MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 3
Due on Thursday, 2/24, in class

1. Chapter 1, #5 (p50)

2. Show that the tangent bundle TM of a smooth n-manifold M is a real vector bundle of rank n over M. What is its transition data?

3. Show that the tangent bundle TS^1 of S^1, defined as in 1.25 (p19), is isomorphic to the trivial real line bundle over S^1. *Hint:* Use Lemma 8.5 in *Lecture Notes*.

4. Suppose that $f: X \longrightarrow M$ is a smooth map and $\pi: V \longrightarrow M$ is a smooth vector bundle. The pullback of V by f, $\pi_1: f^*V \longrightarrow X$, is the vector bundle defined by taking
$$f^*V = \big\{(x,v) \in X \times V : f(x) = \pi(v)\big\} \subset X \times V.$$
Show that f^*V is indeed a smooth submanifold of $X \times V$.

5. Show that the tautological line bundle $\gamma_n \longrightarrow \mathbb{C}P^n$ is indeed a complex line bundle (describe its trivializations). What is its transition data? Why is it non-trivial for $n \geq 1$? (not isomorphic to $\mathbb{C}P^n \times \mathbb{C} \longrightarrow \mathbb{C}P^n$ as line bundle over $\mathbb{C}P^n$). *Hint:* See proof of Lemma 8.4 in *Lecture Notes*.

6. Suppose $k < n$. Show that the map
$$\iota: \mathbb{C}P^k \longrightarrow \mathbb{C}P^n, \qquad [X_0, \ldots, X_k] \longrightarrow [X_0, \ldots, X_k, \underbrace{0, \ldots, 0}_{n-k}],$$
is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$). Show that the normal bundle to this immersion, $\mathcal{N}_\iota$, is isomorphic to
$$(n-k)\gamma_k^* \equiv \underbrace{\gamma_k^* \oplus \ldots \oplus \gamma_k^*}_{n-k},$$
where $\gamma_k \longrightarrow \mathbb{C}P^k$ is the tautological line bundle (isomorphic as complex line bundles).

7. Let $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ be the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$. Show that $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n$ is isomorphic to the line bundle
$$\gamma_n^{*\otimes(n+1)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{n+1},$$
where $\gamma_n \longrightarrow \mathbb{C}P^n$ is the tautological line bundle (isomorphic as complex line bundles).

Hint on the next page

Hint for 6 and 7: There are a number of ways of doing these, including:

(i) construct an isomorphism between the two line bundles;

(ii) show that there exists a short exact sequence of vector bundles

$$0 \longrightarrow \mathbb{C}P^n \times \mathbb{C} \longrightarrow (n+1)\gamma_n^* \longrightarrow T\mathbb{C}P^n \longrightarrow 0$$

and this implies the claim (exact means that at each position the kernel of the outgoing map equals to the image of the incoming map over every point of M);

(iii) use Problems PS1-3b and 2 and 5 above to determine transition data for the two bundles. However, you will need to modify trivializations for one of the line bundles in Problem 7 to arrive at the same transition data;

(iv) show that there exists a holomorphic diffeomorphism between $(n-k)\gamma_k^*$ and a neighborhood of $\iota(\mathbb{C}P^k)$ in $\mathbb{C}P^n$ and that this implies the claim in Problem 6.