MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 11

Problem 1 (15pts)

Suppose M and N are smooth oriented compact connected n-manifolds. If $f: M \longrightarrow N$ is a smooth map, the **degree of** f *is the number* $\deg f \in \mathbb{R}$ *such that*

$$\int_M f^*\omega = (\deg f) \cdot \int_N \omega \qquad \forall\, \omega \in E^n(N).$$

This number is well-defined.

(a) *Show that if $f: M \longrightarrow N$ and $g: N \longrightarrow X$ are smooth maps between smooth oriented compact connected n-manifolds, then*
$$\deg(g \circ f) = (\deg g) \cdot (\deg f).$$

(b) *Show that if $f: M \longrightarrow N$ is a covering projection between smooth oriented compact connected n-manifolds, then $\deg f$ is the degree of f as a covering map (i.e. the number of elements in each fiber).*

(c) *Show that if $f: M \longrightarrow N$ is a smooth map of degree one, then it induces a surjective homomorphism between the fundamental groups of M and N.*

(a) If $\omega \in E^n(X)$, then

$$\int_M (g\circ f)^*\omega = \int_M f^*(g^*\omega) = (\deg f) \cdot \int_N g^*\omega = (\deg f) \cdot (\deg g) \int_X \omega$$
$$= \big((\deg g) \cdot (\deg f)\big) \int_X \omega.$$

Since this equality holds for all $\omega \in E^n(X)$, by definition of the degree of a map

$$\deg(g \circ f) = (\deg g) \cdot (\deg f).$$

(b) Suppose $f: M \longrightarrow N$ is a k-to-1 covering map. Choose $y \in N$, an evenly covered neighborhood U of y in N (which can be assumed to be diffeomorphic to $\mathbb{R}^n$), and an element

$$\omega \in E^n(N) \qquad \text{s.t.} \qquad \operatorname{supp}\omega \subset U \quad \text{and} \quad \int_U \omega = \int_N \omega = 1.$$

Since $f^{-1}(U)$ is the disjoint union of k copies of U (with its orientation) under f and $\operatorname{supp}\omega \subset U$,

$$\int_M f^*\omega = \int_{f^{-1}(U)} f^*\omega = k \cdot \int_U \omega = k = k \cdot \int_N \omega.$$

Thus, the degree of f must be k.

(c) Suppose $x \in M$ and H is the image of the homomorphism

$$f_*\colon \pi_1(M,x) \longrightarrow \pi_1\big(N, f(x)\big).$$

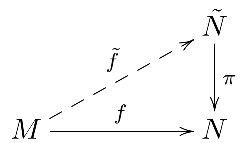
Since N is semi-locally simply connected (being locally Euclidean), there exists a covering map $\pi:\tilde{N} \longrightarrow N$ with $\tilde{N}$ connected, such that

$$\pi_*\big(\pi_1(\tilde{N},z)\big) = H \subset \pi_1\big(N, f(x)\big),$$

for any $z \in \pi^{-1}(f(x))$; see Theorem 82.1 in Munkres. Since N is a smooth oriented manifold, so is $\tilde{N}$. Since

$$f_*\big(\pi_1(M,x)\big) = H \subset H = \pi_*\big(\pi_1(\tilde{N},z)\big),$$

by the Lifting Lemma (Munkres, Lemma 79.1) the map $f: M \longrightarrow N$ lifts over π, i.e. there exists a continuous map $\tilde{f}\colon M \longrightarrow \tilde{N}$ such that diagram



commutes. Since f is smooth and $\tilde{\pi}$ is a smooth covering projection, $\tilde{f}$ is also a smooth map.

If $\tilde{N}$ is compact (or equivalently, $\pi^{-1}(y)$ is finite for any $y \in N$), then degrees of π and $\tilde{f}$ are well-defined integers and

$$1 = \deg f = (\deg \pi)\cdot(\deg \tilde{f})$$

by part (a). By part (b), the degree of π is its degree as a covering map. Since $\deg \tilde{f}$ is an integer, π must be a 1:1-covering map, i.e. a diffeomorphism. Thus,

$$f_*\big(\pi_1(M,x)\big) = H = \pi_*\big(\pi_1(\tilde{N},z)\big) = \pi_1\big(N, f(x)\big),$$

i.e. f_* is surjective.

