

# MAT 531: Topology & Geometry, II Spring 2010

## Final Exam

### Instructions

- Give concise proofs, quoting established facts as appropriate; no treatises.
- The problems are worth 20 points each, but are not necessarily of the same difficulty. Parts of a problem may not carry equal weight.
- Your final exam score will be based on your work on 5 problems: 2 (highest-scoring) from Part I, 2 from Part II, and 1 from Part III. The points earned on the bonus problem will be added to your final-exam score, even if the total exceeds 100. However, *NO extra credit will be awarded for solutions to any of the other three problems.*
- Please start each problem on a new sheet of paper. When you are finished, please assemble your solutions in order and attach them to the cover sheet. Do *not* attach this problem sheet.

### Part I (choose 2 problems from 1, 2, and 3)

1. Let  $M$  and  $N$  be compact oriented connected  $n$ -manifolds and  $f: M \rightarrow N$  a smooth map. Show that there exists a unique number  $(\deg f) \in \mathbb{R}$  such that

$$\int_M f^* \omega = (\deg f) \cdot \int_N \omega \quad \forall \omega \in E^n(N).$$

*Note:* there are two parts to this problem.

2. Let  $X$  and  $Y$  be the vector fields on  $\mathbb{R}^3$  given by

$$X = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}, \quad Y = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(a) Compute  $[X, Y]$ .

(b) Is there a coordinate chart  $\varphi = (x_1, x_2, x_3): U \rightarrow \mathbb{R}^3$  on a neighborhood of the origin in  $\mathbb{R}^3$  such that

$$X|_U = \frac{\partial}{\partial x_1}, \quad Y|_U = \frac{\partial}{\partial x_2}?$$

3. Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere centered at the origin.

(a) Why is the line bundle  $\Lambda^2(TS^2) \rightarrow S^2$  trivial?

(b) Describe an explicit isomorphism  $\Lambda^2(TS^2) \rightarrow S^2 \times \mathbb{R}$  of real line bundles over  $S^2$  (give a formula).

**Part II** (choose 2 problems from 4,5, and 6)

4. Show that there exist a closed 1-form  $\alpha$  on  $\mathbb{R}P^n$  and a smooth function  $f: [0, 1] \rightarrow \mathbb{R}P^n$  so that

$$f(0) = f(1) \quad \text{and} \quad \int_{[0,1]} f^* \alpha \neq 0$$

if and only if  $n=1$ .

5. Let  $M = (S^1 \times \mathbb{R}P^n \times \mathbb{R}P^n) / \sim$ , where  $n \in \mathbb{Z}^+$ ,  $(x, y, z) \sim (-x, z, y)$ , and  $S^1$  is viewed as the unit circle in  $\mathbb{C}$  (so  $x \in \mathbb{C}$ ). Show that  $M$  is not orientable and describe the orientable double cover of  $M$ .  
*Hint:* both parts require some care.

6. Let  $M = \mathbb{R}^3 / \sim$ , where

$$(x, y, z) \sim (x+k, y+m, z+ky+n) \quad \forall (x, y, z) \in \mathbb{R}^3, (k, m, n) \in \mathbb{Z}^3.$$

- (a) Show that this is an equivalence relation and  $M$  is a connected compact orientable 3-manifold.  
(b) Determine the de Rham cohomology of  $M$  (as graded vector space).

**Part III** (choose 1 problem from 7 and 8)

7. Let  $T^3 = S^1 \times S^1 \times S^1$  be the 3-torus and  $X$  the complement of two disjoint closed balls in  $T^3$ . The boundary of  $\bar{X}$  consists of two copies of  $S^2$ ,  $S_0$  and  $S_1$ , which inherit an orientation from the removed closed balls (this orientation is *opposite* to the orientation as the boundary of  $\bar{X}$ ). The boundary of  $S^2 \times [0, 1]$  also consists of two copies of  $S^2$ ,  $S^2 \times 0$  and  $S^2 \times 1$ , which inherit an orientation from the standard orientation of  $S^2$  (on  $S^2 \times 0$  this orientation is opposite to its orientation as boundary of  $S^2 \times [0, 1]$ ). Let  $M$  be the smooth 3-manifold obtained by joining  $\bar{X}$  and  $S^2 \times [0, 1]$  along their boundaries so that  $S_i$  is identified with  $S^2 \times i$  by an orientation-preserving diffeomorphism for  $i=0, 1$ . Show that  $M$  is not orientable and determine its de Rham cohomology (as graded vector space).

8. Let  $X$  be a smooth vector field on a manifold  $M$  and define

$$P: \Gamma(M; TM) \rightarrow \Gamma(M; TM) \quad \text{by} \quad P(Y) = [X, Y].$$

- (a) Show that  $P$  is a first-order differential operator.  
(b) What is the symbol of  $P$ ?  
(c) Under what conditions (on  $M$  and/or  $X$ ) is  $P$  elliptic?

**Bonus Problem**

Let  $\gamma_n \rightarrow \mathbb{C}P^n$  be the tautological complex line bundle, where  $n \geq 1$ . Show that for every  $k \in \mathbb{Z}^+$ , the complex line bundles

$$\gamma_n^{\otimes k} \equiv \underbrace{\gamma_n \otimes \dots \otimes \gamma_n}_k \rightarrow \mathbb{C}P^n \quad \text{and} \quad \gamma_n^{\otimes (-k)} \equiv \underbrace{\gamma_n^* \otimes \dots \otimes \gamma_n^*}_k \rightarrow \mathbb{C}P^n$$

are not trivial (not isomorphic to  $\mathbb{C}P^n \times \mathbb{C}$  as complex line bundles over  $\mathbb{C}P^n$ ).

*Hint:* there is a short solution, but it connects several different things encountered in class and thus requires a solid understanding of what is going on.