of variables $q_1, \dots, q_n$, there exists a partial integral submanifold parameterized by $q_1, \dots, q_n$ such that $H = h(q_1, \dots, q_n)$ on this submanifold.

6*. Consider a contact form α on a $2n+1$-dimensional manifold X. For a point $p \in X$, prove that the bilinear form $d\alpha_p$ can not vanish identically on a $(n+2)$-dimensional vector subspace of T_pX. Prove that the distribution $\alpha = 0$ does not admit any partial integral submanifolds of dimension $n+1$. (That is why α is called maximally non-integrable: it turns out that any hyperplane distribution has partial integral submanifolds of dimension n)