Suppose $\tilde{N}$ is not compact (or equivalently, $\pi^{-1}(y)$ is infinite for any $y \in N$). Let $\omega \in E^n(N)$ be any element such that $\int_N \omega \neq 0$, e.g. an orientation (or volume) form on N. Since $f = \pi \circ \tilde{f}$,

$$\int_M f^*\omega = \int_M (\pi \circ \tilde{f})^*\omega = \int_M \tilde{f}^*\big(\pi^*\omega\big).$$

Since the n-manifold $\tilde{N}$ is connected and not compact, $H^n_{\mathrm{de\,R}}(\tilde{N}) = 0$. Since $\pi^*\omega \in E^n(\tilde{N})$ is closed (being a top form), it must be exact. Therefore, $\tilde{f}^*(\pi^*\omega)$ is also exact and by Stokes Theorem

$$(\deg f)\int_N \omega = \int_M f^*\omega = \int_M \tilde{f}^*\big(\pi^*\omega\big) = 0.$$

Since $\int_N \omega \neq 0$, it follows that $\deg f = 0$, contrary to the assumption.

Note: The fundamental group of any compact manifold is finitely generated (and of an arbitrary manifold is countably generated). Thus, the index of the subgroup H above is countable, and so $\tilde{N}$ is second-countable. Alternatively, one can adapt the proof of the vanishing of the top cohomology of non-compact manifolds to the present situation.

Problem 2 (5pts)

State and prove a Mayer-Vietoris theorem for compactly supported cohomology.

If U is an open subset of a smooth manifold M, a compactly supported form $\alpha\in E^*_c(U)$ determines a compactly supported form on M. Thus, there is an inclusion homomorphism

$$\iota_{M,U}\colon E^*_c(U)\longrightarrow E^*_c(M),$$

which replaces the restriction homomorphism $r_{U,M}\colon E^*(M)\longrightarrow E^*(U)$ going in the opposite direction.

Mayer-Vietoris for compactly supported cohomology: If M is a smooth manifold and $U,V\subset M$ are open subsets such that $M=U\cup V$, then there is a long exact sequence

$$\ldots H^{p-1}_{\mathrm{de\,R};c}(M)\xrightarrow{\delta_c}H^p_{\mathrm{de\,R};c}(U\cap V)\xrightarrow{i}H^p_{\mathrm{de\,R};c}(U)\oplus H^p_{\mathrm{de\,R};c}(V)\xrightarrow{j}H^p_{\mathrm{de\,R};c}(M)\xrightarrow{\delta_c}H^{p+1}_{\mathrm{de\,R};c}(U\cap V)\ldots$$
$$i\big([\kappa]\big)=\big([\iota_{U,U\cap V}\kappa],-[\iota_{V,U\cap V}\kappa]\big),\qquad j\big([\mu],[\eta]\big)=[\iota_{M,U}\mu]+[\iota_{M,V}\eta].$$

By the Snake Lemma, it is sufficient to show that the sequence

$$0\longrightarrow\big(E^*_c(U\cap V),d_{U\cap V}\big)\xrightarrow{i}\big(E^*_c(U)\oplus E^*_c(V),d_U\oplus d_V\big)\xrightarrow{j}\big(E^*_c(M),d_M\big)\longrightarrow 0$$
$$i(\kappa)=\big(\iota_{U,U\cap V}\kappa,-\iota_{V,U\cap V}\kappa\big),\qquad j\big(\mu,\eta\big)=\iota_{M,U}\mu+\iota_{M,V}\eta,$$

is an exact sequence of co-chain complexes. It is immediate that i and j commute with the differentials (the forms are just extended by 0), i is injective (same reason), and $j\circ i=0$. If $\iota_{M,U}\mu+\iota_{M,V}\eta=0$,

$$\operatorname{supp}\mu=\operatorname{supp}\eta\subset U\cap V\quad\Longrightarrow\quad \mu|_{U\cap V}=-\eta|_{U\cap V}\in E^*_c(U\cap V)\quad\Longrightarrow\quad (\mu,\eta)=i\big(\mu|_{U\cap V}\big).$$

