MAT 531: Topology&Geometry, II
Spring 2010

Problem Set 8
Due on Thursday, 4/15, in class

1. Suppose X is a topological space and $\mathcal{P}=\{S_U;\rho_{U,V}\}$ is a presheaf on X. Let

$$\bar{S}_U=\Big\{(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}: U_\alpha\subset U \text{ open},\; U=\bigcup_{\alpha\in\mathcal{A}}U_\alpha;\, f_\alpha\in S_{U_\alpha};$$
$$\forall\,\alpha,\beta\in\mathcal{A},\, p\in U_\alpha\cap U_\beta\;\exists\,W\subset U_\alpha\cap U_\beta \text{ open s.t. } p\in W,\; \rho_{W,U_\alpha}f_\alpha=\rho_{W,U_\beta}f_\beta\Big\}\Big/\sim,$$

where $$(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\sim(U'_\beta,f'_\beta)_{\beta\in\mathcal{A}'} \quad\text{if}\quad \forall\,\alpha\in\mathcal{A},\,\beta\in\mathcal{A}',\,p\in U_\alpha\cap U'_\beta$$
$$\exists\,W\subset U_\alpha\cap U'_\beta \text{ s.t. } p\in W,\; \rho_{W,U_\alpha}f_\alpha=\rho_{W,U'_\beta}f'_\beta.$$

Whenever $U\subset V$ are open subsets of X, the homomorphisms $\rho_{U,V}$ induce homomorphisms

$$\bar{\rho}_{U,V}:\bar{S}_V\longrightarrow\bar{S}_U,$$

so that $\bar{\mathcal{P}}\equiv\{\bar{S}_X;\bar{\rho}_{U,V}\}$ is a presheaf on X.
(a) Show that if $\mathcal{P}$ is a complete presheaf, then $\bar{\mathcal{P}}$ is isomorphic to $\mathcal{P}$.
(b) Show that $\bar{\mathcal{P}}$ is necessarily a complete presheaf.
(c) If $\mathcal{R}$ is a subsheaf of $\mathcal{S}$, show that

$$\alpha(\mathcal{S}/\mathcal{R})\approx\overline{\alpha(\mathcal{S})/\alpha(\mathcal{R})}.$$

Hint: You may want to use Chapter 5, #2,5 (p216).
Note: The presheaf $\overline{\mathcal{P}}$ is isomorphic to $\alpha(\beta(\mathcal{P}))$, where α and β are as in Subsection 5.6; $\overline{\mathcal{P}}$ contains $\mathcal{P}$ iff $\mathcal{P}$ satisfies the uniqueness property for complete presheafs.

2. We have defined Čech cohomology for sheafs or presheafs of K-modules. All such objects are abelian. The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheafs or presheafs of non-abelian groups as well. The main example of interest is the sheaf $\mathcal{S}$ of germs of smooth (or continuous) functions to a Lie group G.[1] If $\underline{U}=\{U_\alpha\}$ is an open cover, $f\in\check{C}^0(\underline{U};\mathcal{S})$, and $g\in\check{C}^1(\underline{U};\mathcal{S})$, define

$$\mathrm{d}_0f\in\check{C}^1(\underline{U};\mathcal{S}) \qquad\text{by}\qquad (\mathrm{d}_0f)_{\alpha_0\alpha_1}=f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}},$$
$$\mathrm{d}_1g\in\check{C}^2(\underline{U};\mathcal{S}) \quad\text{by}\quad (\mathrm{d}_1g)_{\alpha_0\alpha_1\alpha_2}=g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_2}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}},$$

where for all $\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}$, $f\in\check{C}^0(\underline{U};\mathcal{S})$, $g\in\check{C}^1(\underline{U};\mathcal{S})$, and $h\in\check{C}^2(\underline{U};\mathcal{S})$,

$$f_{\alpha_0}\in\Gamma(U_{\alpha_0};\mathcal{S}),\qquad g_{\alpha_0\alpha_1}\in\Gamma(U_{\alpha_0}\cap U_{\alpha_1};\mathcal{S}),\qquad h_{\alpha_0\alpha_1\alpha_2}\in\Gamma(U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2};\mathcal{S}).$$

Define an action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ by

$$\{f*g\}_{\alpha_0\alpha_1}=f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\in\Gamma(U_{\alpha_0}\cap U_{\alpha_1};\mathcal{S}).$$

