MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 4
Due on March, 3/03, in class

1. Chapter 1, #13ad (p51)

2. Chapter 1, #22 (p51). *Hint:* this is 2-3 lines

3. Chapter 1, #17 (p51). *Hint:* only slightly longer

4. Let V be the vector field on $\mathbb{R}^3$ given by

$$V(x,y,z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Explicitly describe and sketch the flow of V. *Hint:* an easy MAT 303/305 problem

5. Suppose X and Y are smooth vector fields on a manifold M. Show that for every $p\in M$ and $f\in C^\infty(M)$,

$$\lim_{s,t\longrightarrow 0} \frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(p))\big))\big) - f(p)}{s\,t} = [X,Y]_p f \in \mathbb{R}.$$

Do not forget to explain why the limit exists.
Note: This means that the extent to which the flows $\{X_t\}$ of X and $\{Y_s\}$ of Y do not commute (i.e. the rate of change in the "difference" between $Y_s\circ X_t$ and $X_t\circ Y_s$) is measured by $[X,Y]$.

6. Let U and V be the vector fields on $\mathbb{R}^3$ given by

$$U(x,y,z) = \frac{\partial}{\partial x} \qquad \text{and} \qquad V(x,y,z) = F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z},$$

where F and G are smooth functions on $\mathbb{R}^3$. Show that there exists a proper[1] foliation of $\mathbb{R}^3$ by 2-dimensional embedded submanifolds such that the vector fields U and V everywhere span the tangent spaces of these submanifolds if and only if

$$F(x,y,z) = f(y,z)\,e^{h(x,y,z)} \qquad \text{and} \qquad G(x,y,z) = g(y,z)\,e^{h(x,y,z)}$$

for some $f,g\in C^\infty(\mathbb{R}^2)$ and $h\in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

[1] in the sense of Definition 10.4 in *Lecture Notes*