It remains to show that j is surjective. Let $\{\psi_U,\psi_V\}$ be a partition of unity on M subordinate to $\{U,V\}$. Then, for every $\gamma\in E^*(M)$,

$$\psi_U\gamma|_U\in E^*_c(U),\quad \psi_V\gamma|_V\in E^*_c(V),\quad \gamma=\psi_U\gamma+\psi_V\gamma=j\big(\psi_U\gamma,\psi_V\gamma\big);$$

so j is surjective.

Problem 3 (5+10+10pts)

Let M be an oriented n-manifold, possibly non-compact.

(a) *Show that the pairing*

$$H^*_{\mathrm{de\,R}}(M)\otimes H^*_{\mathrm{de\,R};c}(M)\longrightarrow\mathbb{R},\qquad [\alpha]\otimes[\beta]\longrightarrow\int_M\alpha\wedge\beta,$$

is well-defined.

(b) *Show that the above pairing is nondegenerate if $M=\mathbb{R}^n$.*

(c) *Suppose that M admits a cover $\{U_i\}_{i=1,\ldots,m}$ such that every intersection $U_{i_1}\cap\ldots\cap U_{i_k}$ is either empty or diffeomorphic to $\mathbb{R}^n$. Show that the above pairing is nondegenerate.*

(a) If $\alpha\in E^*(M)$ and $\beta\in E^*_c(M)$, $\alpha\wedge\beta\in E^*_c(M)$. Since M is oriented, $\int_M \alpha\wedge\beta$ exists (only the homogeneous part of $\alpha\wedge\beta$ of degree n contributes to the integral). If in addition $\alpha\in\ker d$, then

$$\alpha\wedge d\beta=\pm d(\alpha\wedge\beta) \qquad\Longrightarrow\qquad \int_M \alpha\wedge d\beta=\pm\int_{\partial M}\alpha\wedge\beta=0$$

by Stokes Theorem because $\alpha\wedge\beta\in E^*_c(M)$ and $\partial M=0$. If $\beta\in\ker d\cap E^*_c(M)$, then

$$d\alpha\wedge\beta=d(\alpha\wedge\beta) \qquad\Longrightarrow\qquad \int_M d\alpha\wedge\beta=\int_{\partial M}\alpha\wedge\beta=0$$

by Stokes Theorem because $\alpha\wedge\beta\in E^*_c(M)$. Thus, the homomorphism

$$(\ker d)\otimes\big((\ker d)\cap E^*_c(M)\big)\longrightarrow\mathbb{R}, \qquad \alpha\otimes\beta\longrightarrow\int_M\alpha\wedge\beta,$$

vanishes on $(\ker d)\otimes(dE^*_c(M))\oplus(dE^*(M))\otimes((\ker d)\cap E^*_c(M))$ and thus induces a well-defined homomorphism on the quotient

$$H^*_{\mathrm{de\,R}}(M)\otimes H^*_{\mathrm{de\,R};c}(M)=\frac{(\ker d)\otimes((\ker d)\cap E^*_c(M))}{(\ker d)\otimes(dE^*_c(M))\oplus(dE^*(M))\otimes((\ker d)\cap E^*_c(M))}\longrightarrow\mathbb{R},$$
$$[\alpha]\otimes[\beta]\longrightarrow\int_M\alpha\wedge\beta.$$

