MAT 531: Topology&Geometry, II
Spring 2010

Midterm

Give concise proofs, quoting established facts as appropriate; no treatises. You can do the problems and parts of problems in any order. You do not need to copy the statements of the problems. Please write legibly.

Problem 1 (15pts)

Let $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow Z$ be smooth maps between smooth manifolds. State the chain rule for the differential of the map $g \circ f : M \longrightarrow Z$ and obtain it directly from the relevant definitions (state the relevant definition(s); you do not need to show that they define the required objects).

Problem 2 (20pts)

Let M be a smooth manifold and $p \in M$ a fixed point of a smooth map $f\colon M \longrightarrow M$, i.e. $f(p)=p$. Show that if all eigenvalues of the linear transformation

$$\mathrm{d}_p f\colon T_pM \longrightarrow T_pM$$

are different from 1 (so $\mathrm{d}_p f(v) \neq v$ for all $v \in T_pM - 0$), then p is an isolated fixed point (has a neighborhood that contains no other fixed point).

Problem 3 (20pts)

Let $\alpha = \mathrm{d}x_1 + f\mathrm{d}x_2$ be a smooth 1-form on $\mathbb{R}^3$ (so $f \in C^\infty(\mathbb{R}^3)$). Show that for every $p \in \mathbb{R}^3$ there exists a diffeomorphism

$$\varphi = (y_1, y_2, y_3)\colon U \longrightarrow V$$

from a neighborhood U of p to an open subset V of $\mathbb{R}^3$ such that $\alpha|_U = g\mathrm{d}y_1$ for some $g \in C^\infty(U)$ if and only if f does not depend on x_3.

Problem 4 (20pts)

Let $D \subset \mathbb{R}^2$ be the closed unit disk centered at the origin.
(a) State Stokes' Theorem (for integration of top forms on manifold; no singular chains) for D.
(b) Show that it reduces to Green's theorem of calculus (if you do not remember what the latter says, make sure your final statement is in calculus notation).

Problem 5 (25pts)

(a) State the usual definition of the tautological line bundle γ_n over the real projective space $\mathbb{R}P^n$, making clear the topology on the total space and the projection map.
(b) Show that $\gamma_1 \longrightarrow \mathbb{R}P^1$ is isomorphic to the line bundle formed by projecting the infinite Mobius Band to the circle S^1.
(c) Show that the line bundle $\gamma_n \longrightarrow \mathbb{R}P^n$ is not orientable (for $n \geq 1$).