(a) Show that under this action $\check{C}^0(\underline{U};\mathcal{S})$ maps $\ker\mathrm{d}_1$ into itself.
(b) Show that for every Čech 1-cocycle g (i.e. $g\in\ker\mathrm{d}_1$) for an open cover $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$,

$$g_{\alpha\alpha}=e|_{U_\alpha},\quad g_{\alpha\beta}g_{\beta\alpha}=e|_{U_\alpha\cap U_\beta},\quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=e|_{U_\alpha\cap U_\beta\cap U_\gamma},\qquad\forall\,\alpha,\beta,\gamma\in\mathcal{A},$$

[1] This means that G is a smooth manifold and a group so that the group operations are smooth. Examples include $O(k)$, $SO(k)$, $U(k)$, $SU(k)$.

where e is the "zero" (or "identity") section of $\mathcal{S}$ (i.e. $e(m)$ is the identity element of the group $\mathcal{S}_m$ for every $m\in M$).
By part (a), we can define

$$\check{H}^0(\underline{U};\mathcal{S}) = \ker \mathrm{d}_0 \qquad \text{and} \qquad \check{H}^1(\underline{U};\mathcal{S}) = \ker \mathrm{d}_1 / \check{C}^0(\underline{U};\mathcal{S}).$$

The first set is a group being the kernel of a group homomorphism. If $\underline{U}' = \{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$, any refining map $\mu\colon \mathcal{A}' \longrightarrow \mathcal{A}$ induces group homomorphisms

$$\mu_p^*\colon \check{C}^p(\underline{U};\mathcal{S}) \longrightarrow \check{C}^p(\underline{U}';\mathcal{S}),$$

which commute with d_0, d_1, and the action of $\check{C}^0(\cdot;\mathcal{S})$ on $\check{C}^1(\cdot;\mathcal{S})$, similarly to Section 5.33. Thus, μ induces a group homomorphism and a map

$$R^0_{\underline{U}',\underline{U}}\colon \check{H}^0(\underline{U};\mathcal{S}) \longrightarrow \check{H}^0(\underline{U}';\mathcal{S}) \qquad \text{and} \qquad R^1_{\underline{U}',\underline{U}}\colon \check{H}^1(\underline{U};\mathcal{S}) \longrightarrow \check{H}^1(\underline{U}';\mathcal{S}).$$

(c) Show that these maps are independent of the choice of μ.
Thus, we can again define $\check{H}^0(M;\mathcal{S})$ and $\check{H}^1(M;\mathcal{S})$ by taking the direct limit of all $\check{H}^0(\underline{U};\mathcal{S})$ and $\check{H}^1(\underline{U};\mathcal{S})$ over open covers of M. The first set is a group, while the second need not be (unless $\mathcal{S}$ is a sheaf of abelian groups). These sets will be denoted by $\check{H}^0(M;G)$ and $\check{H}^1(M;G)$ if $\mathcal{S}$ is the sheaf of germs of smooth (or continuous) functions into a Lie group G. As in the abelian case, $\check{H}^0(M;\mathcal{S})$ is the space of global sections of $\mathcal{S}$.
(d) Show that there is a natural correspondence

$$\big\{\text{isomorphism classes of rank-}k \text{ real vector bundles over } M\big\} \longleftrightarrow \check{H}^1\big(M;O(k)\big).$$

(e) What are the analogues of these statements for complex vector bundles? (state them and indicate the changes in the argument; do not re-write the entire solution).
Hint: For (d) and (e), you might want to look over Sections 3 and 5 in *Notes on Vector Bundles.* Do not forget that $\check{H}^1(M;\mathcal{S})$ is a *direct limit.*

3. (a) Show that the set of isomorphism classes of line bundles on M forms an abelian group under the tensor product (i.e. satisfies 3 properties for a group and another for abelian). Show that in the real case all nontrivial elements are of order two.
(b) Show that the correspondence

$$\big\{\text{isomorphism classes of real line bundles over } M\big\} \longleftrightarrow \check{H}^1(M;\mathbb{Z}_2)$$

of the previous problem is a group isomorphism.
(c) Show that there is a natural group isomorphism

$$\big\{\text{isomorphism classes of complex line bundles over } M\big\} \longleftrightarrow \check{H}^2(M;\mathbb{Z}).$$

Hint: ses/les

Note: The groups $\check{H}^1\big(M;\mathbb{Z}_2\big)$ and $\check{H}^2\big(M;\mathbb{Z}\big)$ are naturally isomorphic to the singular cohomology groups $H^1\big(M;\mathbb{Z}_2\big)$ and $H^2\big(M;\mathbb{Z}\big)$. The image of a real line bundle L

$$w_1(L) \in H^1(M;\mathbb{Z}_2)$$

is *the first Stiefel-Whitney class of* L; the image of a complex line bundle

$$c_1(L) \in H^2(M;\mathbb{Z})$$

is *the first Chern class of* L. However, this is not how these *characteristic classes* are normally defined.