(b) $H^0_{\mathrm{de\,R}}(\mathbb{R}^n)$ is generated by the constant function 1 on $\mathbb{R}^n$, while $H^n_{\mathrm{de\,R};c}(\mathbb{R}^n)$ is generated by $[\eta]$ for any $\eta\in E^n_c(\mathbb{R}^n)$ with nonzero integral on $\mathbb{R}^n$. The pairing of these two elements is nonzero:

$$[1]\otimes[\eta]\longrightarrow\int_{\mathbb{R}^n}1\wedge\eta=\int_{\mathbb{R}^n}\eta\neq0.$$

Thus, the pairing

$$H^0_{\mathrm{de\,R}}(\mathbb{R}^n)\otimes H^n_{\mathrm{de\,R};c}(\mathbb{R}^n)=\mathbb{R}\longrightarrow\mathbb{R}, \qquad [\alpha]\otimes[\beta]\longrightarrow\int_{\mathbb{R}^n}\alpha\wedge\beta,$$

is nonzero and thus nondegenerate (since both vector spaces are one-dimensional). Since $H^p_{\mathrm{de\,R}}(\mathbb{R}^n)=0$ for $p\neq0$, it remains to show that $H^q_{\mathrm{de\,R};c}(\mathbb{R}^n)=0$ for $q\neq n$. This is immediate for $q>n$ (because $E^q(\mathbb{R}^n)=0$ in this case) and easy for $q=0$ (done in class).

Thus, we need to show that for every

$$\alpha\equiv\sum_I f_I dx_I\in E^q_c(\mathbb{R}^n)$$

with $q=1,\ldots,n-1$ and $d\alpha=0$, there exists $\beta\in E^{q-1}_c(\mathbb{R}^n)$ such that $\alpha=d\beta$. Since $\alpha\in E^q_c(\mathbb{R}^n)$, there exists $A>0$ such that $\alpha|_x=0$ if $|x|\geq A$. By Warner 4.18 (Poincare Lemma and its proof), $\alpha=d(\iota_X\tilde\alpha)$, where ι_X is the contraction (Warner 2.11),

$$X=\sum_{i=1}^{i=n}x_i\frac{\partial}{\partial x_i}\equiv r\frac{\partial}{\partial r}, \quad \tilde\alpha_x=\sum_I\bigg(\int_0^1 t^{q-1}f_I(tx)dt\bigg)dx_I=\sum_I\bigg(\int_0^{|x|}t^{q-1}f_I(tx/|x|)dt\bigg)\frac{dx_I}{|x|^q} \quad \text{if } x\neq0$$
$$=\sum_I\bigg(\int_0^A t^{q-1}f_I(tx/|x|)dt\bigg)\frac{dx_I}{|x|^q} \quad \text{if } |x|\geq A.$$

Since $\iota_X(\iota_X\tilde{\alpha})=0$ ($\iota_X\tilde{\alpha}$ vanishes if any input is X), $\iota_X\tilde{\alpha}=r^*\beta'$ on $\mathbb{R}^n-B_A(0)$, where $r\colon \mathbb{R}^n-0 \longrightarrow S^{n-1}$ is the usual retraction and $\beta'\in E^{q-1}(S^{n-1})$ is given by

$$\beta'_x = \sum_I \bigg(\int_0^A t^{q-1} f_I(tx)dt \bigg) \iota_X dx_I\,.$$

Since $d(r^*\beta')=\alpha=0$ on the sphere S_A^{n-1} of radius A and r^* is injective, $d\beta'=0\in E^q(S^{n-1})$. If $q=1$, it follows that β' is a constant function with some value C on S^{n-1}; thus, the 0-form $\beta\equiv\iota_X\tilde{\alpha}-C$ is supported in $\bar{B}_A(0)$ (because $\iota_X\tilde{\alpha}=r^*\beta'=C$ outside of $\bar{B}_A(0)$) and $\alpha=d\beta$. Suppose instead $2\le q\le n-1$. Since $H^{q-1}_{\mathrm{de\,R}}(S^{n-1})=0$ and $d\beta'=0$, $\beta'=d\gamma'$ for some $\gamma'\in E^{q-2}(S^{n-1})$. Choose a smooth function $\eta\colon \mathbb{R}^n\longrightarrow[0,1]$ such that

