MAT 531: Topology&Geometry, II
Spring 2010

Problem Set 4
Due on Thursday, 2/25, in class

1. Chapter 1, #13ad (p51)

2. Suppose X and Y are smooth vector fields on a manifold M. Show that for every $p \in M$ and $f \in C^\infty(M)$,
$$\lim_{s,t \longrightarrow 0} \frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(p))\big))\big) - f(p)}{s\,t} = [X,Y]_p f \in \mathbb{R}.$$
Do not forget to explain why the limit exists.
Note: This means that the extent to which the flows $\{X_t\}$ of X and $\{Y_s\}$ of Y do not commute (i.e. the rate of change in the "difference" between $Y_s \circ X_t$ and $X_t \circ Y_s$) is measured by $[X,Y]$.

3. Let U and V be the vector fields on $\mathbb{R}^3$ given by
$$U(x,y,z) = \frac{\partial}{\partial x} \qquad \text{and} \qquad V(x,y,z) = F(x,y,z)\frac{\partial}{\partial y} + G(x,y,z)\frac{\partial}{\partial z},$$
where F and G are smooth functions on $\mathbb{R}^3$. Show that there exists a smooth 2-dimensional foliation $\mathcal{F}$ on $\mathbb{R}^3$ such that the vector fields U and V are everywhere tangent to $\mathcal{F}$[1] if and only if
$$F(x,y,z) = f(y,z)\,e^{h(x,y,z)} \qquad \text{and} \qquad G(x,y,z) = g(y,z)\,e^{h(x,y,z)}$$
for some $f,g \in C^\infty(\mathbb{R}^2)$ and $h \in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

4. Chapter 2, #13 (p79)

5. Let $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ be the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$. Show that $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n$ is isomorphic to the line bundle
$$\gamma_n^{*\otimes(n+1)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{n+1},$$
where $\gamma_n \longrightarrow \mathbb{C}P^n$ is the tautological line bundle (isomorphic as complex line bundles).
Hint: There are a number of ways of doing this, including:
   (i) construct an isomorphism between the two line bundles;
   (ii) use Problems PS1-3b, PS2-3, and PS2-5 to determine transition data for $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n$ and $\gamma_k^{*\otimes(n+1)}$. However, you will need to modify trivializations for one of the line bundles to arrive at the same transition data.
   (iii) show that there exists a short exact sequence of vector bundles
$$0 \longrightarrow \mathbb{C}P^n \times \mathbb{C} \longrightarrow (n+1)\gamma_n^* \longrightarrow T\mathbb{C}P^n \longrightarrow 0$$
   and this implies the claim (exact means that at each position the kernel of the outgoing map equals to the image of the incoming map over every point of M.)

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[1] This means that $\mathcal{F}$ is a collection of *embedded* submanifolds of $\mathbb{R}^3$ that partition $\mathbb{R}^3$ such that the tangent bundles of the submanifolds are spanned by U and V.