$$\eta|_{B_{A/2}(0)}\equiv 0, \qquad \eta|_{\mathbb{R}^n-B_A(0)}\equiv 1,$$

and let $\gamma=\eta\cdot r^*\gamma'\in E^{q-2}(\mathbb{R}^n)$; even though r is not defined at $0\in\mathbb{R}^n$, γ is well-defined because $\eta\cdot r^*\gamma'$ vanishes on $B_{A/2}(0)-0$ and thus extends by 0 over the origin. With $\beta=\iota_X\tilde{\alpha}-d\gamma\in E^{q-1}(\mathbb{R}^n)$, $\alpha=d\beta$. Since $\iota_X\tilde{\alpha}=r^*\beta'=r^*d\gamma'=d\gamma$ outside of $B_A(0)$, β is supported in $\bar{B}_A(0)$ in this case as well.

(c) We prove by induction on m that $H^*(M)\equiv H^*_{\mathrm{de\,R}}(M)$ and $H^*_c(M)\equiv H^*_{\mathrm{de\,R};c}(M)$ are finite-dimensional (actually of dimension at most m) and that the homomorphism

$$H^p(M)\longrightarrow H_c^{n-p}(M)^*, \qquad [\alpha]\longrightarrow \int_M \alpha\wedge\cdot\,,$$

induced by the pairing is an isomorphism; the latter is equivalent to the pairing being non-degenerate (when the vector spaces are finite-dimensional). Part (b) is the $m=1$ case.

Suppose $m\ge 2$ and both statements hold for all oriented n-manifolds admitting good covers as above with at most $m-1$ elements. Let

$$U=U_1\cup U_2\cup\ldots\cup U_{m-1}, \qquad V=U_m\,.$$

By our inductive assumption, H^* and H^*_c of U, V, and

$$U\cap V=(U_1\cap U_m)\cup(U_2\cap U_m)\cup\ldots\cup(U_{m-1}\cap U_m)$$

are finite-dimensional and dual to each other via the Poincare pairing. We have two MV long exact sequence for $M=U\cup V$:

$$\begin{array}{ccccccccccc}
\longrightarrow & H^{p-1}(U)\oplus H^{p-1}(V) & \xrightarrow{g} & H^{p-1}(U\cap V) & \xrightarrow{(-1)^p\delta} & H^p(M) & \xrightarrow{f} & H^p(U)\oplus H^p(V) & \xrightarrow{g} & H^p(U\cap V) & \longrightarrow \\
 & \otimes \quad \otimes & & \otimes & & \otimes & & \otimes \quad \otimes & & \otimes & \\
\longleftarrow & H_c^{q+1}(U)\oplus H_c^{q+1}(V) & \xleftarrow{i} & H_c^{q+1}(U\cap V) & \xleftarrow{\delta_c} & H_c^q(M) & \xleftarrow{j} & H_c^q(U)\oplus H_c^q(V) & \xleftarrow{i} & H_c^q(U\cap V) & \longleftarrow \\
 & \Big\downarrow{\scriptstyle \langle,\rangle=\int_U+\int_V} & & \Big\downarrow{\scriptstyle \langle,\rangle=\int_{U\cap V}} & & \Big\downarrow{\scriptstyle \langle,\rangle=\int_M} & & \Big\downarrow{\scriptstyle \langle,\rangle=\int_U+\int_V} & & \Big\downarrow{\scriptstyle \langle,\rangle=\int_{U\cap V}} & \\
= & \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} & =
\end{array}$$

where $p+q=n$, the homomorphisms f, g, and δ in the top row are as in Problem 2a on PS7, and the homomorphisms i, j, and δ_c are as in Problem 2 above. The entire diagram commutes, i.e.

$$\big\langle f([\alpha]),([\mu],[\eta])\big\rangle=\big\langle[\alpha],j([\mu],[\eta])\big\rangle, \qquad \big\langle g([\beta],[\gamma]),[\kappa]\big\rangle=\big\langle([\beta],[\gamma]),i([\kappa])\big\rangle,$$
$$(-1)^{p+1}\big\langle\delta([\omega]),[\theta]\big\rangle=\big\langle[\omega],\delta_c([\theta])\big\rangle \quad \forall\,[\omega]\in H^p(M).$$

The first two identities are immediate from the definitions of f, g, i, and j:

$$\begin{aligned}
\big\langle f([\alpha]),([\mu],[\eta])\big\rangle &= \int_U \alpha|_U\wedge\mu + \int_V \alpha|_V\wedge\eta \\
&= \int_M \alpha\wedge\big(\iota_{M,U}\mu+\iota_{M,V}\eta\big) = \big\langle[\alpha], j([\mu],[\eta])\big\rangle\,, \\
\big\langle g([\beta],[\gamma]),[\kappa]\big\rangle &= \int_{U\cap V}\big(\beta|_{U\cap V}-\gamma|_{U\cap V}\big)\wedge\kappa \\
&= \int_U \beta\wedge(\iota_{U,U\cap V}\kappa)+\int_V\beta\wedge(-\iota_{V,U\cap V}\kappa) = \big\langle([\beta],[\gamma]), i([\kappa])\big\rangle;
\end{aligned}$$

the middle equalities above hold because μ, η, and κ are extended by 0 outside of U, V, and $U\cap V$, respectively. For the last identity, we need explicit expressions for δ and δ_c. Let $\{\psi_U,\psi_V\}$ be a partition of unity subordinate to $\{U,V\}$. By Problem 2a on PS7,

$$\delta\big([\omega]\big) = \big[\iota_{M,U\cap V}(d\psi_V\wedge\omega)\big].$$

Similarly, since $\theta=\iota_{M,U}(\psi_U\theta)+\iota_{M,V}(\psi_V\theta)$ and $\psi_U+\psi_V=1$,

$$\delta_c\big([\theta]\big) = \big[d\psi_U\wedge\theta|_{U\cap V}\big] = -\big[d\psi_V\wedge\theta|_{U\cap V}\big].$$

Thus,

$$(-1)^{p+1}\big\langle\delta([\omega]),[\theta]\big\rangle = (-1)^{p+1}\int_M \iota_{M,U\cap V}(d\psi_V\wedge\omega)\wedge\theta = \int_{U\cap V}\omega\wedge(d\psi_U\wedge\theta)|_{U\cap V} = \big\langle[\omega],\delta_c([\theta])\big\rangle\,,$$

if $[\omega]\in H^p(M)$.

Taking the dual of the middle row in the above diagram, we thus obtain a commutative diagram of two exact sequences

$$\begin{array}{ccccccccccccc}
\longrightarrow & H^{p-1}(U)\oplus H^{p-1}(V) & \xrightarrow{g} & H^{p-1}(U\cap V) & \xrightarrow{(-1)^p\delta} & H^p(M) & \xrightarrow{f} & H^p(U)\oplus H^p(V) & \xrightarrow{g} & H^p(U\cap V) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_c^{q+1}(U)^*\oplus H_c^{q+1}(V)^* & \xrightarrow{i^*} & H_c^{q+1}(U\cap V)^* & \xrightarrow{\delta_c^*} & H_c^q(M)^* & \xrightarrow{j^*} & H_c^q(U)^*\oplus H_c^q(V)^* & \xrightarrow{i^*} & H_c^q(U\cap V)^* & \longrightarrow
\end{array}$$

with the vertical maps induced by the pairing $\langle,\rangle$. By the inductive assumption, the second and fourth vector spaces in each row are finite-dimensional; since the rows are exact, so are the middle vector spaces in each row. The first, second, fourth, and fifth vertical arrows are isomorphisms by the inductive assumption; since the rows are exact, so is the middle vertical arrow by the *Five Lemma*. Thus, every oriented n-manifold admitting a good cover with at most m elements has finite-dimensional H^* and H_c^* and satisfies Poincare duality between the two cohomologies.